PRINCIPLES

OF

THE MECHANICS

OF

MACHINERY AND ENGINEERING.

SECTION I.

PHORONOMY; OR, THE PURE MATHEMATICAL SCIENCE OF MOTION.

CHAPTER I.

SIMPLE MOTION.

§ 1. *Rest and Motion.*—Every body occupies a certain position in space; a body is at rest when it does not change its position; and is in motion, on the other hand, when it successively passes from one position into others. The rest and motion of a body are either absolute or relative, according as we refer its position to a space which itself is at rest or in motion, or considered to be in either state.

Upon the earth there is no rest, for all bodies upon the earth share in its motion about the sun, and about its own axis; but if we suppose the earth to be at rest, then all those terrestrial bodies are at rest which, with reference to the earth, do not change their position.

§ 2. *Kinds of Motion.*—The continual succession of positions which a body in its motion gradually occupies, forms a space which is called the trajectory, or path of the moving body. The path of a moving point is a line; that of a geometrical body, is another body; but by this latter is generally understood that line which a certain point of the body, viz. the centre, describes in its motion.

When the path is a straight line, the motion is rectilinear; when a curved line, the motion is curvilinear.

With reference to time, motion is uniform or variable.

§ 3. A motion is uniform, when equal spaces are described by it in equal and arbitrarily small times; and variable, when this equality does not take place. When the spaces, described in equal times, increase continuously with the time, a variable motion is called—accelerated; and when the spaces decrease,—retarded.

Periodic differs from uniform motion in this, that equal spaces are described within certain intervals only, which are called periods.

The apparent diurnal revolution of the fixed stars, and the progressive motion of the hands of a watch are instances of uniform motion. Falling and upwardly projected bodies, the sinking of the surface of water by its flow from vessels are instances of variable motion. The oscillations of a pendulum, the play of the piston of a steam engine, &c., are illustrations of periodic motion.

§ 4. *Uniform Motion.*—Velocity is the rapidity or magnitude of a motion. The greater the space is which a body describes in a given time, the more rapid is its motion, and the greater is its velocity. In uniform motion, the velocity is invariable; and in a variable one, it changes at every instant. The measure of the velocity at any determinate point of time, is the path which a body actually describes, or would describe in a unit of time or second, if from that moment the motion were to become uniform, and the velocity to remain invariable. In general this measure is called simply—velocity.

§ 5. When a body describes the path σ at each particle of time, and a second consists of very many (n) such particles; the path during one second is the velocity, or rather the measure of the velocity:

$$c = n \cdot \sigma.$$

After a time, t (seconds) $n \cdot t$ particles have elapsed, but in each a space σ has been described; the whole space, therefore, which corresponds to the time t is:

$$s = n \cdot t \cdot \sigma = n \cdot \sigma \cdot t, \text{ i.e.}$$

$$\text{I. } s = ct.$$

so that in uniform motion, the space (s) is a product of the velocity (c) and the time (t).

Inversely

$$\text{II. } c = \frac{s}{t} \text{ and III. } t = \frac{s}{c}.$$

Example 1. A locomotive, going with a velocity of 30 feet per second, passes over in 2 hours = 120 min. = 7200 seconds, a space (s) $30 \times 7200 = 216000$ feet.—2. If a time, $3\frac{1}{2}$ minutes = 210 seconds, be required to raise up a ton weight from a shaft 1200 feet deep, its mean velocity (v) must be taken $= \frac{1200}{210} = \frac{40}{7} = 5\frac{5}{7} = 5,714 \ldots$ feet.—3. A horse, moving with a velocity of 6 feet per second, requires to perform 24,000 feet, a time $\frac{24000}{6} = 4000$ seconds.

§ 6. If two different uniform motions be compared with each other, the following is arrived at:

The spaces are $s = c\,t$ and $s_1 = c_1 t_1$, therefore their ratio is $\frac{s}{s_1}$

$= \frac{c\,t}{c_1 t_1}$. If $t_1 = t$, then $\frac{s}{s_1} = \frac{c}{c_1}$, or if $c_1 = c$, then $\frac{s}{s_1} = \frac{t}{t_1}$, lastly if $s_1 = s$, $\frac{c}{c_1} = \frac{t_1}{t}$.

The spaces described in different uniform motions in equal times are to each other as the velocities; the spaces described with equal velocities are as the times; and lastly, the velocities corresponding to equal spaces are inversely proportional to the times.

§ 7. *Uniformly variable Motion.*—A motion is uniformly variable when its velocity either increases or diminishes by a certain amount in equal and arbitrarily small times. It is either uniformly accelerated or uniformly retarded, according as in the first a gradual increase, or in the second a gradual diminution of velocity takes place.

In *vacuo*, the motion of a falling body is uniformly accelerated; were the air to exert no influence upon it, the motion of a body vertically projected would be uniformly retarded.a

§ 8. Any change in the strength or magnitude of the velocity of a body is called acceleration; it is either positive or negative, according as there is increase or diminution of the velocity. The greater this increase or diminution within a given time, the greater is the acceleration. In uniformly variable motion, the acceleration is invariable, and may be measured by the increase or diminution of velocity which takes place in a second of time. In every other motion, the measure of the acceleration is the increase or diminution which a body would acquire if, from the moment in which the acceleration begins, it lose its variability and the motion pass into a uniformly variable one. Diminution of velocity is termed *retardation.*

The measure is very commonly called the *velocity.*

§ 9. If the velocity of a uniformly accelerated motion increase ($\varkappa$) in infinitely small particles of time, and a second of time is made up of such particles, the increment of velocity, or the acceleration, in one second is:

$$p = n\,\varkappa,$$

and the increment after t seconds $= n\,t\,.\,\varkappa = n\,\varkappa\,.\,t = pt$.

If the initial velocity (the moment from which the time is counted) $= c$, the terminal velocity, *i. e.* the velocity acquired after the time (t) is:

$$v = c + pt.$$

For motion, commencing without velocity, $c = 0$, therefore $v = p\,t$, and for uniformly retarded motion, having a negative acceleration (p), $v = c - p\,t$.

Example 1. The acceleration of a body falling freely in vacuo $= 32\,.\,2$ feet; it acquires, therefore, after 3 seconds, a velocity $v = pt = 32\,.\,2 \times 3 = 96\,.\,6$ feet.—2. A sphere rolling down an inclined plane, with an initial velocity $c = 25$ feet, acquires in its course, at each second, 5 feet additional velocity; its velocity, therefore, after $2\frac{1}{2}$ seconds: $v = 25 + 5 \times 2.5 = 25 + 12.5 = 37.5$ feet, &c.; proceeding from the last point uniformly, it will pass over 37.5 feet in every second.—3. A locomotive going with a 30 feet velocity is so retarded, that in each second it loses 3.5 feet of velocity; its acceleration is -3.5; its velocity, therefore, after 6 seconds is $v = 30 - 3.5 \times 6 = 30 - 21 = 9$ feet.

§ 10. *Uniformly accelerated Motion.*—The velocity of every motion may be regarded as invariable within a small particle of time τ. We may, therefore, put the space described in such time $\sigma = v \cdot \tau$, and we obtain the whole space in the finite time t, by the measurement of these small spaces. Now for all these small spaces, the time τ is one and the same; this sum, therefore, may be put equal to the product of these particles of time, and the sum of the velocities corresponding to equal intervals.

In uniformly accelerated motion, the sum $(0 + v)$ of the velocities, in the first and last moment, is as great as the sum $p\tau + (v - p\tau)$ of the velocities in the second and last but one; also, equal to the sum $2p\tau + (v - 2p\tau)$ of the velocities in the third and last but two; and this sum is equal to the terminal velocity v. Here, therefore, the sum of all the velocities is equal to the product $\left(v \cdot \frac{n}{2}\right)$ of the terminal velocity v, and half the number of all the particles of time. The space described is the product $\left(v \cdot \frac{n}{2} \cdot \tau\right)$ of the terminal velocity v, and half the number and magnitude of the particles. Now the magnitude (τ) of such a particle, multiplied by the number, gives the time t; the space, therefore, described in the time t with a uniformly accelerated motion is $s = \frac{vt}{2}$.

The space, therefore, described in uniformly accelerated motion is as in uniform motion when, in the latter case, its velocity is half as great as the terminal velocity of the former.

Example 1. If a body in 10 seconds has acquired a velocity v by uniformly accelerated motion of 26 feet, the space described in that time is $s = \frac{26 \times 10}{2} = 130$ feet.—2. A carriage which, in its accelerated motion, goes over 25 feet in $2\frac{1}{4}$ seconds, proceeds at the end with a velocity $v = \frac{2 \times 25}{2.25} = \frac{50 \times 4}{9} = 22.22$.

§ 11. The two fundamental formulæ of uniformly accelerated motion:

$$\text{I. } v = pt \text{ and II. } s = \frac{vt}{2},$$

which express that the velocity is a product of the acceleration and the time; and the space, half the velocity and the time; include two other principal formulæ which are obtained, if from both equations v be eliminated once, and t twice. It follows that:

$$\text{III. } s = \frac{pt^2}{2} \text{ and IV. } s = \frac{v^2}{2p}.$$

From this, the space described is a product of half the acceleration, and the square of the time; and it is also the quotient of the square of the velocity by twice the acceleration.

These four formulæ give, by inversion, after one or other of the magnitudes contained have been separated, eight other formulæ.

Example 1. A body moving with an acceleration of 15,625 feet, describes in $1\frac{1}{2}$ second a space $\frac{15.625 \times (1.5)^2}{2} = 15.625 \times \frac{9}{8} = 17.578$ feet.—2. A body transported with an acceleration $p = 4.5$ into a velocity $v = 16.5$ feet, has described a space $s = \frac{(16.5)^2}{2.4,5} = 30.25$ feet.

§ 12. By a comparison of two uniformly accelerated motions, we arrive at the following:

The velocities are $v = pt$ and $v_1 = p_1 t_1$, the spaces on the other hand are $s = \frac{pt^2}{2}$ and $s_1 = \frac{p_1 t_1^2}{2}$; from this it follows:

$$\frac{v}{v_1} = \frac{pt}{p_1 t_1} \text{ and } \frac{s}{s_1} = \frac{pt^2}{p_1 t_1^2} = \frac{vt}{v_1 t_1} = \frac{v^2 p_1}{v_1^2 p}.$$

If we put $t_1 = t$, we have $\frac{s}{s_1} = \frac{v}{v_1} = \frac{p}{p_1}$; the spaces described are to each other as the terminal velocities; or, as the accelerations.

If further, we take $p_1 = p$, it gives $\frac{v}{v_1} = \frac{t}{t_1}$ and $\frac{s}{s_1} = \frac{t^2}{t_1^2} = \frac{v^2}{v_1^2}$; so that, in like accelerations, and also in one and the same uniformly accelerated motion, the terminal velocities are proportional to the times and the spaces described to the squares of the times, as also to the squares of the terminal velocities.

Further, if $v_1 = v$ gives $\frac{p}{p_1} = \frac{t_1}{t}$, and $\frac{s}{s_1} = \frac{t}{t_1}$; in equal velocities the accelerations are inversely, and the spaces directly proportional to the times.

Lastly, $s_1 = s$ gives $\frac{p}{p_1} = \frac{t_1^2}{t^2} = \frac{v^2}{v_1^2}$; with equal spaces, the accelerations are inversely as the squares of the times, and directly as the squares of the terminal velocities.

