CHAPTER V.

ON THE RESISTANCES OF FRICTION AND RIGIDITY.

§ 154. We have hitherto assumed that two bodies can only act upon each other by forces at right angles to the plane of contact. If the surfaces at the point of contact were perfectly mathematical, *i. e.* not interrupted by the smallest irregular elevations or depressions, this law would also be fully confirmed by experience; but because everybody possesses a certain degree of elasticity or softness, and because the surface of every body, even if it is smoothed or polished in a high degree, has still some small elevations or indentures, and in consequence of the porosity of matter, no continuity; therefore, by the reciprocal action of two bodies in contact, reciprocal impressions and partial penetration of the parts take place at the point of contact, by which an adhesion of the two bodies is caused, which can only be overcome by a distinct force, whose direction coincides with the plane of contact.

This adhesion, produced by the impression and partial penetration of the bodies in contact, and the resistance on the plane of contact arising from it, has obtained the name of friction. Friction presents itself in the motion of bodies as a passive power or resistance, because it only impedes and checks motion, but never produces nor promotes it. It is introduced into investigations in mechanics as a force which is opposed to every motion, whose direction lies in the plane of contact of two bodies. In whatever direction we move forward a body resting on a horizontal or inclined plane, friction will always act opposite to the direction of motion; for example, it will impede the ascent as much as the descent of a body on an inclined plane. The smallest addition of force produces motion in a system of forces in equilibrium, so long as friction is not called into action; but when the same exerts its effect, a greater addition of force, dependent on the friction, is required to disturb the equilibrium.

§ 155. On overcoming friction, the parts in contact are compressed, and those which protrude, bent down, torn, or broken off, &c. Friction is not only dependent on the roughness or smoothness of the surfaces in contact, but also on the physical properties of the bodies themselves. Hard metals, for instance, cause less friction than soft. We can, however, lay down no general rules *à priori* of the dependence of friction on the physical properties of bodies; it is, on the contrary, necessary to make experiments on friction with bodies of different substances in order to find out the friction which takes place under various circumstances between bodies of the same substance.

The unguents which are applied to the rubbing surfaces exert a particular influence upon the friction and on the abrasions arising from the contact of bodies. The pores are filled up and other asperities diminisbed, and in general, the further penetration of the bodies prevented by the fluid or semi-fluid unguents, such as oil, tallow, fat, soap, &c., for which reason these occasion a considerable diminution of friction.*

Friction must not, however, be confounded with adhesion, *i. e.* with that holding together of two bodies, which takes place when they come into contact at many points without reciprocal pressure. Adhesion increases with the size of the surface in contact, and is independent of the pressure, whilst the contrary is the case with friction. If the pressure be slight, the adhesion will be considerable in proportion to the friction; but if the pressure be considerable, then it will constitute but a small part of the friction, and, therefore, generally may be neglected. Unguents, like all fluid bodies, increase the adhesion, because they increase the number of the points of contact.

§ 156. *Kinds of Friction.*—Two kinds of friction are distinguishable, viz., the rolling and the sliding. Sliding friction is that kind of resistance which is given out when a body so moves that all its points describe parallel lines. Rolling friction, on the other hand, is that resistance which arises from rolling, *i. e.* that motion of a body which moves progressively and rolls at the same time, and whose point of contact describes as great a space upon the body in motion as upon the body at rest. A body M supporting itself upon the plane HR, Fig. 158, for instance, moves sliding over the plane, and consequently has to overcome sliding friction when its points, A, B, C describe parallel spaces AA_1, BB_1, CC_1, &c., and therefore all these

Fig. 158.

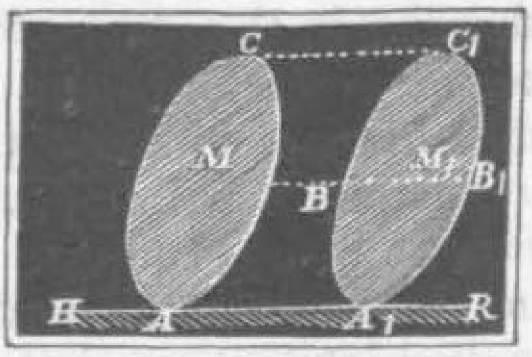
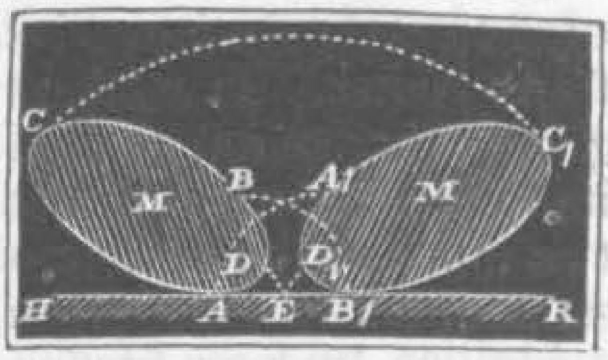


Fig. 159.



points of the moving body come into contact with others of the support. The body M, Fig. 159, on the other hand, rolls upon the plane HR, and has to overcome rolling friction, when the points A, B, &c., of the surface so move that the space $AB_1 = AB = A_1B_1$, likewise $AD = AE$, and $B_1E = B_1D_1$, &c.

The friction of axles is a particular kind of sliding friction, which arises when a cylindrical axle revolves in its bearing. We distinguish two kinds of axles, the gudgeon and the pivot. The gudgeon rubs against its support or envelop, wbilst its other points always successively come into contact with the same points of the support.

* The "anti-attrition metal," composed of copper 1 lb., antimony 2 lbs., and tin 3 lbs., (afterwards tempered or softened by re-melting with more tin,) now very generally used by machinists, in the United States, performs the same office, and prevents the heating of gudgeons, boxes, &c.—AM. ED.

The pivot, on the other hand, presses with its circular base against its support, where its points revolve in concentric circles.

Further, particular frictions arise when a body oscillates upon a sharp edge, as in the balance, or when a vibrating body reposes upon a point, as in the magnetic needle.

Lastly, we distinguish the friction of quiescence which is to be overcome, when a body at rest is put into motion, from the friction of motion which opposes itself to the transmission of motion.

§ 157. *Laws of Friction.*—The general laws to which friction is subject, are the following:

1. Friction is proportional to the normal pressure between the rubbing bodies. If a body be pressed against another by a double force, the friction is as great again; three times the pressure gives three times the friction. If in small pressures this law varies from observation, it must be attributed to the proportionately greater effect of adhesion.

2. Friction is independent of the extent of the surfaces of contact. The greater the surfaces are, the greater is the number of parts which rub against each other; the smaller the pressure, the less the friction of each part; the sum of the frictions of all the parts is the same for a greater as for a less surface, in so far as the pressure and the other circumstances remain the same. If the side surfaces of a parallelopipedical brick are of the same quality, the force necessary to push it along a horizontal plane is the same, whether it rest upon the least, the mean, or the greatest surface. With very large side surfaces and with small pressures, this law has exceptions, in consequence of the effect of adhesion.

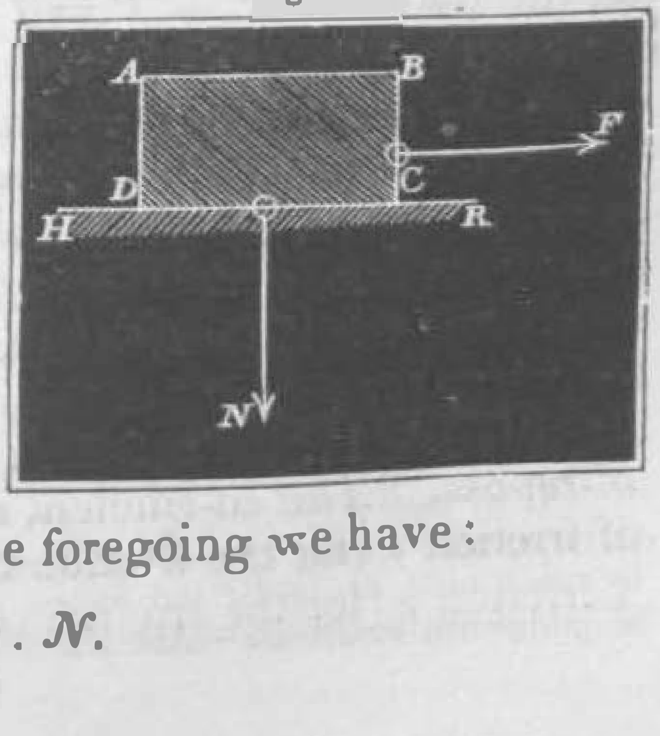
3. The friction of quiescence is indeed generally greater than that of motion; the last, however, is independent of the velocity; it is the same in small as in great velocities.