§ 13. For a uniformly accelerated motion commencing with a velocity (c) we have § 9:

$$\text{I. } v = c + pt,$$

and as the space ct belongs to the invariable velocity (c), and the space $\frac{pt^2}{2}$ to the acceleration p:

$$\text{II. } s = ct + \frac{pt^2}{2}.$$

If we eliminate p from both equations, we have:

$$\text{III. } s = \frac{c+v}{2} t,$$

and substituting the value of t,

$$\text{IV. } s = \frac{v^2 - c^2}{2p}.$$

Example 1. A body propelled with an initial velocity $c = 3$ feet, and with an acceleration $p = 5$ feet, describes in 7 seconds, a space $s = 3.7 + 5.\frac{7^2}{2} = 21 + 122.5 =$

143.5 feet.—2. Another body which in 3 minutes $=$ 180 seconds, changes its velocity from $2\frac{1}{2}$ feet into $7\frac{1}{2}$ feet, performs in this time a distance $\frac{2.5 + 7.5}{2} \cdot 180 = 900$ feet.

§ 14. For a uniformly retarded motion with an initial velocity c, these formulæ are applicable:

$$\text{I. } v = c - p\,t,$$

$$\text{II. } s = c\,t - \frac{p\,t^2}{2},$$

$$\text{III. } s = \frac{c + v}{2} \cdot t,$$

$$\text{IV. } s = \frac{c^2 - v^2}{2p},$$

they are derived from the former §, when p is made negative. Whilst in uniformly accelerated motion, the velocity increases without limit, in a uniformly retarded one, the velocity at a certain point of time becomes null, and afterwards negative, *i. e.* it goes on in an inverse direction.

If in the first formula we put $v = 0, p\,t = c$, the time at which the velocity becomes null is, $t = \frac{c}{p}$; if we substitute this value of t in the second equation, we have the space which the body has described at the point of time $= \frac{c^2}{2p}$.

If the time be greater than $\frac{c}{p}$, the space is less than $\frac{c^2}{2p}$; if it be $= \frac{2c}{p}$, the space becomes null, and the body returns to the point from which it set out. If the time be greater than $\frac{2c}{p}$, then s becomes negative, and the body is on the opposite side of its initial point.

Example. A body which rolls up an inclined plane with an initial velocity of 40 ft., by which it suffers a retardation of 8 feet per second, ascends only $\frac{40}{8} = 5$ seconds and $\frac{40^2}{2.8}$ $= 100$ feet in height, then rolls back and returns after 10 seconds with a velocity of 40 feet to its initial point; and after 12 seconds, arrives at a distance $40 \times 12 - 4 \times (12)^2$ $= 96$ feet below this point if the plane extend itself backwards.

§ 15. *Free Descent of Bodies.*—The free or vertical descent of bodies in vacuo, offers the most important example of uniformly accelerated motion. The acceleration of this motion brought about by gravity is designated by the letter g, and has the mean value of:

9,81 metres
30,20 Paris feet.
32,22 English feet.
31,03 Vienna feet.
31,25 Prussian feet.

If either of these values of g be substituted in the formula:

$$v = gt,\ s = g\,\frac{t^2}{2} \text{ and } s = \frac{v^2}{2g},\ v = \sqrt{2gs},$$

all questions, with reference to the free descent of bodies, may be answered. For the English measure:

$$v = 32{,}2a\ t = 8{,}02 \sqrt{s};\ s = 16{,}1a\ t^2 = .0155\ v^2$$
$$\text{and } t = 0{,}031\ v = 0{,}249 \sqrt{s}.$$

Example 1. A body acquires in its free descent of 4 seconds a velocity $v = 32.2 \times 4 = 128.8$ feet, and describes in this time a space $s = 15.625 \times 4^2 = 250$ feet.—2. A body falling from a height $s = 9$ feet, has a velocity $v = 8.02 \sqrt{9} = 24.06$ feet.—3. A body projected vertically with a velocity of 10 feet ascends to a height $s = 0.016 \times 10^2 = 1.6$ feet, and requires for it a time $t = 0.031 \times 10 = 0.3$, or about one-third of a second.

§ 16. The following table will show the relations of the motion to the time in the free descent of bodies.

Time in seconds.	0	1	2	3	4	5	6	7	8	9	10
Velocity.	0	$1g$	$2g$	$3g$	$4g$	$5g$	$6g$	$7g$	$8g$	$9g$	$10g$
Space.	0	$1\frac{g}{2}$	$4\frac{g}{2}$	$9\frac{g}{2}$	$16\frac{g}{2}$	$25\frac{g}{2}$	$36\frac{g}{2}$	$49\frac{g}{2}$	$64\frac{g}{2}$	$81\frac{g}{2}$	$100\frac{g}{2}$
Difference.	0	$1\frac{g}{2}$	$3\frac{g}{2}$	$5\frac{g}{2}$	$7\frac{g}{2}$	$9\frac{g}{2}$	$11\frac{g}{2}$	$13\frac{g}{2}$	$15\frac{g}{2}$	$17\frac{g}{2}$	$19\frac{g}{2}$

The last horizontal column of this table gives the spaces which the freely falling body describes in the single seconds. We see that these spaces are to each other as the odd numbers 1, 3, 5, 7, &c., whilst the times and velocities are as the natural numbers 1, 2, 3, 4, &c., and the spaces fallen through as their squares 1, 4, 9, 16, &c. For example, the velocity after six seconds, is $6g = 193{,}2$ feet, that is, the body would, if it proceeded from this time uniformly upon an horizontal plane, offering no impediment, pass over in each second a space $6g = 193{,}2$ feet. This space it describes in the course of the following and seventh second, but not in reality, for according to the last column it amounts to $13.\frac{g}{2} = 13 \times 16{,}1 = 209{,}3$ feet, in the eighth second it is $15.\frac{g}{2} = 15 . 16{,}1 = 241$ feet, &c.

Remark.—Many writers designate the space of 16 feet, which a body freely descending will describe in one second, by g, and term it properly the acceleration of gravity. They have then for the free descent of bodies, the following formula:

$$v = 2gt = 2\sqrt{g\,s},$$
$$s = gt^2 = \frac{v^2}{4g},$$
$$t = \frac{v}{2g} = \sqrt{\frac{s}{g}}$$

This custom, which is met with in Germany only, is disappearing by degrees, and in consequence of its being frequently misunderstood, and the many mistakes which arise therefrom, this is much to be desired.

§ 17. If the free descent of a body go on with a certain initial velocity (c) the formulæ are of the following kind:

$v = c + gt = c + 32{,}2\,t$; also $v = \sqrt{c^2 + 2\,g\,s} = \sqrt{c^2 + 64{,}4.s}$;

$s = ct + g\frac{t^2}{2} = ct + 16.1\,t^2$, also $s = \frac{v^2 - c^2}{2g} = 0{,}0155\,(v^2 - c^2)$.

If, on the other hand, the body be projected vertically to a height with the velocity c, then:

$v = c - gt = c - 32{,}2\,t$; also $v = \sqrt{c^2 - 2\,g\,s} = \sqrt{c^2 - 64{,}4s}$;

$s = ct - g\frac{t^2}{2} = ct - 16{,}1\,t^2$; also $s = \frac{c^2 - v^2}{2g} = 0{,}0155\,(c^2 - v^2)$.

If we consider a given velocity c as the terminal velocity acquired by a free descent, then the corresponding space fallen through $\frac{c^2}{2g}$ $= 0{,}0155\,.\,c^2$ is called the height due to the velocity. By the introduction of this quantity, some of the foregoing formulæ may be expressed more simply. If the heighte $\left(\frac{c^2}{2g}\right)$ due to the initial velocity c be put $= h$, and that due to the terminal velocitye $\frac{v^2}{2g} = h_1$, we have the following for falling bodies:

$$h_1 = h + s,\ s = h_1 - h;$$

and for ascendinge $\quad h_1 = h - s,\ s = h - h_1$.

The space of fall or ascent is, therefore, equal to the difference of the heights due to velocity.

Example. The velocities are 5 and 11 feet, the heights due to velocitye $= 0{,}0155 \cdot (5)^2 =$ 0,3875 feet, and $0{,}0155\,.\,11^2 = 1{,}875$ feet; the space which is described during the passage from one velocity to the other: $s = 1{,}8755 - 0{,}3875 = 1{,}4880$ feet.

§ 18. From the formulae $s = h - h_1$ it also follows that a body vertically projected has at each point that velocity which it would have, but in an inverse direction, were it to have fallen from the height still remaining to that point, and which it then actually possesses in its following descent.

Example. A body is thrown up with a 15 feet velocity, and strikes in its rise against an elastic impediment, which for the moment throws it back with the same velocity with which it struck. How great then is this velocity, and the time of ascending and descending? To the velocity ($c = 15$ ft.) corresponds the height of ascent $h = 3{,}49$ ft.; the height due to velocity at the moment of impact is $h_1 = 3{,}49 - 2{,}00 = 1{,}49$, and consequently this velocitye $= 8{,}02\sqrt{1{,}49} = 9{,}652$ ft. The time to attain the whole height (3,49 ft.) is $t = 0{,}032.\,v = 0{,}032.\,15 = 0{,}480''$, for the height 1,49 ft. $t_1 = 0{,}032\,.\,10 = 0{,}320''$; there remains then for the time required to rise to the height of 2 ft., or the time from the commencement to impact: $t - t_1 = 0{,}480 - 0{,}320 = 0{,}160''$, and the whole time of rising and falling $= 2\,.\,0{,}160 = 0{,}320''$. This is also but the $\frac{0{,}320}{0{,}960}$th $= 3$d part of the time, which would be necessary for rising and falling if the body were to rise and fall unimpeded. This fall finds its application in the forging of hot iron, because in the gradual cooling of that metal it is desirable that the blows of the hammer follow as quickly as possible in a short time. When the hammer is thrown back by an elastic arrangement, it will give in the same time, in the proportions above laid down, thrice the number of blows to what it would give were its rise umimpeded.

Remark 1. The transformation of the velocity into height due to velocity, and the reverse is very often required in practical mechanics, and especially in hydraulics. A table where this is set down is of great use to the practical man.

Remark 2. The foregoing formulæ are only strictly correct for a free descent in vacuo; they may be used with tolerable accuracy for a fall in air, if the falling bodies have a weight great in proportion to their volume, and if the velocities do not come out very great. For the rest, they are also used under other circumstances and relations in many other descents, as will hereafter be shown.

§ 19. *Variable Motions in Particular.*—For variable motions especially, in which the periodic are also included, the formulæ

$$1.\ \varkappa = p\,\tau, \text{ and}$$
$$2.\ \sigma = v\,\tau$$

hold good:—the increment of velocity ($\varkappa$) acquired in a very small time τ (element of time), is a product of the acceleration p and this time; and the space σ described in the element of time τ is a product of the velocity (v) and the time τ. By inversion:

$$3.\ p = \frac{\varkappa}{\tau} \text{ and}$$

$$4.\ v = \frac{\sigma}{\tau}$$

Acceleration is the quotient of the increment of velocity by the element of the time τ in which it is acquired. Velocity is the ratio of the element of space to that of the time.

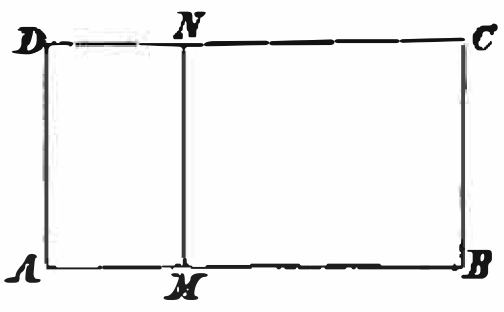
The two last formulæ may be used for the measurement of the acceleration and velocity. *Ex.* From the motion given by the formula $s = at^2$ when a is the space described after the first second, it follows: if t increase by τ and s by σ, $s + \sigma = a\,(t + \tau)^2$. Now $(t + \tau)^2 = t^2 + 2\,t\,\tau + \tau^2$, or because τ is small $= t^2 + 2\,t\,\tau$, it therefore follows $s + \sigma = at^2 + 2\,at\,\tau$, or $\sigma = 2\,at\,\tau$; lastly, $v = \frac{\sigma}{\tau} = 2\,at$. By the same hypotheses, we learn from the last formula $v + \varkappa = 2\,a\,(t + \tau) = 2\,a\,t + 2\,a\,\tau$, so that $\varkappa = 2\,a\,\tau$ and the acceleration $p = \frac{\varkappa}{\tau} = 2\,a$. We have, therefore, in this way found from the formulæ for the spaces, formulæ for the velocity and acceleration.

§ 20. The velocity $c = \frac{s}{t}$ differs from the velocity $v = \frac{\sigma}{\tau}$ of an element of time, and is given when the space, which in a certain time or period of a periodic motion is described, is divided by the time itself. This is called the mean velocity, and may be also regarded as that velocity which a body must have in order to describe uniformly in a given time (t) a given space (s), which, in reality, is described variably. So, for example, in uniformly variable motion, the velocity is equal to half the sum $\left(\frac{c + v}{2}\right)$ of the initial and terminal velocities: for, according to § 13, the space is equal to this sum $\left(\frac{c + v}{2}\right)$ multiplied by the time (t).

While a handle turns uniformly in a circle, the load attached to it, the piston of an air or water-pump, for instance, moves variably up and down; the velocity of this at its lowest and highest point is at a minimum, viz., null; at half the height a maximum, viz., equal to the velocity of the handle. In half a revolution, the mean velocity equals the whole height of ascent, *i. e.*, the diameter of the circle which the handle describes, divided by the time of half a revolution. The diameter $= 2r$ and the time $= t$, then the mean velocity of the load $= \frac{2r}{t}$. The handle in this time describes the semicircle πr; its velocity, therefore, $= \frac{\pi r}{t}$, and consequently the mean velocity of the load $= \frac{2}{\pi} = \frac{2}{3.141} = 0{,}6366$ times as great as the invariable velocity of the handle.

§ 21. *Graphical Representation.*—The laws of motion found above may be expressed by geometrical figures, or, as it is said, graphically represented. Graphical representations especially facilitate the conception, sustain the thoughts, prevent mistakes, and serve not unfrequently for the discovery of a quantity, and on that account are of great use in mechanics.

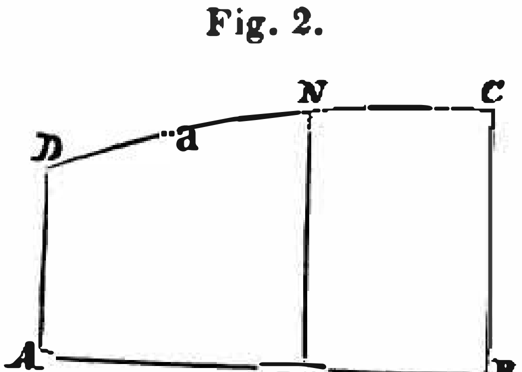
Fig. 1.



In uniform motion the space (s) is the product (ct) of the velocity and the time, and in geometry the area of a rectangular figure is the product of the height and base. We can, therefore, represent the space (s) uniformly described by a rectangle $ABCD$, Fig. 1, whose base AB is the time (t) and whose height ($AD = BC$) is the velocity (c), provided that the time be expressed in the same unit of length as the velocity, and that the second of time and the foot be represented by one and the same line.

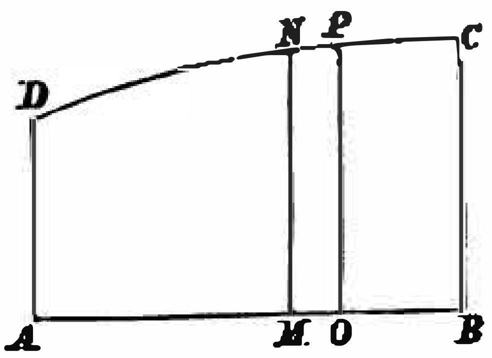
§ 22. Whilst, in uniform motion, the velocity (MN) at any other time (AM) of the motion is one and the same, it differs at every instant in a variable one; this motion, therefore, can only be represented by a quadrilateral figure $ABCD$, Fig. 2, which has AB the time for base, and for the other limits, three lines AD, BC, CD, of which the first two are equal to the initial and terminal velocities, and the last is determined by the extremity (N) of the different velocities at the intervals (M). The line CD is either straight or curved, according to the different kinds of variable motion from the commencement, ascending or descending, or lastly concave or convex towards the base. But in every case the area of this figure must be put equal to the variably described space (s); for each area of space $ABCD$, Fig. 3, may be considered as decom-

Fig. 2.



posable into many small rectangular strips, like *M O P N*, of which each is a product of a part (*M O*) of the base, and its corresponding height (*M N* or *O P*), and the spaces described in a certain time composed of particles of which each is a product of that particle and its corresponding velocity.

Fig. 3.



§ 23. In uniformly variable motion, the increase or diminution ($v-c$) of the velocity ($=pt$, § 13) is proportional to the time. If in the Figures 4 and 5, the line *D E* be drawn

Fig. 4.

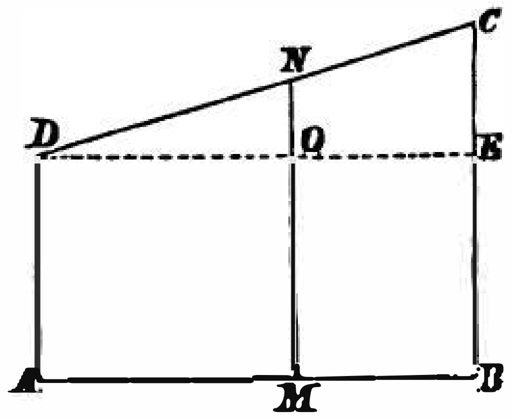
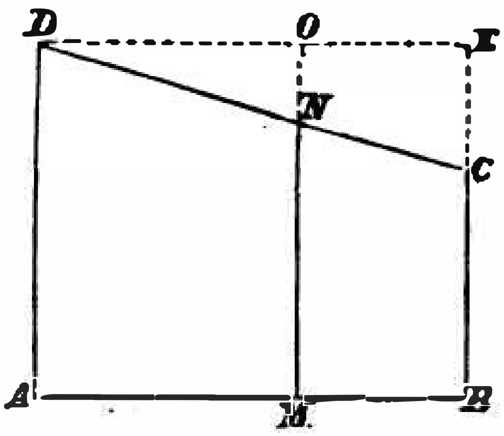


Fig. 5.



parallel to the base *A B*, and *B E* and *M O* = to the initial velocity *A D* be cut off from the lines *M N* and *B C*, there remain the lines *C E* and *N O* for the increase or diminution of the velocity, for which from the above we have the proportion

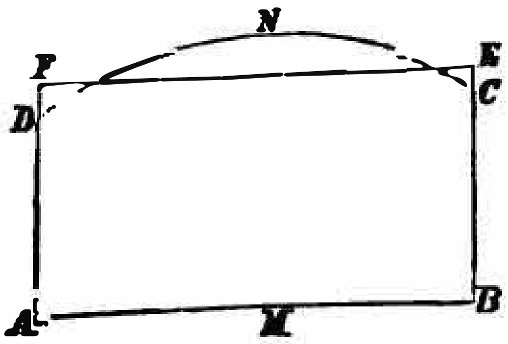
$$N\,O : C\,E = D\,O : D\,E.$$

Such a proportion requires that *N* as well as each point of the line *C D* lie in the straight line connecting *C* and *D*, and also that the line *C D* limiting the different velocities (*M N*) be a right line.

In consequence of this, the uniformly accelerated and uniformly retarded space described may be represented by the area of a trapezium *A B C D*, which has for the height *A B*, the time (t), and for the parallel bases the initial and terminal velocities *A D* and *B C*. The formula of § 13, $s = \frac{c+v}{2} \cdot t$ is in perfect accordance with this. In uniformly accelerated motion, the fourth side *D C* ascends from its initial point, and in uniformly retarded motion descends. In a uniformly accelerated motion beginning with a velocity null, the trapezium becomes a triangle whose area is $\frac{1}{2}\,B\,C \times A\,B = \frac{1}{2}\,ct$.

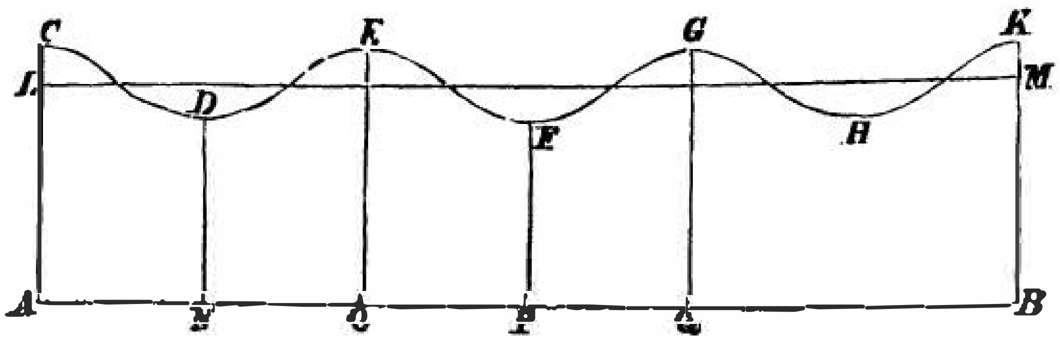
§ 24. The *mean velocity* of a variable motion is the quotient of the *space divided by the time;* multiplied, therefore by the time, it gives as a product the trajectory, and consequently may be also considered as the height *A F* = *B E* of the parallelogram *A B E F* Fig. 6, which has the time (t) for the base *A B*, and an area equal to the four-sided figure *A B C N D* which measures the trajectory or space

Fig. 6.



passed through. The mean velocity is, therefore, likewise obtained by transforming the four-sided figure *A B C N D* into a parallelogram *A B E F* of the same length. Its determination is of importance, particularly in periodic motions, which occur in nearly all machines. The law for these motions is represented by a curved line *C D E F G H K*, Fig. 7. If the line *L M* running parallel

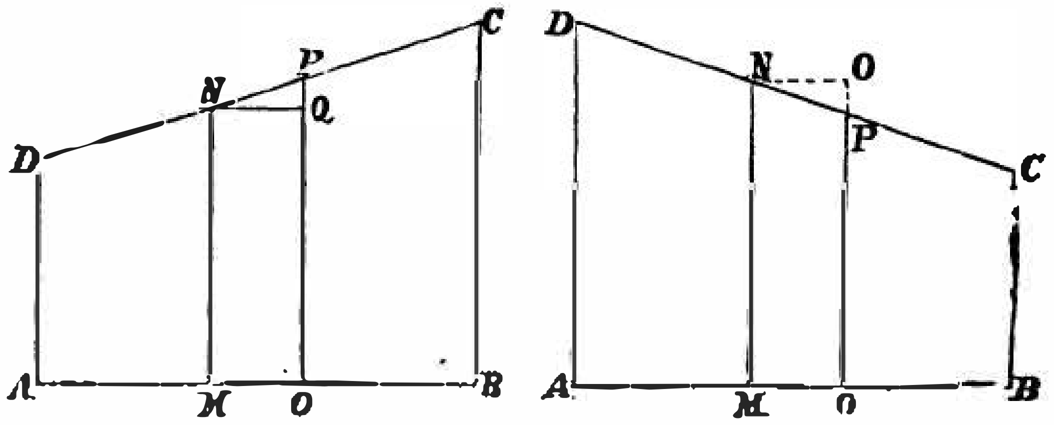
Fig. 7.



with *A B* cuts off the same space as the curved line, and is, as it were, the axis round which *C D E F* . . . . coils itself, then the distance *A L* = *B M* between the two parallel lines *A B* and *L M* is the *mean* velocity of the periodic motion, whilst *A C*, *O E*, *B K*, &c., is the *maximum*, and *N D*, *P F*, &c., the *minimum* velocity of a period *A O*, *O Q*, *Q B*, &c.