4. The friction of greased surfaces is generally less than that of ungreased, and depends less on the rubbing bodies than on the unguents.

5. The friction of gudgeons revolving on their bearings is less than the common sliding friction; the friction of rolling is in most cases so small, that it need hardly be taken into account in comparison with the sliding friction.

§ 158. *Co-efficient of Friction.*—From the first law laid down in the former paragraph, the following may be deduced. A body AC, Fig. 160, presses against its support, first with the force N, and requires to draw it along, *i. e.* to overcome its friction, the exertion of a certain force F, and secondly with the force N_1, and requires the force F_1 to cause it to pass from a state of rest into one of motion. From the foregoing we have:

Fig. 160.



$$\frac{F}{F_1} = \frac{N}{N_1}, \text{ and therefore } F = \frac{F_1}{N_1} \cdot N.$$

If, by experiment, we have found the friction F_1 corresponding to a certain pressure N_1, we hence find, if the rubbing bodies, and the other circumstances are the same, the friction F corresponding to another pressure N when we multiply this pressure by the ratio $\left(\frac{F_1}{N_1}\right)$ of the values F_1 and N_1 corresponding to the first observation.

The ratio of the friction to the pressure, or the friction for a pressure = unity, a pound, for instance, is called the *co-efficient of friction*, and will in the sequel be expressed by f, wherefore we may generally put $F = f \, . \, N$.

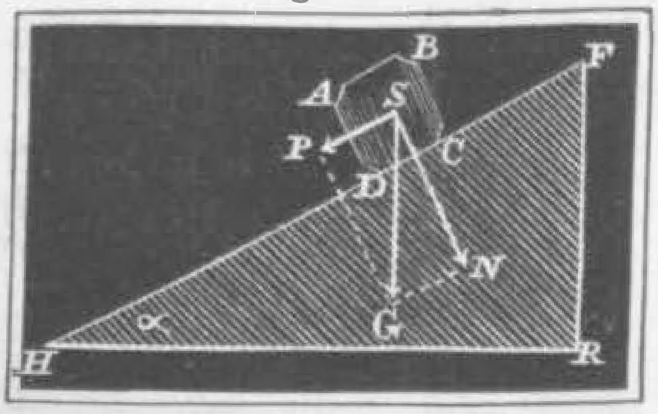
The co-efficient of friction is different for different substances and different conditions of friction, and must therefore be found out by experiment for each particular case.

When a body AC is drawn a distance s over a surface, there is a mechanical effect Fs to perform; the mechanical effect or work required to overcome friction is, therefore, fNs, equal to the product of the co-efficient of friction, the normal pressure, and the distance along the plane of contact. When the plane is also moving, we must then understand by s the relative distance.

Example.—1. If by a pressure of 260 lbs., the friction amounts to 91 lbs., the corresponding co-efficient of friction is $f = \frac{91}{260} = \frac{7}{20} = 0{,}35$.—2. To draw a sledge of 500 lbs. weight along a horizontal and very smooth surface of snow, the co-efficient of friction is $f = 0{,}04$, the required force $F = 0{,}04 \, . \, 500 = 20$ lbs.—3. If the co-efficient of friction of a cart drawn over a paved road is 0,45 and the load amounts to 500 lbs., the mechanical effect required to draw it 480 feet is $= f Ns = 0{,}45 \, . \, 500 \, . \, 480 = 108000$ ft. lbs.

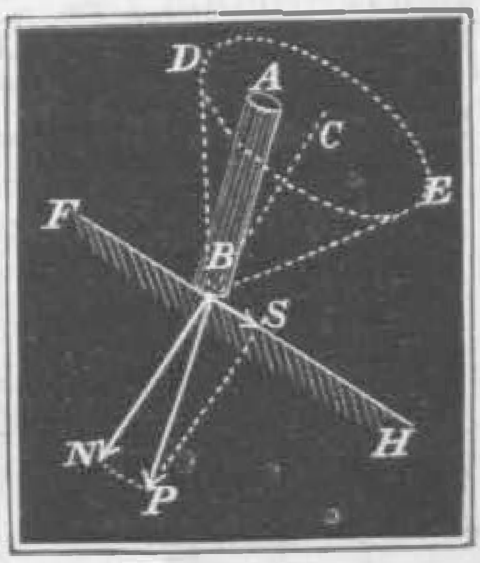
§ 159. *The Angle of Friction and the Cone of Friction.*—A body AC, Fig. 161, lies on an inclined plane FH, whose angle of inclination $FHR = \alpha$, its weight G resolves itself into the normal pressure $N = G \cos. \alpha$ and into the parallel force $P = G \sin. \alpha$. From the first force there arises the friction $F = f G \cos. \alpha$, which is opposed to every motion upon the plane, wherefore the force to push it upwards on the plane $= F + P = f G \cos. \alpha + G \sin. \alpha = (\sin. \alpha + f \cos. \alpha) G$, on the other hand, the force to push it downwards is $= F - P = (f \cos. \alpha - \sin. \alpha) G$; the last force is null, *i. e.* the body is sustained by its friction on the plane, when $\sin. \alpha = f \cos. \alpha$, *i. e.* when the $\tang. \alpha = f$. As long as the inclined plane has an angle of inclination, whose tangent is less than f, the body remains at rest on the plane, but when the tangent of this angle is a little greater than f, the body immediately begins to slide down. The angle, whose tangent is equal to the co-efficient of friction, is called the angle of friction or the angle of repose. The co-efficient of friction is given by observing the angle of friction ρ (for the friction of repose), when f is put $= \tang. \rho$.

Fig. 161.



In consequence of friction, the surface FH, Fig. 162, reacts not only against the normal pressure N of another body AB; but also against its oblique pressure P, when the deviation $NBP = \phi$ of the direction of this pressure from the normal BN does not exeeed the angle of friction, for since the force P gives the normal pressure $BN = P \cdot cos. \phi$, and the lateral or tangential pressure $BS = S = P\, sin. \phi$, and there arises from the normal pressure $P\, cos. \phi$ the friction $f\, P\, cos. \phi$ opposed to every motion in the plane FH, S will therefore be unable to give rise to motion, and will remain in equilibrium so long as $f\, P\, cos. \phi > P\, sin. \phi$, or $f\, cos. \phi > sin. \phi$, *i. e.* $tang. \phi$ is $< f$, or $\phi < \rho$. If the angle of repose $CBD = \rho$ be made to revolve about the normal CB, it will describe a cone, which we may call the cone of friction or resistance. The cone of resistance includes all those directions of force by which a perfect counteraction of the oblique pressure takes place.

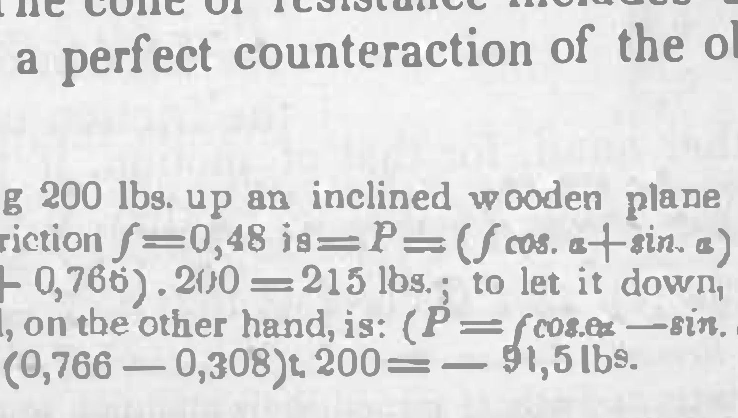
Fig. 162.



Example. To draw a filled cask weighing 200 lbs. up an inclined wooden plane of 50°, the force required with a co-efficient of friction $f = 0{,}48$ is $= P = (f\, cos.\, a + sin.\, a)\, G = (0{,}48\, cos.\, 50° + sin.\, 50°) \,.\, 200 = (0{,}308 + 0{,}766) \,.\, 200 = 215$ lbs.; to let it down, or to prevent its sliding down, the force required, on the other hand, is: $(P = f\, cos. a - sin.\, a)\, G = -(sin.\, 50° - 0{,}48 \,.\, cos.\, 50°)\, 200 = -(0{,}766 - 0{,}308) \,.\, 200 = -91{,}5$ lbs.

§ 160. *Experiments on Friction.*—Experiments on friction have been made by many philosophers, the most extensive of which, and on the greatest scale, are those of *Coulomb* and *Morin.* To find out the co-efficients of friction for sliding motion, these two made use of a sledge sliding on a horizontal surface, which was pulled forward by a cord, passing over a fixed pulley, from which weights were suspended, as in Fig. 163, where AB represents the way, CD the sledge, E the pulley, and G the weight. To obtain the co-efficient of friction for different substances, the surfaces in contact, not only of the sledge, but also of the way forming the support, were covered with the smoothest possible pieces of the substances under experiment, such as wood, iron, &c. &c. The co-efficients of the friction of repose were given by the weight which was necessary to cause the sledge to pass from a state of rest into one of motion, and the co-efficient of the friction of motion by the time t, which the sledge required to pass over a certain space s. If G be the weight of the sledge, and P the weight required to draw it, we have the friction $= f\, G$, the motive force $= P - f\, G$, and the mass $M = \frac{P + G}{g}$, it therefore follows from § 65, that the acceleration of the uniformly accele-

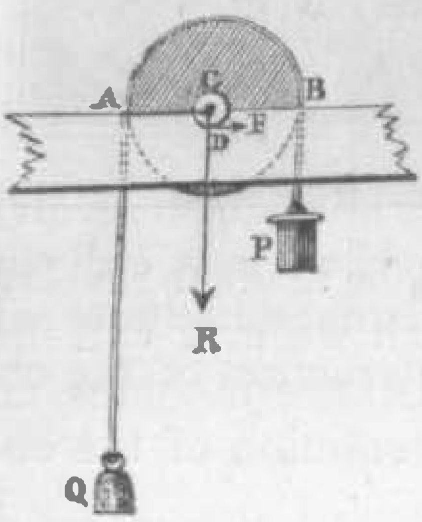
Fig. 163.



rated motion arising, is : $p = \frac{P - f G}{P + G} g$, and inversely, the co-efficient of friction $f = \frac{P}{G} - \frac{P + G}{G} \cdot \frac{p}{g}$. But $s = \frac{1}{2} p t^2$ (§ 11), therefore, $p = \frac{2 s}{t^2}$, and $f = \frac{P}{G} - \frac{P + G}{G} \cdot \frac{2 s}{g t^2}$.

To measure the co-efficient of friction, for axle friction, a fixed pulley ACB, Fig. 164, is made use of, over which a cord passes, which is stretched by the weights P and Q. From the sum of the weights, the pressure $P + Q$ is given, and from their difference $P - Q$ the force at the circumference of the pulley, which is in equilibrium with the friction of the axle, $F = f(P + Q)$, if now $CA = a$ the radius of the pulley, and $CD = r$ that of the axle, we have from the equality of moments $(P - Q)\, a = Fr = f(P + Q)\, r$, and therefore the friction of repose; $f = \frac{P - Q}{P + Q} \cdot \frac{a}{r}$, on the other hand, for that of motion, if the weight P falls a space s in the time (t), and Q rises as much, $f = \left(\frac{P - Q}{P + Q} - \frac{2 s}{g t^2}\right) \cdot \frac{a}{r}$.

Fig. 164.



Remark. Before Coulomb, Amontons, Camus, Bülffinger, Muschenbroek, Ferguson, Vince, and others turned their attention to and made experiments on friction. The results of all these investigations are of little value in practice, because they were conducted upon too small a scale. The experiments of Ximenes, which were made about the same time as those of Coulomb, also fail in this respect. The results are to be found in a work, "Teoria e Pratica delle resistenze de' solidi ne' loro attriti," Pisa, 1782. The experiments of Coulomb are fully described in his work, "Théorie des Machines simples," 1821. The latest experiments upon friction are those of Rennie and Morin. Rennie used for his experiments partly, a sledge upon a horizontal surface, and partly upon an inclined plane, from which the bodies were allowed to slide down, and by which the amount of the friction was deduced from the angle of friction. Rennie's experiments extend to substances of various kinds met with in practice, as cloth, leather, wood, stones, and metals; they give important results upon the abrasion of bodies, but from the apparatus and the mode of conducting these experiments, we cannot rely upon them for that accuracy which those of Morin appear to have attained. The experiments of Rennie are to be found in the "Philosophical Transactions" of 1818. The most extensive experiments, and promising a high degree of accuracy, have been completed by Morin, although it cannot be denied that they leave some doubts and uncertainties, and somewhat to be desired. This is not the place to describe the methods and apparatus of these experiments; we can only refer to the author's writings, "Nouvelles Expériences sur le Frottement," par Morin. An excellent article on Friction, and a full description of all the experiments upon it, especially those of Morin, is given by Brix in the "Transactions of the Society for the Promotion of Manufacturing Industry in Prussia," 16 and 17 Jahrgang, Berlin, 1837–8.*

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* A series of experiments on the resistance of friction, particularly as applied to railway cars, will be found in "Wood's Treatise on Railways," 2d ed., 1832, chap. 6; Vid. Smith's Am. ed., pp. 171–228.—Am. Ed.

§ 161. The following tables contain a condensed summary of the *co-efficients of friction* the most useful in practice.

TABLE I.

CO-EFFICIENTS OF THE FRICTION OF REPOSE.

NAMES OF BODIES.		Nature of the surfaces and unguents.							
		Dry.	Damped with water.	With olive oil.	Lard.	Tallow.	Dry soap.	Polished and greasy.	Greasy and wetted.
Wood upon wood -	least,	0,30	0,65	—	—	0,14	0,22	0,30	—
	mean,	0,50	0,68	—	0,21	0,19	0,36	0,35	—
	greatest,	0,70	0,71	—	—	0,25	0,44	0,40	—
	values,								
Metal upon metal -	least,	0,15	—	0,11					
	mean,	0,18	—	0,12	0,10	0,11	—	0,15	—
	greatest,	0,24	—	0,16					
	values,								
Wood upon metal - - - - - - - - -		0,60	0,65	0,10	0,12	0,12	—	0,10	—
Hempen ropes, twisted or matted, upon wood	least,	0,50							
	mean,	0,63	0,87						
	greatest,	0,80							
	values,								
Thick sole leather, upon wood or iron	high at the edges,	0,43	0,62	0,12	—	—	—	—	0,27
	flat or smooth,	0,62	0,80	0,13					
Black strap leather, upon pulleys	of wood,	0,47	—	—	—	—	—	0,28	0,38
	of iron,	0,54							
Stones or bricks upon stones or bricks, smooth worked	least,	0,67							
	greatest,	0,75							
	value,								
Stones and wrought iron	least,	0,42							
	greatest,	0,49							
	value,								
Oak upon muschekalk - - - - - - - -		0,64							

TABLE II.