§ 25. The acceleration also, or the increase of velocity during a second of time may be easily shown in the figure. In the case of uniformly variable motion it remains unchangeable; it is hence the difference *P Q*, Figs. 8 and 9, between two velocities *O P* and *M N*,

Fig. 8. Fig. 9.



the one of which appertains to a longer time by one second (*M O*) than the other. If the motion is not uniformly variable, and the line of velocity *C D* therefore a curve, then for each second of time (*M*) the acceleration varies, and is, consequently, not the real difference *P Q* between the two velocities *O P* and *M N* = *O Q* Figs. 10 and 11; but it is the increase *R Q* of the velocity *M N*, which would occur if commencing at the moment *M* the motion became uniformly accelerated, and the curved line of velocity *N C* passed into the straight line *N E*. Now the tangent or line of contact *N E*, is that straight line in the direction of which a curve *D N* proceeds, when from a certain point (*N*) it ceases to change its direction; hence the new line of velocity coincides with the tangent, and the perpendicular line *O R* which cuts it, is accordingly the velocity which would take place after the lapse of a second, supposing the motion to have become uniformly accelerated from the commencement of that period, and lastly,

Fig. 10.

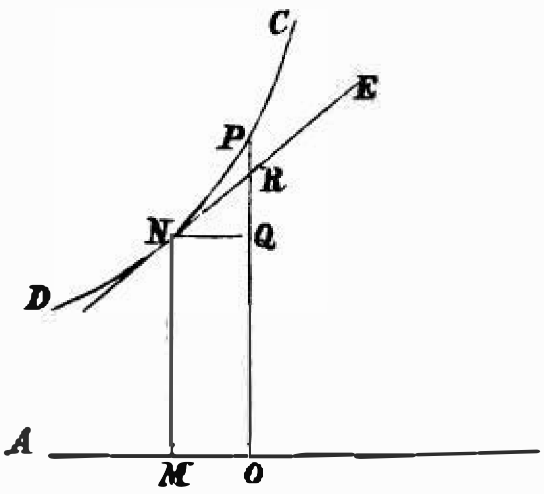
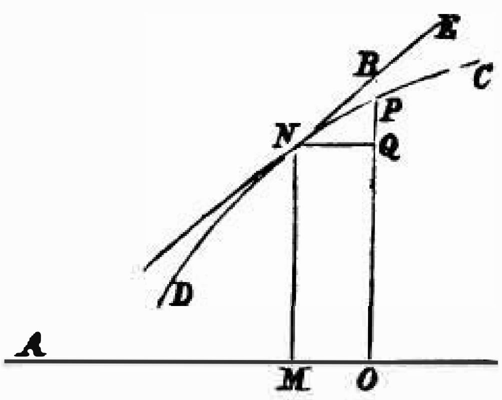


Fig. 11.



the difference *R Q* between this velocity and the primary velocity (*M N*) is the acceleration for that moment which is determined by the point *M* in the time line *A B*.

CHAPTER II.

COMPOUND MOTION.

§ 26. *Compound Motion.*—One and the same body may at the same time have two or more motions; every (relative) motion consists of the motion within a certain space, and of the motion of this space within, or in relation to, a second space. Each point upon the surface of the earth has thus two motions, for it revolves daily once round the axis of the earth, and simultaneously with the earth once yearly round the sun. A person walking on board a ship has two motions in relation to the shore, his own motion and that of the water; water flowing from a hole in the bottom or side of a vessel, whilst the latter is moving along in a carriage, has two motions, the motion from the vessel and the motion with the vessel, &c.

Hence we distinguish *simple* and *compound* motion. Those rectilinear motions are called *simple*, of which other rectilinear or curvilinear motions, consequently called compound, are made up, or may be imagined to be made up.

The combination of several simple motions to form one single motion, and the resolution of a compound motion into several simple motions, will be treated of in the sequel.

§ 27. When simple motions occur in the direction of one and the same straight line, their sum or difference gives the resulting compound motion, the former, when the motions take place in the same direction; the latter, when their directions are opposite. The truth of this axiom becomes directly obvious, when the contemporary spaces of the simple motions are united into one. The contemporary spaces $c_1 t$ and $c_2 t$ correspond with the uniform motions and their velocities c_1 and c_2; if these motions go on in the same direction, then after t seconds the space becomes $s = c_1 t + c_2 t = (c_1 + c_2) t$, and consequently the resulting velocity with which the compound motion proceeds is the sum of the velocities of the simple motions. When the directions of both motions are opposite, then $s = c_1 t - c_2 t = (c_1 - c_2) t$, here, therefore, the resulting velocity is equal to the difference of the simple velocities.

Example 1. To a person moving with a velocity of four feet upon the deck of a ship, in the same direction with the motion of the ship itself, which has a velocity of six feet, the objects on the shore appear to pass by with a velocity of $4 + 6 = 10$ feet.—2. The water which flows from the lateral opening of a vessel with a velocity of 25 feet, whilst the vessel containing it is moved in an opposite direction with a velocity of 10 feet, has, in relation to the other objects at rest, only a velocity of $25 - 10 = 15$ feet.

§ 28. The same relations obtain with variable motions. If one and the same body have, in addition to the primary velocities, c_1 and c_2 the constant accelerations p_1 and p_2, then the corresponding spaces are $c_1 t_1$, $c_2 t_1$, $p_1 \frac{t^2}{2}$, $p_2 \frac{t^2}{2}$, if the velocities and the accelerations are in the same direction, the whole space corresponding to these simple motions, will be:

$$s = (c_1 + c_2) t + (p_1 + p_2) \frac{t^2}{2}.$$

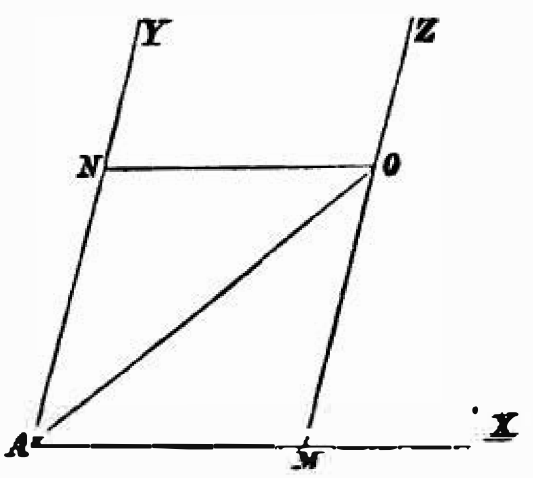
If $c_1 + c_2 = c$ and $p_1 + p_2 = p$, we then obtain $s = ct + p \frac{t^2}{2}$, and it follows, consequently, that not only the velocity of the resulting or compound motion is made up of the sum of the simple velocities, but that also the sum of the accelerations of the simple motions gives the resulting acceleration.

Example. A magnet falls more quickly to the earth than another body, when a mass of iron is immediately below it. The acceleration which the magnet experiences, in consequence of this iron, may be considered invariable when the height from which it falls is small and the mass of iron very considerable, viz., an extensive layer of magnetic iron ore. If this acceleration were 5 feet, then the magnet would fall with an increased velocity of $31{,}25 + 5 = 36{,}25$ feet in the first second, therefore it would fall $18\frac{1}{8}$ feet instead of $15\frac{5}{8}$ feet.

§ 29. *Parallelogram of the Velocities.*—If a body has at the same time two motions differing from each other in direction, it will assume a medium direction between them; and if these motions are of different kinds, viz., the one, uniform, and the other uniformly increasing, then the direction will vary in every part of the motion, and the motion itself become curvilinear.

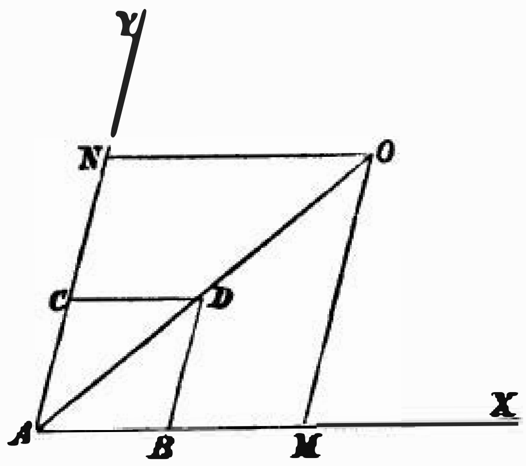
The place O, Fig. 12, which a body moving simultaneously in the directions AX and AY will occupy after a certain time (t), is found when the fourth corner of the parallelogram $A\ M\ O\ N$, determined by the contemporaneous trajectories $A\ M = x$ and $A\ N = y$, as well as by the angle $X\ A\ Y$, or the distance by which the directions of motion deviate from each other, is known. The correctness of this mode of procedure becomes evident when the trajectories x and y are supposed described one after the other, and not at the same time. In compliance with the one motion, the body describes the trajectory $A\ M = x$; and in compliance with the other, the trajectory proceeding from M in the direction of $A\ Y$, therefore in a line $M\ Z$ parallel to $A\ Y$, or the trajectory $A\ N = y$. If $M\ O = A\ N$, then O is the position of the body corresponding to both motions x and y simultaneously, which, in accordance with the construction, is the fourth corner of the parallelogram $A\ M\ O\ N$. We may likewise imagine that the space $A\ M = x$ is passed over in a line $A\ X$, which with all its points proceeds at the same time in the direction $A\ Y$, and therefore carries with it M in a parallel direction to $A\ Y$, and causes this point to perform the trajectory $M\ O = A\ N = y$.

Fig. 12.



§ 30. If both the motions in the directions $A\ X$ and $A\ Y$ take place uniformly and with the velocities c_1 and c_2, then the spaces will become after a certain time (t) : $x = c_1\ t$ and $y = c_2\ t$; their relationship $\frac{y}{x} = \frac{c_2}{c_1}$ is, therefore, the same at all times, a peculiarity which is only proper to the straight line $A\ O$, Fig. 13. Hence it follows that the compound motion proceeds in a straight line. If, with the velocities $A\ B = c_1$ and $A\ C = c_2$, the parallelogram $A\ B\ C\ D$ is constructed, its fourth corner gives the position D, in which the body will be placed after the lapse of one second. But as the resulting motion is rectilinear, it follows that it must always occur in the direction of the diagonal of that parallelogram which is constructed by the velocities. If the trajectory $A\ O$ which is actually passed through in the time t be $= s$, then, on account of the similarity of the triangles $A\ M\ O$ and $A\ B\ D$, we have: $\frac{s}{x} = \frac{A\ D}{A\ B}$, and it consequently follows that this trajectory $s = \frac{x\ .\ A\ D}{A\ B}$ $= \frac{c_1\ t\ .\ A\ D}{c_1} = A\ D\ .\ t$. In accordance with the last equation, the trajectory in the diagonal is proportional to the time (t), the motion itself consequently uniform, and the diagonal $A\ D$ its velocity.

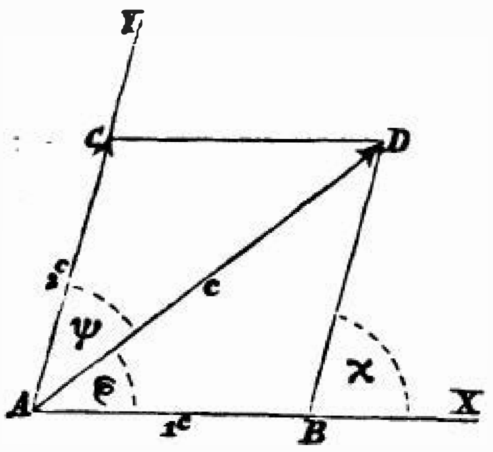
Fig. 13.



The diagonal, therefore, of a parallelogram formed by two velocities, *and the angle which they make with each other, gives the direction and magnitude of the actually resulting motion.* This parallelogram is called the *parallelogram of velocities*, the simple velocities are called the *components*, and the compound velocity the *resultant.*

§ 31. By the use of trigonometrical formulæ, the direction and magnitude of the mean velocity may be ascertained by calculation. The resolution of one of the equal triangles, viz., ABD, of which the parallelogram of velocities $ABDC$ (Fig. 14) is composed, gives the mean velocity $AD = c$ by means of the components $AB = c_1$ and $AC = c_2$, and the angle $BAC = \alpha$ formed by their directions by the formula: $c = \sqrt{c_1^2 + c_2^2 + 2\, c_1 c_2 \cos.\, \alpha}$, and the angle $BAD = \phi$, included by the mean velocity, and the velocity c_1 is expressed by the formula $\sin.\, \phi = \frac{c_2 \sin.\, \alpha}{c}$, or $\tang.\, \phi = \frac{c_2 \sin.\, \alpha}{c_1 + c_2 \cos.\, \alpha}$. If the velocities c_1 and c_2 are equal, and their parallelogram consequently a rhombus, then we obtain in a more simple form, in consequence of the diagonals being at right angles to each other:

Fig. 14.



$$c = 2\, c_1 \cos.\, \tfrac{1}{2}\alpha \text{ and } \phi = \tfrac{1}{2}\alpha.$$

Lastly, if the velocities enclose a right angle, then likewise we obtain more simply:

$$c = \sqrt{c_1^2 + c_2^2} \text{ and } \tang.\, \phi = \frac{c_2}{c_1}.$$

Example 1. The water flowing from a vessel or a machine has a velocity $c_1 = 25$ ft., whilst the vessel is moved with a velocity $c_2 = 19$ feet in a direction forming an angle $a^\circ = 130^\circ$ with the direction of the flowing water. What is the direction and magnitude of the resultant, or as it is also called, the absolute velocity?

The required resulting velocity is $c = \sqrt{25^2 + 19^2 + 2 \,.\, 25 \,.\, 19 \cos.\, 130^\circ} = \sqrt{625 + 361 - 50.19 \cos.\, 50^\circ} = \sqrt{986 - 950 \cos.\, 50^\circ} = \sqrt{986 - 610{,}7} = \sqrt{375{,}3} =$ 19,37 feet.

Moreover, $\sin.\, \phi = \frac{19 \sin.\, 130^\circ}{19{,}37} = 0{,}9808 \sin.\, 50^\circ = 0{,}7513$, and consequently the angle by which the resultant differs from the velocity c_1 $\phi = 48^\circ, 42'$ and the angle which it makes with the direction of motion of the vessel: $\alpha - \phi = 81^\circ 18'$.

2. If the former velocities were acting at right angles to each other, then $\cos.\, \alpha = \cos.\, 90^\circ = 0$, thence the mean velocity $c = \sqrt{986} = 31{,}40$ feet; for its direction we should have $\tang.\, \phi = \frac{19}{25} = 0.76$, and consequently, its deviation from the first velocity: $\phi = 37^\circ{,}14'$.

§ 32. Any given velocity may be supposed to consist of two components, and can consequently be resolved into them, in accordance with certain conditions. If, for instance, the angles $DAB = \phi$ and $DAC = \psi$, Fig. 14, are given, and enclose the velocities required together with the mean velocity $AD = c$, then draw through the terminal point D other lines which represent the degrees corresponding to c, parallel to the directions AX and AY: the points of section will then cut off the required velocities $AB = c_1$ and $AC = c_2$.

Trigonometry expresses these velocities by the formula $c_1 = \frac{c\ sin.\ \psi}{sin.\ (\phi + \psi)}$, $c_2 = \frac{c\ sin.\ \phi}{sin.\ (\phi + \psi)}$. In the usual practical cases, the two velocities are at right angles to each other, and then $\phi + \psi = 90^\circ$, $sin.\ (\phi + \psi) = 1$, and it follows:

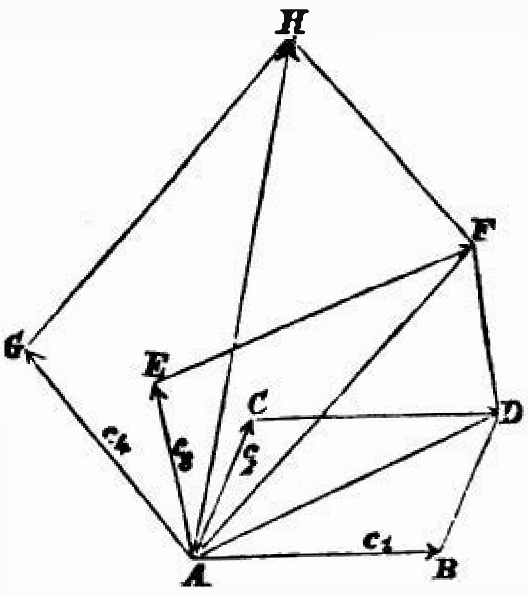
$$c_1 = c\ cos.\ \phi \text{ and } c_2 = c\ sin.\ \phi.$$

Therefore, with one component (c_1) and its angle of direction (ϕ), the direction and magnitude of the other component may be estimated. Lastly, from the velocities c, c_1 and c_2 alone their angles of direction may be determined, as the three angles of a triangle may be computed by the three sides.

Example. Suppose velocity $c = 10$ feet is to be resolved into two components which deviate from its direction by the angle $\phi = 65^\circ$ and $\psi = 70^\circ$. These velocities will be: $c_1 = \frac{10\ sin.\ 70^\circ}{sin.\ 135^\circ} = \frac{9,397}{sin.\ 45^\circ} = 13,29$ feet and $c_2 = \frac{10\ sin.\ 65^\circ}{sin.\ 135^\circ} = \frac{9,063}{0,7071} = 12,81$ ft.

§ 33. *Composition and Resolution of Velocities.*—By repeated application of the parallelogram of velocities, any number of velocities may be reduced to one. By constructing the parallelogram *A B D C*, Fig. 15, the mean velocity *A D* to c_1 and c_2 is obtained; by constructing the parallelogram *A D F E*, we get the mean velocity *A F* to *A D* and *A E* = c_3; and in like manner by constructing the parallelogram *A F H G*, the mean velocity *A H* = c to *A F* and *A G* = c_4 is obtained, and thus the mean of c_1, c_2, c_3, and c_4.

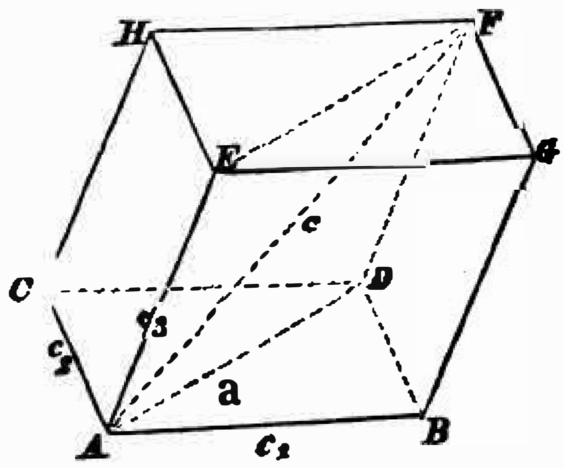
Fig. 15.



The simplest method of obtaining the mean velocity in question, is by the construction of a polygon *A B D F H*, the sides of which *A B*, *B D*, *D F*, and *F H*, are drawn parallel and equal to the given velocities c_1, c_2, c_3, and c_4; the last side *A H* is then always the resultantvelocity.

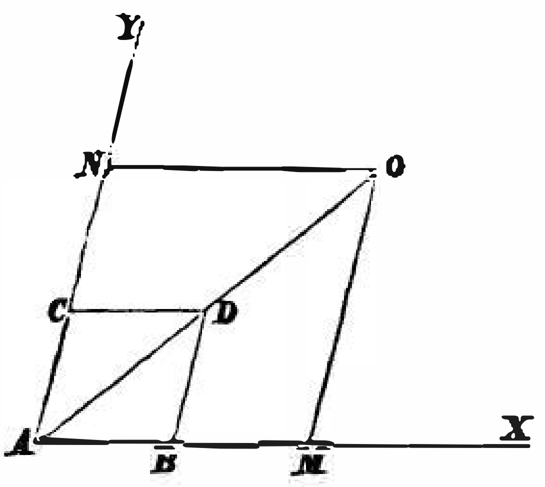
In the case, also, in which the velocities are not in the same plane, the mean velocity may be ascertained by repeated application of the parallelogram of velocities. The mean velocity *A F* = c (Fig. 16) of three velocities *A B* = c_1, *A C* = c_2 and *A E* = c_3, which are not in the same plane, is the diagonal of a parallelepipedon *B C G H*, the sides of which are equal to these velocities. The *parallelepipedon of velocities* is, therefore, also a term in general use.

Fig. 16.



§ 34. *Composition of the Accelerations.*—Two uniformly accelerated motions, beginning with null velocity, produce, when combined, a uniformly accelerated motion in a straight line. If the accelerations of these motions, proceeding in the directions *A X* and *A Y* (Fig. 17) are p_1 and p_2, then, at the close of the time t, the spaces will be $A M = x = \frac{p_1 t^2}{2}$, and $A N = y = \frac{p_2 t^2}{2}$, and their relation $\frac{x}{y} = \frac{p_1 t^2}{p_2 t^2} = \frac{p_1}{p_2}$ is in no way dependent upon the time; and, consequently, the trajectory *A O* of the compound motion is rectilinear. If *A B* is made $= p_1$ and $B D = A C = p_2$, we obtain a parallelogram *A B D C*, which is similar to the parallelogram *AMON*, and from which $\frac{A O}{A D} = \frac{A M}{A B} = \frac{\frac{1}{2} p_1 t^2}{p_1} = \frac{1a}{2} \cdot t^2$; therefore, $A O = \frac{1}{2} A D \cdot t^2$. From this equation it appears that the trajectory *A O* of the compound motion is proportional to the square of the time; the motion itself, therefore, uniformly accelerated, and its acceleration is the diagonal *A D* of the parallelogram constructed by the simple accelerations p_1 and p_2.