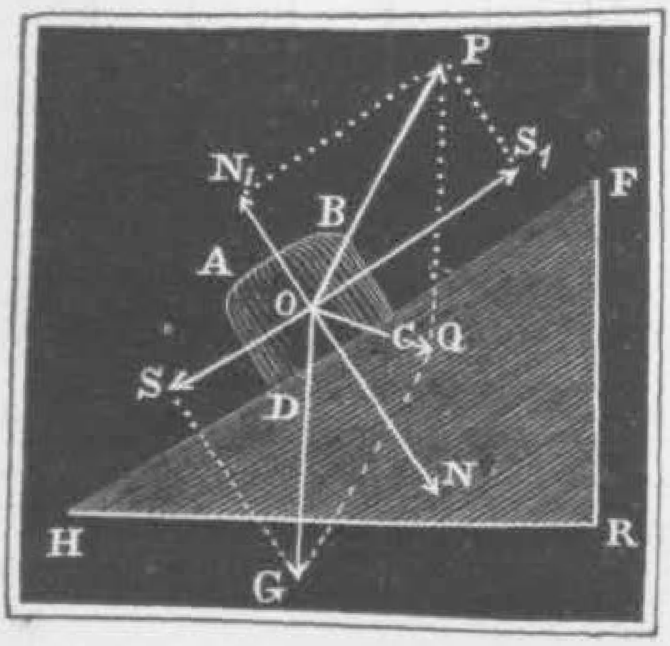
CO-EFFICIENTS OF THE FRICTION OF MOTION.

NAMES OF BODIES.		Nature of the surfaces and unguents.								
		Dry.	Water.	Olive oil.	Lard.	Tallow.	Lard and black lead.	Polished and greasy.	Dry soap.	Greasy and wetted.
Wood upon wood	least,	0,20	—	—	0,06	0,06	—	—	0,14	0,08
	mean,	0,36	0,25	—	0,07	0,07	—	—	0,15	0,12
	greatest, value,	0,48	—	—	0,07	0,08	—	—	0,16	0,15
Metal upon metal	least,	0,15	—	0,06	0,07	0,07	0,06	0,12	—	0,11
	mean,	0,18	0,31	0,07	0,09	0,09	0,08	0,15	0,20	0,13
	greatest, value,	0,24	—	0,08	0,11	0,11	0,09	0,17	—	0,17
Wood upon metal	least,	0,20	—	0,05	0,07	0,06	—	—	—	0,10
	mean,	0,42	0,24	0,06	0,07	0,08	0,08	0,10	0,20	0,14
	greatest, value,	0,62	—	0,08	0,08	0,10	—	—	—	0,16
Hemp, cords, twists, &c.	on wood,	0,45	0,33							
	on iron,	—	—	0,15	—	0,19				
Sole leather, smooth, upon wood or metal	raw,	0,54	0,36	0,16	—	0,20				
	compressed,	0,30	—							
	greasy,	—	0,25							
The same, high at the edges, &c.	dry,	0,34	0,31	0,14	—	0,14				
	greasy,	—	0,24							

Remark. The co-efficients of friction for porous masses will be given in the Second Part, in the theory of the pressure of earth.

§ 162. *Inclined Plane.*—The theory of sliding friction has its chief application in the investigation of the equilibrium of a body AC, on an inclined plane FH, Fig. 165. If in accordance with § 135, $FHR = \alpha$, the angle of inclination of the inclined plane, and $POS_1 = \beta$, the angle which the force P makes with the inclined plane, we have the normal force arising from the weight G of the body $N = G \cos. \alpha$, on the other hand, the force for sliding down $= S = G \sin. \alpha$, further the force N_1 with which P strives to draw the body down the plane is $= P \sin. \beta$,

Fig. 165.



and the force S_1 with which it pushes the body up the plane $= P \cos. \beta$. The remaining normal pressure is: $N - N_1 = G \cos. \alpha - P \sin. \beta$, consequently the friction $F = f(G \cos. \alpha - P \sin. \beta)$. If it be required to find the force P drawing the body up the plane, then there will be friction to overcome, and it must therefore be $S_1 = S + F$, *i. e.*

$$P \cos. \beta = G \sin. \alpha + f(G \cos. \alpha - P \sin. \beta).$$

But if the force, which is to prevent the body from sliding down is to be determined, then friction comes to its assistance, and the force is: $S_1 + F = S$, *i. e.* $P \cos. \beta + f(G \cos. \alpha - P \sin. \beta) = G \sin. \alpha$.

From this the force may be determined:

For the first case: $P = \dfrac{\sin. \alpha + f \cos. \alpha}{\cos. \beta + f \sin. \beta} \cdot G$,

For the second: $P = \dfrac{\sin. \alpha - f \cos. \alpha}{\cos. \beta - f \sin. \beta} \cdot G$.

If the angle of friction ρ be introduced, whilst we put $f = tang. \rho = \dfrac{\sin. \rho}{\cos. \rho}$, we shall obtain $P = \dfrac{\sin. \alpha \cdot \cos. \rho \pm \cos. \alpha . \sin. \rho}{\sin. \beta . \cos. \rho \pm \cos. \beta . \sin. \rho} \cdot G$, or from the known rules of trigonometry: $P = \dfrac{\sin. (\alpha \pm \rho)}{\cos. (\beta \mp \rho)} \cdot G$, and the upper signs are to be taken, when motion is to be brought about; the lower, on the other hand, when motion is to be impeded.

The last formula is found by a simple application of the parallelogram of forces. Since a body counteracts that force of another body, which deviates by the angle of friction ρ from the normal to its surface (§ 159), equilibrium in the foregoing case can subsist if the resultant $OQ = Q$ of the components P and G makes with the normal ON the angle $NOQ = \rho$. If now we put in the general formula $\dfrac{P}{G} = \dfrac{\text{sine } GOQ}{\sin. POQ}$, $GOK = GON + NOQ = \alpha + \rho$, and $POQ = POS_1 + S_1OQ = \beta + 90^\circ - \rho$, we then have $\dfrac{P}{G} = \dfrac{\sin. (\alpha + \rho)}{\sin. \left(\beta - \rho + \frac{\pi}{2}\right)} = \dfrac{\sin. (\alpha + \rho)}{\cos. (\beta - \rho)}$, and for a negative value of ρ: $\dfrac{P}{G} = \dfrac{\sin. (\alpha - \rho)}{\cos. (\beta + \rho)}$, quite in accordance with the above.

If the body reposes on a horizontal plane $\alpha = 0$, therefore, the force to push P forward is: $P = \dfrac{f G}{\cos. \beta + f \sin. \beta} = \dfrac{f G \sin. \rho}{\cos. (\beta - \rho)}$.

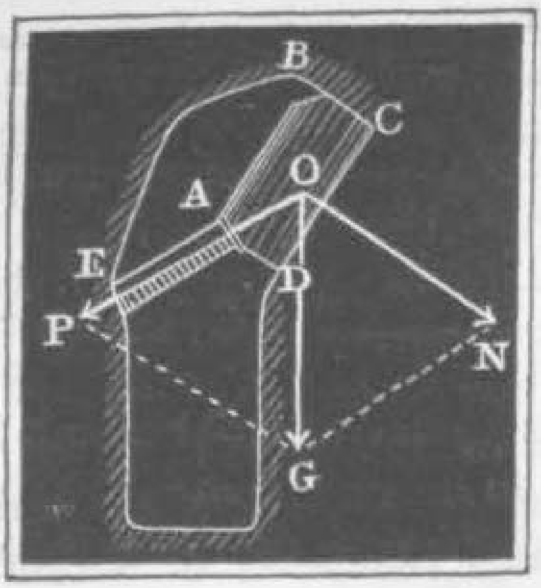
If the force acts parallel to the inclined plane, then $\beta = 0$, and therefore, $P = (\sin. \alpha \pm f \cos. \alpha) G, = \dfrac{(\sin, \alpha \pm \rho)}{\cos. \rho} \cdot G$. (compare § 159). If the force acts horizontally $\beta = -\alpha$; $\cos. \beta = \cos$

α and $sin.\ \beta = -\ sin\ \alpha$, therefore, $P = \frac{(sin.\ \alpha \pm f\ cos.\ \alpha}{cos.\ \alpha \pm f\ sin.\ \alpha}.\ G =$
$\frac{tang.\ \alpha \pm f}{1 \pm f\ tang.\ \alpha}.\ G$, also $= tang.\ (\alpha \pm \rho)\ G$.

Again, the force to push a body upwards is least when the denominator $cos.\ (\beta - \rho)$ is greatest, viz. $= 1$, therefore, $\beta - \rho = 0$, *i.e.* $\beta = \rho$. When, therefore the direction of force deviates by the angle of friction from the inclined plane, the force itself is the least and $=$ $sin.\ (\alpha + \rho)\ .\ G$.

Example. What pressure on the axis has the prop *AE*, Fig. 166, to sustain, in order to prevent a block of stone (a wall) *ABCD*, of 5000 lbs. weight from slipping down the inclined plane *CD*, supposing the angle of the prop to the horizon to be 35°, that of the inclined plane *CD*, 50°, and the coefficient of friction $f = 0{,}75$? Here $G = 5000$, $\alpha = 50°$, $\beta\ 35° - 50° = -15°$, and $f = 0{,}75$; therefore the formula gives:

Fig. 166.

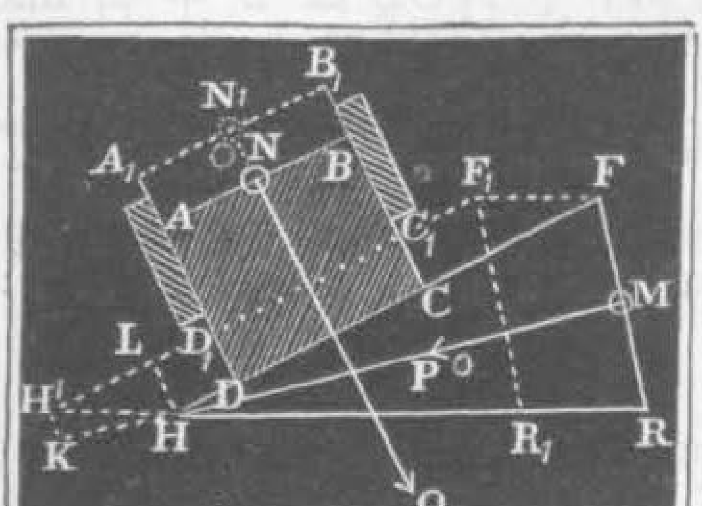


$$P = \frac{sin.\ \alpha - f\ cos.\ \alpha}{cos.\ \beta - f\ sin.\ \beta}\ .\ G = \frac{sin.\ 50° - 0{,}75\ cos.\ 50°}{cos.\ 15° + 0{,}75\ sin.\ 15°}\ .\ 5000$$
$$= \frac{0{,}766 - 0{,}482}{0{,}966 + 0{,}194}\ .\ 5000 = \frac{1420}{1{,}160} = 1224 \text{ lbs.}$$

If the prop were horizontal, we should have $\beta = -50°$, and $tang.\ \rho = 0{,}75$; hence $\rho = 36°\ 52'$; lastly, $P = G\ tang.\ (\alpha - \rho) = 5000\ tang.\ (50° - 36°\ 52') = 5000\ tang.\ 13°\ 8' = 5000\ .\ 0{,}2333 = 1166$ lbs. To push up the same wall upon the supporting one by a horizontal force, under otherwise similar circumstances, a force P would be necessary $= G\ tang.\ (\alpha + \rho) = 5000\ tang.\ 86°\ 52' = 5000 .\ 18{,}2676 = 91338$ lbs.

§ 163. *Wedge.*—In the wedge, friction exerts a considerable influence upon the statical relations. The section of a wedge forms an isosceles triangle *FHR*, Fig. 167, with the edge $FHR = \alpha$, the force P acts at right angles to the back and the weight Q at right angles to the side FH. If we drive the wedge upon the base HR a space $s = FF_1 = HH_1 = RR_1$, the weight Q is raised through a space $CC_1 = DD_1 = HL = HH_1\ .\ sin.\ HH_1L = s\ sin.\ \alpha$, and force passes over $HK = HH_1\ .$

Fig. 167.



$cos.\ H_1HK = s\ cos.\ \frac{\alpha}{2}$; according to the principle of virtual velocities, and without regard to friction, $P\ .\ HK = Q\ .\ DD_1$, *i.e.* $P\ s\ cos.$ $\frac{\alpha}{2} = Qs\ sin.\ \alpha$, therefore $P = \frac{Q\ sin.\ \alpha}{cos.\ \frac{\alpha}{2}} = \frac{2\ Q\ sin.\ \frac{\alpha}{2}\ .\ cos.\ \frac{\alpha}{2}}{cos.\ \frac{\alpha}{2}} = 2$

$Q\ sin. \frac{\alpha}{2}$, which also follows from the formula in § 137, if we put in it $sin. \beta = 1$, and $cos. (\alpha - \delta) = cos. \frac{\alpha}{2}$.

There are now, however, three frictions which come into play, viz., the friction against the sides HF and HR, and the friction of the body $ABCD$ in its constrained motion. As the directions of the force on both sides of the wedge deviate equally, the pressure against both is equal, namely $= Q$, and the friction arising $= f\,Q$. The spaces of these frictions, however, are different. For the friction upon HR : $s = HH_1$, for that upon $HF = H_1L = s\ cos. \alpha$; accordingly the mechanical effects of both frictions are: $= f\,Q\,s + f\,Q\,s\ cos. \alpha = f\,Q\,s\,(1 + cos. \alpha) = 2\,f\,Q\,s \left(cos. \frac{\alpha}{2}\right)^2$. Lastly, the friction between CD and FH presses upon the body $ABCD$ at right angles to its direction, and there produces the friction $f_1 . f\,Q$, if f_1 represent the co-efficient of friction for its constrained motion. This friction, however, has the same space as the weight Q, viz., $DD_1 = s\ sin. \alpha$; and to it corresponds the mechanical effect $f_1\,f\,Q\,s\ sin. \alpha$. In order now to find the extreme limits of the condition of equilibrium, we must put the mechanical effect of the force P equal to that of the weight Q, plus the mechanical effects of the friction, therefore,

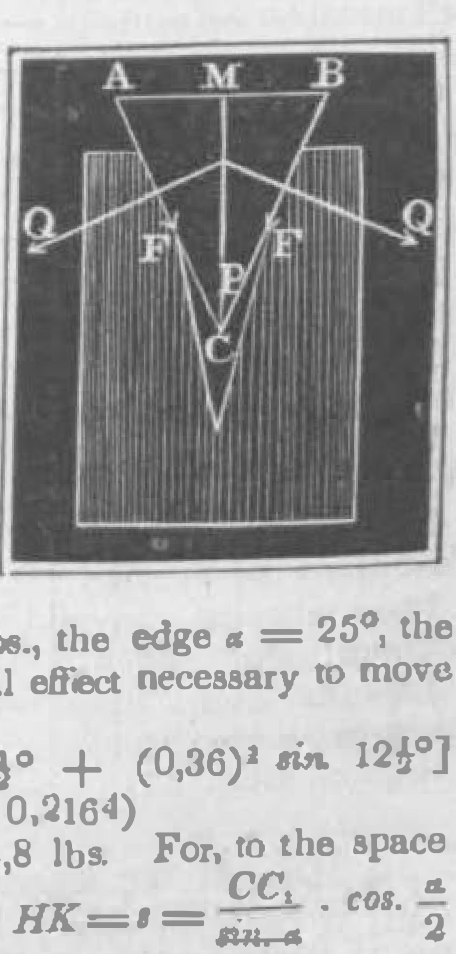
$$Ps\ cos. \frac{\alpha}{2} = Qs\ sin. \alpha + 2\,Q\,f\,s \left(cos. \frac{\alpha}{2}\right)^2 + ff_1\ Qs\ .\ sin. \alpha,$$

and we obtain the force:

$$P = 2\,Q \left(sin. \frac{\alpha}{2} + f\ cos. \frac{\alpha}{2} + ff_1\ sin. \frac{\alpha}{2}\right).$$

In a wedge ABC, Fig. 168, as it is used for the splitting asunder and compression of bodies, the force at the back corresponding to the normal pressure Q against the sides AC and BC, is $P = 2\,Q \left(sin. \frac{\alpha}{2} + f\ cos. \frac{\alpha}{2}\right)$, which is given if we put the sum of the vertical components of Q and $F = f\,Q$, *i. e.* $2\,V_1 = 2\,Q\ sin. \frac{\alpha}{2}$ and $2\,V_2 = 2\,f\,Q\ cos. \frac{\alpha}{2}$ equivalent to the force P.