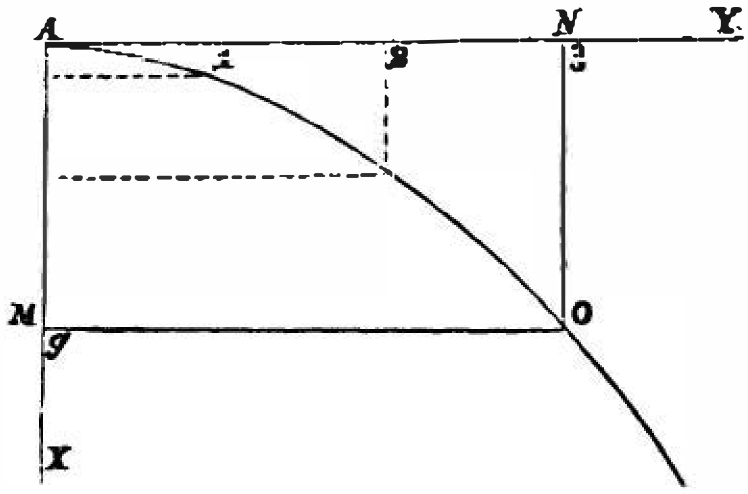
Fig. 17.



In the same manner, therefore, as velocities can be composed or resolved by the parallelogram of velocities, and, according to precisely the same rules, accelerations may be united into one, or broken up into several others by a parallelogram, which is called the *parallelogram of accelerations.*

§ 35. *Combination of Velocity and Acceleration.*—By the *combination* of a *uniform* with a *uniformly accelerated motion*, an entirely variable motion is produced when the directions of the motions do not coincide. In a certain time t, with the velocity c in the direction *A Y*, (Fig. 18,) the trajectory $A N = y = c\, t$ will be described, and in the same time with an unchangeable acceleration, and a direction *A X* at right angles to the former, the trajectory $A M = x = \frac{p t^2}{2}$ will be described, and the body arrives at the terminal point O of the parallelogram composed of $y = c\, t$ and $x = \frac{p t^2}{2}$. With the aid of these formulæ, the position of the body can be determined for any given time, but it is not always in one and the same straight line, for if from the first equation we take $t = \frac{y}{c}$, and place this value in the second, we obtain the equation $x = \frac{p y^2}{2 c^2}$. In accordance with this, the trajectories (x) in the direction of the second motion do not cor-

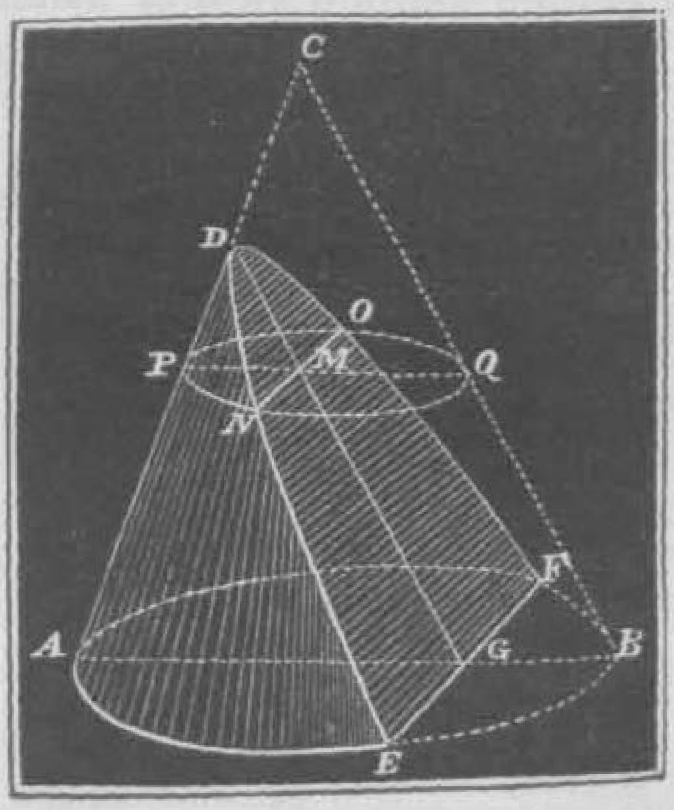
Fig. 18.



respond with those in the first, but with the squares (y^2) of those in the firste and, consequently, the trajectory of the body is not rectilinear, but is a certain curved line, known in geometry by the name of *parabola.*

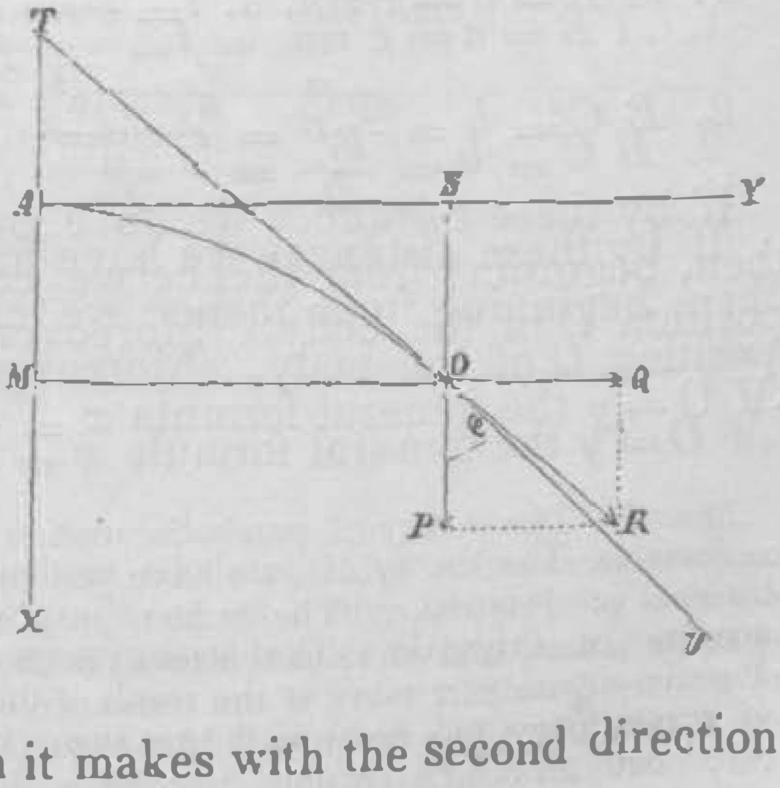
Remark. Let $A\,B\,C$, Fig. 19, be a cone with a circular base $A\,E\,B\,F$, let $D\,E\,F$ be a section of it parallel to the side $B\,C$ and at right angles to the section $A\,B\,C$, and let $O\,P\,N\,Q$ be a second section parallel to the base, and, consequently, also circular. Then let $E\,F$ be the line of section between the base and the first section, and $O\,N$ that between both sections; imagine, then, in the triangular section $A\,B\,C$, the parallel diameters $A\,B$ and $P\,Q$, and in the section $D\,E\,F$, the axis $D\,G$. Then, for the half chord of the circle, $M\,N = M\,O$, the equation applies $M\,N^2 = P\,M \times M\,Q$; but $M\,Q = B\,G$ and for $P\,M$ we have the proportion $P\,M : M\,D = A\,G : D\,G$; hence, it follows $M\,N^2 = B\,G \times \frac{D\,M \times A\,G}{D\,G}$. But, in like manner, $G\,E^2 = B\,G \times A\,G$; if one equation is divided by the other, we obtain, therefore, $\frac{D\,M}{D\,G} = \frac{M\,N^2}{G\,E^2}$; the portions cut off from the axis (*abscissæ*) bear, therefore, the same proportion to each other as the squares of the corresponding perpendiculars (*ordinates*). This law agrees completely with the law for motion found above; this motion, therefore, takes place in a curved line $D\,N\,E$, which can be shown to be a section of the cone (Conic Section).

Fig. 19.



§ 36. *Parabolic Motion.*—In order thoroughly to comprehend motion produced by the combination of velocity and acceleration, we must be able also to indicate the ***direction, velocity*** and ***the space passed through*** during any length of time (t). The velocity parallel to $A\,Y$ is invariable and $= c$, that which is parallel to $A\,X$ is variable and $= pt$. If with these velocities, $O\,Q = c$ and $O\,P = p\,t$, the parallelogram $O\,P\,R\,Q$ is constructed, Fig. 20, we obtain in its diagonal $O\,R$ the mean, or that velocity with which the body at O follows the parabolic curve $A\,O\,U$. This velocity itself is $v = \sqrt{c^2 + pt^2}$.

Fig. 20.



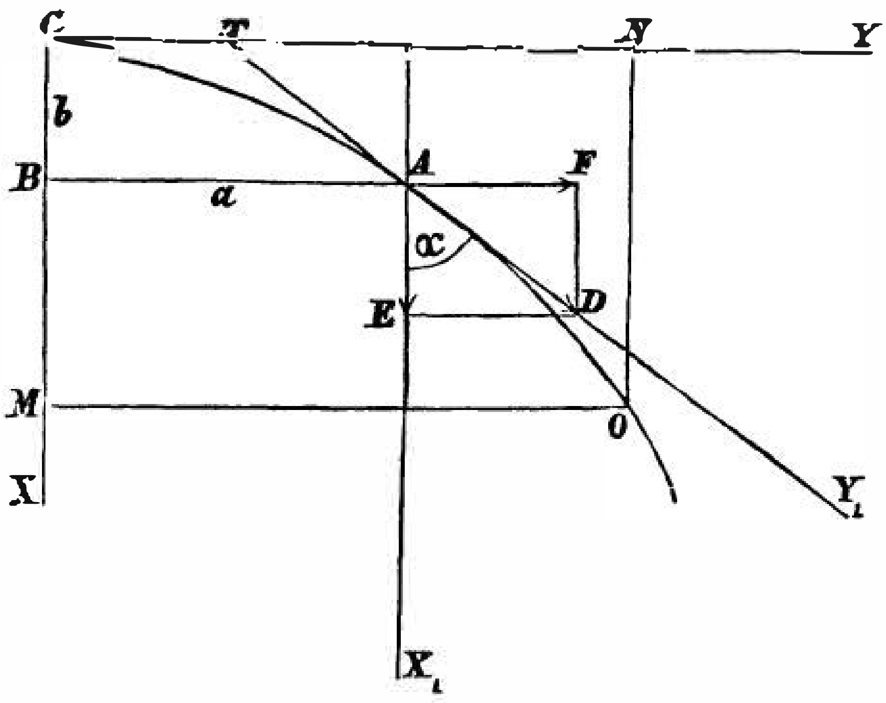
In like manner, $O\,R$ is the tangent or direction in which the body at O proceeds for a single instant, and we obtain for the angle $P\,O\,R = X\,T\,O = \phi$ which it makes with the second direction (axis) $A\,X$, the formula *tang.* $\phi = \frac{O\,Q}{O\,P} = \frac{c}{p\,t}$.

In order, lastly, to find the space passed through, or the curve $A\,O$

$= s$ we can apply the equation $\sigma = v\tau$ (§ 19); according to which we can calculate minute portions of it, which may be considered as its elements. The higher branches of geometry supply us with a complicated formula for calculating the parabolic curve.