Fig. 168.



Example. The load of the wedge Q in Fig. 167 = 650 lbs., the edge $\alpha = 25°$, the co-efficient of friction $f_1 = f = 0,36$. Required, the mechanical effect necessary to move the load Q forward about $\frac{1}{2}$ a foot.

The force is $P = 2\ .\ 650\ [sin.\ 12\frac{1}{2}° + 0,36\ cos.\ 12\frac{1}{2}° + (0,36)^2\ sin.\ 12\frac{1}{2}°]$
$= 1300t. (0,2164 + 0,36t. 0,9763 + 0,1296t. 0,2164)$
$= 1300 . (0,2164 + 0,3515 + 0,0281) = 1300 . 0,5960 = 774,8$ lbs. For, to the space of the load $CC_1 = \frac{1}{2}$ foot, corresponds the space of the force $HK = s = \frac{CC_1}{sin. \alpha} . cos. \frac{\alpha}{2} = \frac{CC_1}{2\ sin. \frac{\alpha}{2}} = \frac{1}{4 . 0,2164} = 1,155$ feet; therefore the mechanical effect and weight is

$Ps = 774{,}8 \cdot 1{,}155 = 895$ ft. lbs. Without regard to friction, it would only be $650 \cdot \frac{1}{2} = 325$ ft. lbs. In consequence of friction, the mechanical effect expended would be nearly tripled.

§ 164. *Axle Friction.*—In axles, the friction of motion only is of importance, on which account experiments on this only exist.

From the following table very important results for practice may be drawn, with axles of wrought or cast iron, moving in bearings of cast iron or brass, coated with oil, tallow or hog's lard, the co-efficient of friction is

By continuous greasing = 0,054,
In the usual manner = 0,070 to 0,080.

The values found by *Coulomb* vary partially from the annexed.

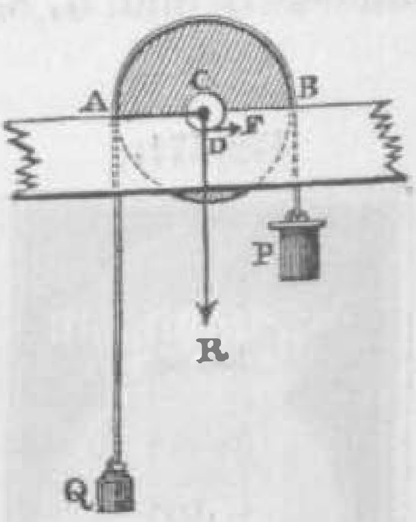
TABLE III.

CO-EFFICIENTS OF AXLE FRICTION, FROM MORIN.

NAMES OF THE BODIES.	Nature of the surfaces and unguents.							
	Dry or a little greasy.	Greasy and wetted with water.	Greased and wetted with water.	Oil, tallow, or lard. In the usual way.	Oil, tallow, or lard. Continuously.	Very soft and purified carriage grease.	Hog's lard with plumbago.	Fatty.
Bell metal on the same	—	—	—	0,097	—	—	—	—
Cast iron upon bell metal	—	—	—	—	0,049			
Wrought iron upon bell metal	0,251	0,189	—	0,075	0,054	0,090	0,111	—
Wrought iron upon cast iron .e.e. . . .	—	—	—	0,075	0,054	—	—	—
Cast iron upon cast iron	—	0,137	0,079	0,075	0,054	—	—	0,137
Cast iron upon bell metal	0,194	0,161	—	0,075	0,054	0,065	—	0,166
Wrought iron upon lignum vitæ	0,188	—	—	0,125				
Cast iron upon lignum vitæ	0,185	—	—	0,100	0,092	—	0,109	0,140
Lignum vitæ upon cast iron	—	—	—	0,116	—	—	—	0,153
Lignum vitæ upon lignum vitæ	—	—	—	—	0,070			

§ 165. If we know the pressure R between an axle and its bearing, and if further the radius r of the axle, Fig. 169, be given, the mechanical effect which the friction of the axle counteracts in every revolution may be calculated. The friction $F = fR$, the space corresponding to it, the circumference $2\pi r$ of the axle; it therefore follows that the mechanical effect lost by friction in each revolution is $= fR \cdot 2\pi r = 2\pi f R r$. If the axle makes one revolution per minute u, the mechanical effect expended in each second

Fig. 169.

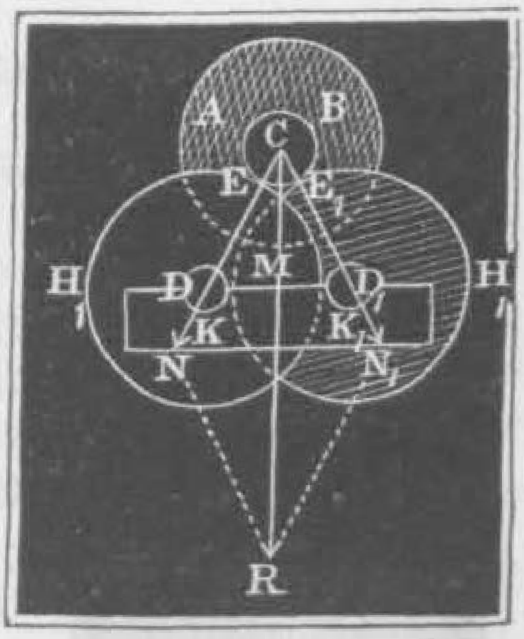


$$= 2\pi f R r \cdot \frac{u}{60} = \frac{\pi u f R r}{30} = 0{,}105 e u f R r.$$

The mechanical effect consumed by friction increases, therefore, with the pressure on the axle, in proportion to the radius of the axle and the number of revolutions. It is, therefore, a rule in practice, not to augment unnecessarily the pressure on the axis in rotating machines by heavy weights, to make the axles no stronger than the solidity required for durability, and likewise not to make a great many revolutions in a minute, at least, not unless other circumstances require it.

By the application of friction wheels, which are substituted for the bearings, the mechanical effect of friction is much diminished. In Fig. 170, AB is a wheel which reposes by its axle CEE_1 on the circumferences EH, E_1H_1 lying close to each other of the friction wheels revolving about D and D_1. From the given pressure R of the wheel, there follow the pressures $N = N_1 = \dfrac{R}{2\cos.\frac{a}{2}}$, if a be the angle DCD_1 which the central lines, or *lines of pressure*, CD and CD_1 make between them. From the rolling friction between the axle C and the circumferences of the wheels, these latter revolve with the axle, and there arise at the bearings D and D_1 the frictions fN and fN_1, which together amount to $\dfrac{fR}{\cos.\frac{a}{2}}$. If the radius of the wheel

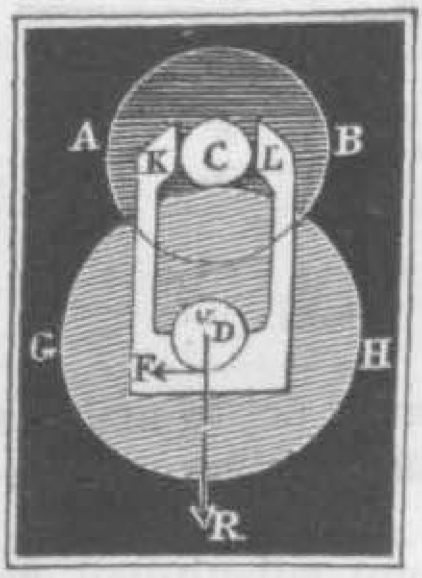
Fig. 170.



$DE = D_1E_1$ be represented by a_1, and that of the axle $DK = D_1K_1$ by r_1, we shall have the force at the circumference of the wheels, or at the circumference of the axle C resting upon these, which is requisite to overcome $\dfrac{fR}{\cos.\frac{a}{2}}$: $F_1 = \dfrac{r_1}{a_1} \cdot \dfrac{fR}{\cos.\frac{a}{2}}$, whilst it will be $= fR$, if the axle C rest immediately in a socket. If we disregard the weights of the friction wheels, the mechanical effect of the friction by the application of these wheels is $\dfrac{r_1}{a_1\cos.\frac{a}{2}}$ times as great as without them.

If we oppose to the pressure of the axle R a single friction wheel GH, Fig. 171, and prevent any accidental lateral forces, by the fixed cheeks K and L, $\alpha = 0$, $cos. \frac{\alpha}{2} = 1$, and the above relations $= \frac{r_1}{a_1}$.

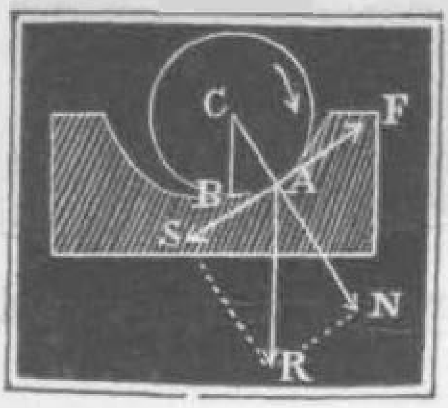
Fig. 171.



Example. A wheel weighs 30000 lbs., its radius $a = 16$ ft. and that of its axle $r = 5$ inches, what is the amount of force at the circumference of the wheel necessary to overcome the friction of the axle, and to maintain it in uniform motion, and what is the corresponding expenditure of mechanical effect if it makes 5 revolutions a minute? We may assume the co-efficient of friction f here $= 0{,}075$, wherefore the friction $f R = 0{,}075 . 30000 = 2250$ lbs. Since the diameter of the wheel is $\frac{16 . 12}{5} = \frac{192}{5} = 38{,}4$ times as great as the diameter of the axle or the arm of the friction, the axle friction reduced to the circumference of the wheel $= \frac{f R}{38{,}4} = \frac{2250}{38{,}4} = 58{,}59$ lbs. The circumference of the axle is $\frac{2 . 5 . \pi}{12} = 2{,}618$ feet; consequently the path of the friction in one second $= \frac{2{,}618 . 5}{60} = 0{,}2182$ feet, and its mechanical effect during one second $= 0{,}2182 . f R = 0{,}2182 . 2250 = 491$ ft. lbs. If the axles of this wheel rest upon friction wheels whose radii are 5 times as great as those of the axle, and therefore $\frac{r_1}{a_1} = \frac{1}{5}$ the power expended, referred to the circumference of the wheel, will only be $\frac{1}{5} . 38{,}4 = 7{,}68$ ft. lbs., and the mechanical effect of friction expended only $\frac{491}{5} = 98{,}2$ ft. lbs.

§ 166. The friction of an axle ACB, Fig. 172, which presses on its bearing in one point A only, is less than that of a new axle resting on all points of the bearing. If no revolution takes place, the axle then presses on the point B, through which passes the direction of the mean pressure R; but if revolution begins in the direction AB, the axle by its friction will rise just so high in its bearing until the sliding force comes into equilibrium with the friction. The mean pressure R is decomposed into a normal pressure N and a tangential S; N passes into the bearing and gives rise to $F = f N$, acting tangentially; S puts itself in equilibrium with F; S is therefore $= f N$. According to the Pythagorean doctrine, R^2 is $= N^2 + S^2$, therefore R^2 is here $= (1 + f^2) N^2$; inversely the normal pressure $N = \frac{R}{\sqrt{1 + f^2}}$, and the friction $F = \frac{f R}{\sqrt{1 + f^2}}$; or, if the angle of friction ρ be introduced, $f = tang. \rho$:

Fig. 172.



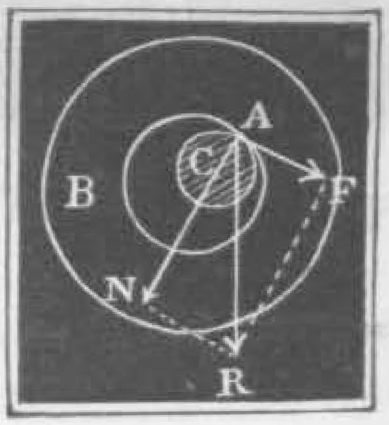
$$F = \frac{tang. \rho}{\sqrt{1 + tang. \rho^2}} . R = tang. \rho \; cos. \rho \; R = R \; sin. \rho.$$

If the axle begins to move, the point of pressure B moves forward in the bearing by the angle $ACB = \rho$ in the opposite direction.

If no forward motion took place, F would be $= f R = R \, tang. \rho = \frac{R \sin. \rho}{\cos. \rho}$; consequently the friction is the $\cos. \rho$ times as great after moving forward as before the motion. Generally, $f = tang. \rho$ not quite $\frac{1}{10}$ and $\cos. \rho > 0{,}995$, therefore the difference is not quite $\frac{5}{1000} = \frac{1}{200}$; therefore, in ordinary cases of application, we need have no regard to the effect of this motion.

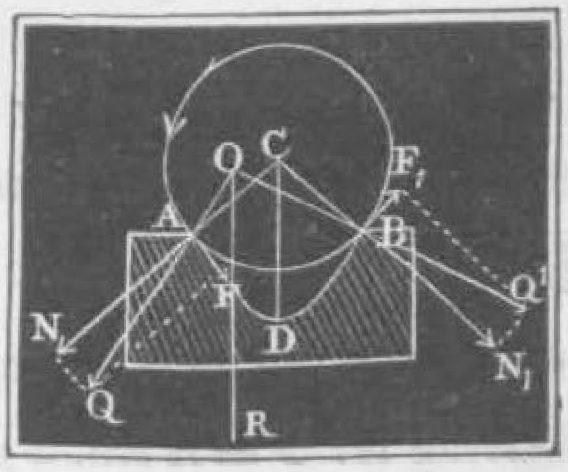
If the wheel AB revolves in a nave or eye, Fig. 173, about a fixed axis AC, the friction is the same as if the axis moves in a roomy nave, only the arm of the friction is the arm of the nave, not that of the fixed axle.

Fig. 173.



§ 167. If the axle lies in a prismatic bearing, there is greater pressure, and consequently more friction, than in a round bearing. If the bearing ADB, Fig. 174, is triangular, the axle lies on two points A and B, and at each there is the same friction to overcome, the mean pressure R is decomposed into two lateral forces Q and Q_1, and each of these gives a normal pressure N and N_1, and each a tangential force $F = f N$ and $F_1 = f N_1$ equal to the friction. According to the former §, these frictions may be put $= Q \sin. \rho$ and $= Q_1 \sin. \rho$, we have then the whole friction $= (Q + Q_1) \sin. \rho$. The forces Q and Q_1 are given by the resolution of a parallelogram constructed from Q and Q_1 with the aid of the mean pressure R, the angle of friction ρ, and the angle $ACB = 2\alpha$, which corresponds to the arc AB, lying in the bearing. If $QOR = ACD - CAO = \alpha - \rho$, $Q_1 OR = BCD + CBO = \alpha + \rho$; lastly, $QOQ_1 = \alpha - \rho + \alpha + \rho = 2\alpha$. The application of the formula § 75, gives:

Fig. 174.



$$Q = \frac{\sin. (\alpha - \rho)}{\sin. 2\alpha} . R \text{ and } Q_1 = \frac{\sin. (\alpha + \rho)}{\sin. 2\alpha} . R;$$

hence the friction sought is

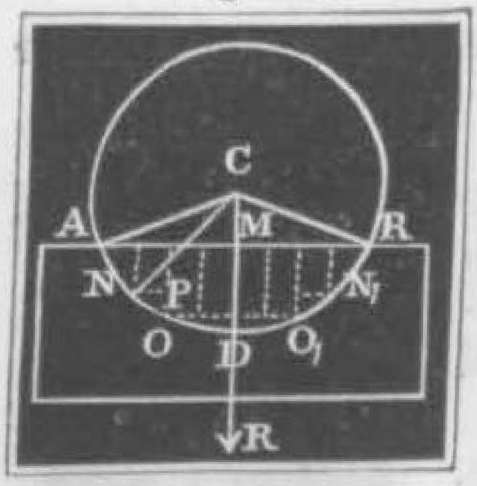
$$F + F_1 = (Q + Q_1) \sin. \rho = (\sin. [\alpha - \rho] + \sin. [\alpha + \rho]) \frac{R \sin. \rho}{\sin. 2\alpha}.$$

But the $\sin. (\alpha - \rho) + \sin. (\alpha + \rho) = 2 \sin. \alpha \cos. \rho$ and $\sin. 2\alpha = 2 \sin. \alpha \cos. \alpha$, therefore $F + F_1 = \frac{2 \sin. \alpha \, R \sin. \rho \cos. \rho}{2 \sin. \alpha \cos. \alpha} = \frac{R \sin. 2\rho}{2 \cos. \alpha}$, which from the smallness of ρ may be put $= \frac{R \sin. \rho}{\cos. \alpha}$. The friction of a triangular bearing is from this $\frac{1}{\cos. \alpha}$ times as great as that of a cylindrical one. If, for example, $ADB = 60°$, ACB

$= 180° - 60° = 120°$, and $ACD = \alpha = 60°$, we then have: $\frac{1}{cos. 60°}$ = twice the friction of that of a cylindrical bearing.