§ 37. As yet we have assumed that the primary directions of motion formed a right angle with each other, and we must now study more closely that case in which the direction of the acceleration makes a certain angle with that of the velocity. If the body (Fig. 21) has in the direction $A\ Y_1$ the velocity c, and in the direction $A\ X_1$, which makes with the first the angle $X_1\ A\ Y_1 = \alpha$ the velocity p, then A is no longer the vertex, and $A\ X_1$ no longer the axis, but only the direction of the axis of the parabola. The vertex C is much more dependent upon the co-ordinates $A\ B = a$ and $B\ C = b$, the latter of which coincides with the axis, and the former is at right angles to it, beginning at the commencing point of the motion A. The velocity $A\ D = c$ is made up of the components $A\ F = c\ sin.\ \alpha$ and $AE = c.\ cos.\ \alpha$. The former of these remains always the same, but the latter must be made equal to the variable velocity $p\ t$, supposing that the body has required the time t to move from the vertex C to the real commencing point A.

Fig. 21.



We have, therefore, $c.\ cos.\ \alpha = p\ t$, consequently $t = \frac{c\ cos.\ \alpha}{p}$, and

1. $A\ B = a = c\ sin.\ \alpha.\ t = \frac{c^2 sin. \alpha cos. \alpha}{p} = \frac{c^2 sin. 2\alpha}{2p}$, and

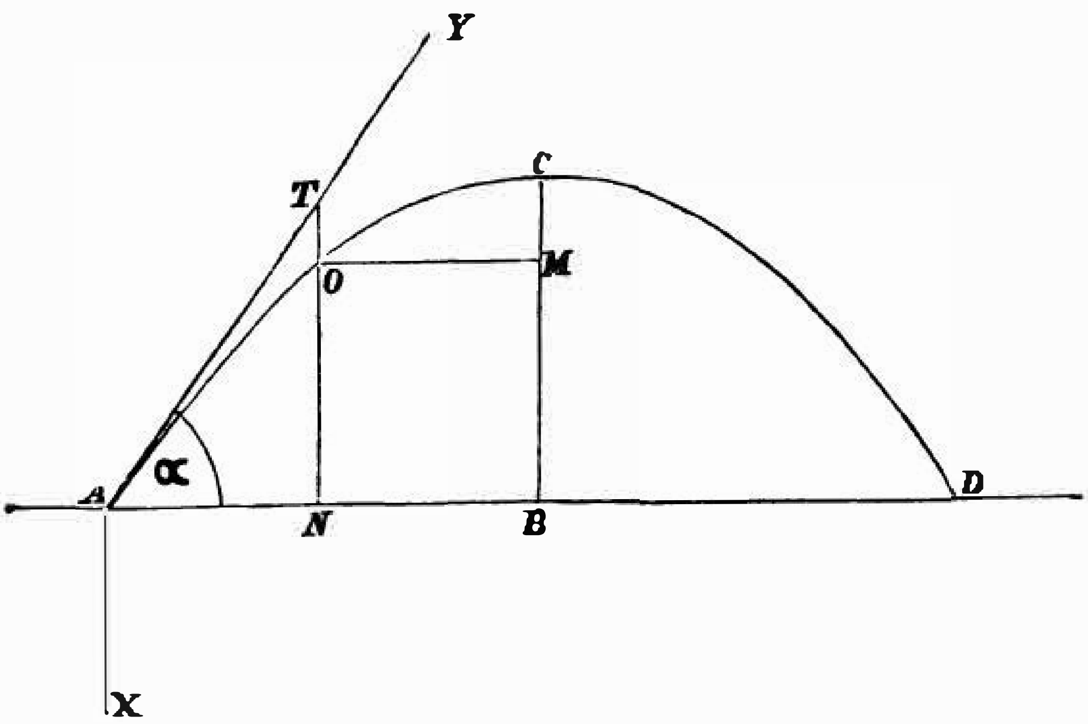
2. $B\ C = b = \frac{p t^2}{2} = \frac{c^2\ cos. \alpha^2}{2p}$.

If by these distances we have found the vertex of the parabola C, then, beginning from thence we can find for any required time the position O of the body. Moreover we have: making $C\ M = x$ and $M\ O = y$ the general formula $x = \frac{py^2}{2\ c^2 sin. \alpha^2}$, also $y = c\ sin.\ \alpha \sqrt{\frac{2x}{p}}$.

Remarks. The theory of parabolic motion produced by an invariable velocity and a constant acceleration, which we have just been considering, finds its application in the doctrine of *Projectiles.* The bodies projected either upwards or downwards would describe a parabolic curve as the result of their primary velocity (c) and the acceleration of gravity ($g = 32.2$ feet), if the resistance of the air could be prevented, or the motion could take place in a vacuum. If the projectile velocity is not great, and the body is very heavy as compared with its volume, then the deviation from the parabola is small enough to be altogether neglected. The most perfect instance of the parabolic course is witnessed in columns of water flowing from vessels or from jets, &c. Bodies shot off, viz., bullets, describe curves which deviate considerably from the parabola in consequence of the great resistance of the air.

§ 38. A body projected at an angle of elevation $Y A D = \alpha$ (Fig. 22), rises to a certain height $B\,C$, which is called the *height of projection*, and it attains the horizontal plane, from which it departed at A,

Fig. 22.



at a distance AD, which is called the *range of projection*. It follows, according to § 37, from the velocity c, the acceleration g, and the angle of elevation, that when p is replaced by g and α^0 by $90^0 + \alpha^0$, therefore $\cos.\ \alpha$ by $\sin.\ \alpha$:

the height of projection is $B\,C = b = \frac{c^2 \sin.\alpha^2}{2g}$, and,

the half of the range of projection $A\,B = a = \frac{c^2 \sin.2\alpha}{2g}$.

On the contrary, the height corresponding to any horizontal distance $A\,N = A\,B - N\,B = a - y$ becomes $N\,O = B\,M = C\,B - C\,M =$

$$b - x = b - \frac{g\,y^2}{2c^2\cos.\alpha^2} = \frac{c^2\sin.\alpha^2}{2\,g} - \frac{g\,y^2}{2c^2\cos.\alpha^2} = h\sin.\alpha^2 - \frac{y^2}{4\,h\cos.\alpha^2},$$

when h represents the height due to velocity $\frac{c^2}{2g}$.

It is evident from the formula for the range of projection, that this will be greatest when $\sin.\ 2\,\alpha = 1$, therefore $2\,\alpha = 90^0$, *i e* $\alpha = 45^0$. A body ascending, therefore, at an angle of elevation of 45^0 attains the greatest range of projection.

Example 1. A jet of water ascending at an angle of elevation of 66° with a velocity of 20 feet, which has therefore a height due to velocity $h = 0{,}016.20^2 = 6{,}4$ feet, attains the height $b = h \sin.\ \alpha^2 = 6{,}4\ (\sin.\ 66^\circ)^2 = 5{,}34$ feet, and has a range of projection $a = 2.\ 6{,}4 \sin.\ 132^\circ = 2.\ 6{,}4 \sin.\ 48^\circ = 9{,}51$ feet. The time which each particle of water requires to perform the whole parabolic curve $A\ C\ D$ is $= \frac{2\ c.\sin\ \alpha}{g} = \frac{2\ .\ 20\ .\sin.66^\circ}{31{,}25} = 1.17$ seconds. The height corresponding to the horizontal distance $A\,N = 3$ feet, is, as $y = \frac{9{,}51}{2} - 3 = 1{,}755$, $NO = 5{,}34 - \frac{1{,}755^2}{4.6{,}4\ (\cos.\ 66^\circ)^2} = 5{,}34 - 0{,}73 = 4{,}61$ feet.

2. The jet of water flowing from a horizontal tube has at a height of $1\frac{3}{4}$ feet a range (half a range of projection) of $5\frac{1}{4}$ feet, what is the velocity of the water?

From the formula $x = \frac{g\,y^2}{2\,c^2} = \frac{y^2}{4\,h}$, it follows $h = \frac{y^2}{4x}$, if x in this case $= 1,75$ and $y = 5,25$, then $h = \frac{5,25^2}{4\,.\,1,75} = 3,937$ feet, and the velocity corresponding to this height is $c = 15,68$ feet.

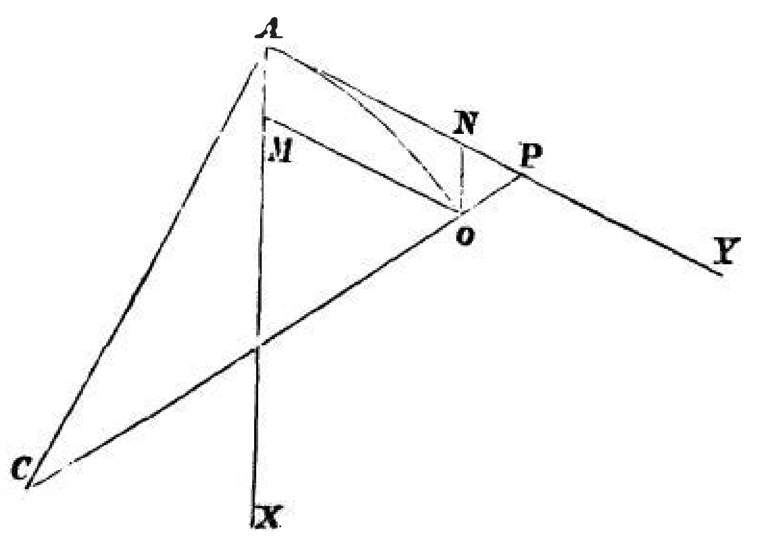
§ 39. *Curved Motions in General.*—By the combination of several velocities and several invariable accelerations, a parabolic motion is likewise produced, for not only the velocities but the accelerations also may be united into a single one; the result is, therefore, the same as if there were only one velocity and one acceleration, *i.e.* only one uniform and one uniformly accelerated motion.

If the accelerations are variable, they can just as well be united into a mean as if they were constant, for it is admissible to consider them invariable within the limits of an infinitely small space of time (τ); and the corresponding motions, therefore, during that space of time, as uniformly accelerated. Of course the resulting acceleration is variable, as are its components themselves. If this resulting acceleration be combined with the given velocity, it is possible to deduce a small parabolic curve, according to which the motion is effected during the small portion of time. If again the velocity and mean acceleration are determined in the same manner for the next small portion of time, we are enabled to obtain a new curve belonging to another parabola; and if this be farther repeated, we at last obtain the whole course.

§ 40. Any minute portion of any curve may be considered as the arc of a circle. The circle to which this arc belongs is called the *circle of curvature*, the radius pertaining to it is the *radius of curvature.* The course of a moving body may, in the same manner, be composed of the arcs of circles, and thus a formula for its radius establishede

Let AM (Fig. 23) be a very small trajectory described with a uniformly accelerated motion $x = \frac{p\tau^2}{2}$ in the direction AX, and let AN be a very small uniformly described trajectory $y = c\tau$, and O the fourth terminating point of the parallelogram constructed from x and y, *i. e.* the point which a body proceeding from A would occupy at the end of the short time (τ). Let AC be drawn at right angles to AY, and let us observe from what point C in this line, a small arc of a circle through A and O can be drawn. On account of the smallness of the arc AO, we may assume that not only CA, but also COP is at right angles to AY; that, therefore, in the small triangle NOP the angle NPO

Fig. 23.



is a right angle. The solution of this triangle gives us $OP = ON$ $sin.\ ONP = AM\,sin.\ XAY = \frac{p\tau^2}{2}\,sin.\ \alpha$, and the tangent $AP =$ $AN + NP = c\tau + \frac{p\tau^2}{2}\cos.\alpha = (c + \frac{p\tau}{2}\cos.\alpha)\tau$, may be made $= c\tau$, because $\frac{p\tau}{2}\cos.\alpha$, on account of the infinitely small factor τ, is inappreciable with respect to c. But now according to the property of the circle $\overline{AP^2} = PO \times (PO + 2\,CO)$, or, as PO vanishes when compared with $2\,CO$, $\overline{AP^2} = PO \times 2\,CO$; we have, therefore, the desired *radius of curvature*,

$$CA = CO = r = \frac{\overline{AP^2}}{2\,PO} = \frac{c^2\tau^2}{p\,\tau^2\,sin.\ \alpha} = \frac{c^2}{p\,sin.\ \alpha}.$$

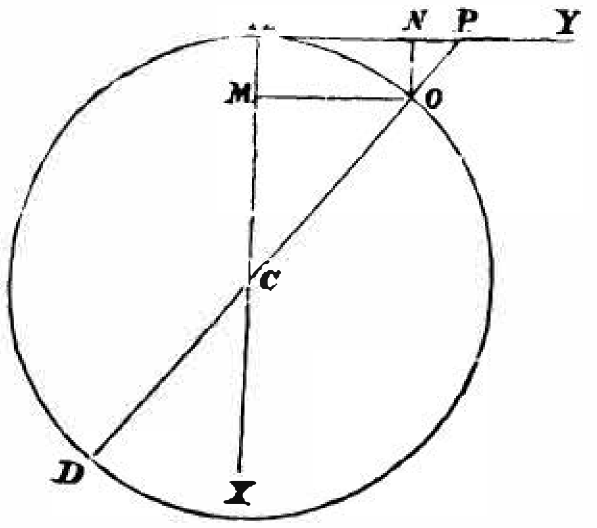
By the aid of the same formula, the radii of curvature of all the elements of curves may be found, when the respective velocities (c) and the acceleration (p) are inserted, and also the angle α which the acceleration makes with the velocity, or with the direction of motion indicated by the line of contact.

Example. For the parabolic path caused by the acceleration of gravity, we have $r = 0{,}031\,\frac{c^2}{sin.\ \alpha}$, and in the vertex of these curves, where $\alpha = 90^\circ$, therefore, $sin.\ \alpha = 1$, it results that $r = 0{,}031\,c^2$. With a velocity of 20 feet, it would therefore be found that $r = 12{,}4$ feet; the farther, however, the body is removed from the vertex, so much the smaller α becomes, and so much the greater, therefore, the radius of curvature.

§ 41. Proceeding from a point A (Fig. 24), where the acceleration is effected at right angles to the direction of motion AY, if, therefore, $\alpha = 90^\circ$, we obtain the radius of curvature $CA = r = \frac{c^2}{p}$, and the velocity at the following point O is composed of c and of $p\,\tau$, hence $v = \sqrt{c^2 + p^2\tau^2} = c + \frac{p^2\tau^2}{2c}$, because τ is infinitely small compared with c. If we make $v = c + \frac{p^2}{2c}\tau\,.\,\tau$, we may then consider $\frac{p^2\tau}{2c}$ as the acceleration, and $\frac{p^2\tau}{2c}\,.\,\tau$ as the corresponding increase of velocity. But as τ is infinitely small, the acceleration $\frac{p^2\tau}{2c}$ becomes also infinitely small, and in one second of time we have an infinitely small increase of velocity, and may therefore consider the motion uniform, and consequently make $v = c$.

If, with the direction of motion, the direction of acceleration also changes, and if these remain constantly at right angles to each other, then we shall always have $v = c$; the velocity of motion, therefore, remains invariably the same as it was at the commencement, namely $= c$. An acceleration such as this, which is always at right angles to the motion, or causes the body to deviate at right angles from the motional direction, is called *normal acceleration*, and we hence know that it alone never causes a change of velocity, but only a deviation from the straight direction. According to the formula above, $r = \frac{c^2}{p}$ we must make the normal acceleration $p = \frac{c^2}{r}$ = *the square of the velocity divided by the respective radius of curvature.*

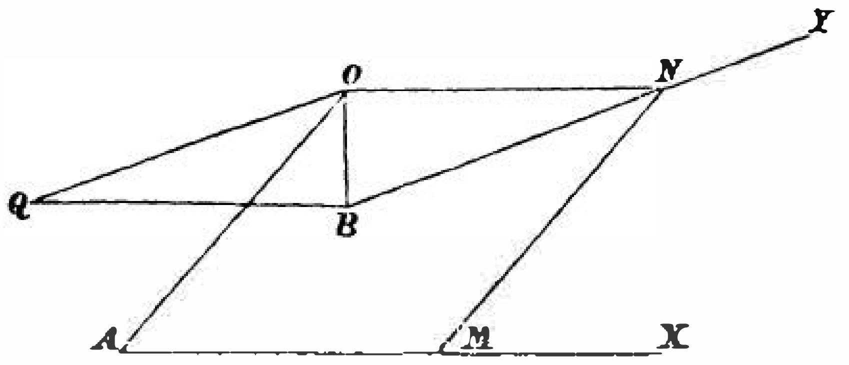
Fig. 24.



In the circle $A\,O\,D$ (Fig. 24) the radius of curvature (r) is the radius of the circle $C A = C O$ itself; hence, when motion occurs in it, the acceleration $p = \frac{c^2}{r}$ is invariable. An invariable acceleration, therefore, which constantly causes the body to deviate at right angles from its motional direction, obliges it to revolve in a circle.

Example. A body which rotates in a circle of 5 feet diameter, in such a manner, that, for each revolution, it requires 5 seconds of time, has a velocity $c = \frac{2\pi r}{t} = \frac{2\pi \cdot 5}{5} = 2 \cdot \pi = 6{,}283$ feet, and a normal acceleration $p = \frac{(6{,}283)^2}{5} = 7{,}896$ feet; viz., in every second it will deviate from a straight line by $\frac{1}{2}\,p = \frac{1}{2} \times 7{,}896 = 3{,}948$ feet.

Fig. 25.



§ 42. In the *simultaneous motions of two bodies*, a constant change is taking place in their relative position, distance, &c., but with the aid of the foregoing formulæ it may be found for any given moment of time.

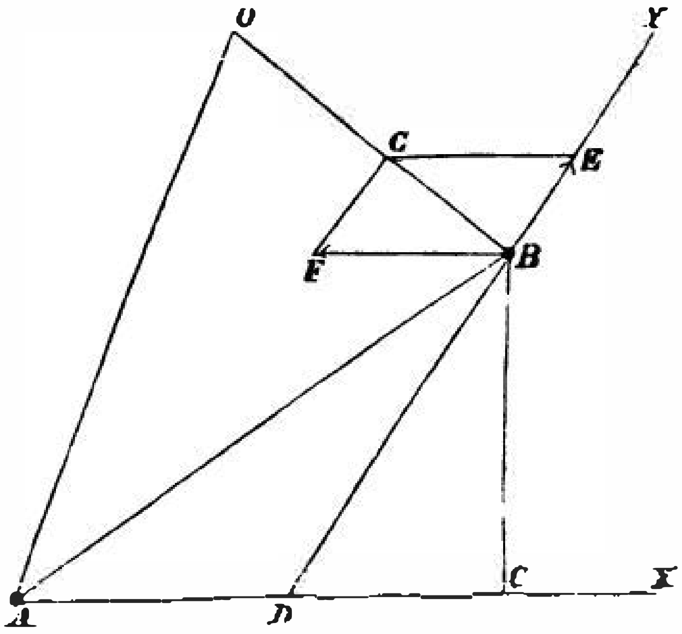
In Fig. 25, let A be the point of application of the one body, B that of the other; the first advances in the direction $A\,X$ in a certain time (t) to M, the second in the direction BY in the same time to N; we then have in this line the relative position and distance of the bodies A and B at the end of this time. If we draw AO parallel with MN, and also make $AO = MN$, then will the line AO likewise give the opposite position of the bodies A and B.

If further we draw ON, we obtain a parallelogram in which ON is also $= AM$. If finally we make BQ parallel and equal to NO, and draw OQ, we have then another parallelogram $BNOQ$, in which one side BN is the absolute path (y) of the second body, and the other side BQ the path (x) of the first body, described in the opposite direction. The fourth corner O is the relative position of the second body, in so far as it is referred to the position of the first body, which is considered as invariable.

The relative position O of a body (B) in motion is also found if we add to the body, besides its own proper motion (BN), that AM of the body (A) to which we refer its position BQ, but in an inverse direction, and then resolve these motions by the parallelogram $BNOQ$ in the usual manner.

§ 43. If the motions of the bodies A and B are uniform, we may substitute for AM and BN the velocities c and c_1, *i. e.* the spaces described in one second. We obtain, therefore, the relative velocity of the one body, when we add to the same in an opposite direction, besides its own absolute velocity, that of the body to which the first velocity is referred. The same relation takes place with the accelerations.

Fig. 26.



Example. A locomotive train sets out upon the line $A\ X$, Fig. 26, from A with a velocity of 35 feet; another simultaneously from B upon the line $B\ Y$, which makes with the former the angle $B\ D\ X = 56^\circ$ with a velocity of 20 feet. If now the initial distances $A\ C = 30{,}000$ feet, and $C\ B = 24{,}000$ feet, how great is the distance $A\ O$ of the two trains at the end of a quarter of an hour? From the absolute velocity $B\ E = c_1 = 20$ feet of the second train, the inverse velocity $B\ F = c = 35$ feet of the first, and the included angle $E\ B\ F = \alpha = 180^\circ - B\ D\ C = 180^\circ - 56^\circ = 124^\circ$. The relative velocity of the second train $B\ G = \sqrt{c^2 + c_1^2 + 2\,c c_1 \cos.\alpha} = \sqrt{35^2 + 20^2 - 2.35.20.\cos. 56^\circ} = \sqrt{1225 + 400 - 1400 \cos. 56^\circ} = \sqrt{1625 - 782{,}9} = \sqrt{842{,}2} = 29{,}02$ feet. For the angle $G\ B\ F = \varphi$, which this makes with the first direction of motion:

$$\sin.\ \varphi = \frac{c_1 \sin. 56^\circ}{29{,}02} = \frac{20.0{,}8290e}{29{,}02};\ Log. \sin.\ \varphi = 0{,}75690 - 1, \text{ hence } \varphi = 34^\circ, 50'.$$

Therefore in 15′ the relative space described is $B\ O = 29{,}02.900 = 26118$ feet, the distance $A\ B = \sqrt{(30000)^2 + (24000)^2} = 38419$ feet. The angle $B\ A\ C = A\ B\ F$, whose tangent $= \frac{24000}{30000} = 0{,}8 = 38^\circ, 40'$, therefore the angle $A\ B\ O = 38^\circ, 40' + \varphi = 38^\circ\ 40' + 34^\circ, 50' = 73^\circ, 30'$, and the distance of the two trains after 15′:

$$A\ O = \sqrt{AB^2 + B\ O^2 - 2\ A\ B\,.\,B\ O \cos.\,A\,B\,O}$$
$$= \sqrt{38419^2 + 26118^2 - 2\,.\,38419\,.\,26118 \cos. 73^\circ, 30'}$$
$$= \sqrt{1588190000} = 39852 \text{ feet.}$$