§ 168. With the aid of the last formula, the friction may be found for a new round bearing which the axle touches at all points.* Let ADB, Fig. 175, be such a bearing. Let us divide the arc ADB, in which lie the axle and the bearing, into many parts, such as AN, NO, &c., which have equal projections on the chord AB, and let us suppose that each of these parts supports an equal amount of the whole pressure R, viz., $= \frac{R}{n}$, (n being the number of parts) of the axle on the bearing.

Fig. 175.



According to the former §, the friction of two opposite parts NO and $N_1O_1 = \frac{R}{n} \cdot \frac{sin. 2\rho}{cos. NCD}$. But $cos. NCD = cos. ONP = \frac{NP}{NO}$, where NP represents the projection of the part NO, and $NP = \frac{chord. AB}{n}$. It therefore follows that the friction corresponding to the parts NO and $N_1O_1 = \frac{R \; sin. 2\rho}{n} \cdot \frac{n \, . \, NO}{chord} = \frac{R \; sin. 2\rho}{chord} . NO$. In order now to find the friction for the whole arc ADB instead of NO, we must put in the arc $AD = \frac{1}{2} ADB$, because the sum of all the frictions is $\frac{R \; sin. 2\rho}{chord}$ times the sum of all the parts of the arc, it follows that the friction in a new bearing is $F = R \; sin. 2\rho . \frac{arc \; AD}{chord \; AB}$, or if we put the angle ACB subtended at the centre by AB, which corresponds to the arc of the bearing, $= 2\alpha°$, therefore the chord $AB = 2 \, AC \, . \, sin. \, \alpha$.

$$F = \frac{R \; sin. 2\rho}{2} \cdot \frac{\alpha}{sin. \alpha}, \text{ or } sin. 2\rho = 2 \; sin. \rho$$

taken approximately

$$F = R \; sin. \rho . \frac{\alpha}{sin. \alpha}.$$

* If a bearing be of an acute wedge-shape it is conceivable that an axle pressed into such wedge-shaped cavity would create an additional friction; but as friction depends on *weight*, not on extent of surface, the demonstrations of this section relative to bearings in cylindrical cavities in which there is no wedging or tightening, but only a distribution of the *weight* over the several parts of the *surface* of contact, the effect of the moving force in a "*new* bearing" will be to relieve the pressure on one part and transfer it to another at the same distance from the centre of motion, or centre of the axle. To a certain extent, that is, to the depth of the semi-diameter of the axle, this surface increases, but if there be a distribution of pressure over a greater number of elements or units of surface, so is there a less amount of pressure on each point rubbed. This law of friction, according to pressure, and not according to surface rubbed, agrees with all experiments.—Am. Ed.

From this the friction is the greater the deeper the axle lies in its bearing. If, for instance, the bearing is half the circumference of the axle, α is then $= \frac{1}{2}\pi$ and $sin.\ \alpha = 1$, we then have $F = \frac{\pi}{2} \cdot R\ sin.\ \rho$, and because $\frac{\pi}{2} = 1{,}57$, therefore 1,57 times as great as that of the free bearing. In an axle which does not rest deep in its bearing, α is small, therefore the $sin.\ \alpha$ may be put $= \alpha - \frac{\alpha^3}{6} = \alpha\left(1 - \frac{\alpha^2}{6}\right)$, whence it follows that:

$F = \left(1 + \frac{\alpha^2}{6}\right) R\ sin.\ \rho$, or $= R\ sin.\ \rho$, if α be very small.

§ 169. The axle pressure R is given generally as the resultant of two forces P and Q, directed at right angles to each other, and is therefore $= \sqrt{P^2 + Q^2}$. Provided we require it only for the determination of the friction $f R = f \sqrt{P^2 + Q^2}$, we may be satisfied with an approximate value of it, partly because the co-efficient f can never be so accurately determined and depends upon so many accidental circumstances, and partly because the whole product of the friction $f\ R$ is mostly only a small part of the remaining forces of the machine resting upon the axle bearing, as the lever, pulley, wheel and axle, &c. The doctrine which teaches us to find an approximate expression of $\sqrt{P^2 + Q^2}$ is known under the name of Poncelet's theorem, and may be developed in the following manner:

$\sqrt{P^2 + Q^2} = P\sqrt{1 + \left(\frac{Q}{P}\right)^2} = P\sqrt{1 + x^2}$, whereby $x = \left(\frac{P}{Q}\right)$, which supposes that P is the smaller force, therefore, x is a mere fraction. We may now put:

$\sqrt{1 + x^2} = \mu + \nu x$, and determine the co-efficients μ and ν, answering certain conditions. The relative error is:

$$y = \frac{\sqrt{1 + x^2} - \mu - \nu x}{\sqrt{1 + x^2}} = 1 - \frac{\mu + \nu x}{\sqrt{1 + x^2}}.$$

For the smallest value of x, viz., $x = 0$, $y = 1 - \mu$, and for the greatest, viz. $x = 1$, we have $y = 1 - \frac{\mu + \nu}{\sqrt{2}}$. If we make these errors, corresponding to the limits of x, equal, we then obtain an equation of condition $\mu = \frac{\mu + \nu}{\sqrt{2}}$, or $\nu = \mu\sqrt{2} - \mu = 0{,}414 \cdot \mu$. If we take $x = \frac{\nu}{\mu}$, the result is, that $y = 1 - \sqrt{\mu^2 + \nu^2} = -(\sqrt{\mu^2 + \nu^2} - 1)$, as a negative error, is greater than any other which arises by assuming $x = \frac{\nu}{\mu} + \Delta$, that is, a little greater, or a little less than $\frac{\nu}{\mu}$; for in the latter case we have

$$y = -\left(\frac{\mu + \nu\left(\frac{\nu}{\mu} \pm \Delta\right)}{\sqrt{1 + \left(\frac{\nu}{\mu} \pm \Delta\right)^2}} - 1\right)$$

$$= -\left(\frac{\mu^2 + \nu^2 \pm \mu\nu\Delta}{\sqrt{\mu^2 + \nu^2 \pm 2\mu\nu\Delta + \mu^2\Delta^2}} - 1\right)$$

$$= -\left(\sqrt{\frac{(\mu^2 + \nu^2 \pm \mu\nu\Delta)^2}{\mu^2 + \nu^2 \pm 2\mu\nu\Delta + \mu^2\Delta^2}} - 1\right)$$

$$= -\left(\sqrt{\mu^2 + \nu^2 - \frac{\mu^4\Delta^2}{\mu^2 + \nu^2} \cdots} - 1\right).$$

If now we make this greatest negative error equal to the greatest positive error, we shall then obtain the following second equation of condition:

$$\sqrt{\mu^2 + \nu^2} - 1 = 1 - \mu; \text{ or } \mu + \sqrt{\mu^2 + \nu^2} = 2.$$

But the first equation gives $\nu = 0{,}414\,\mu$; it, therefore, follows that

$$\mu\,(1 + \sqrt{1 + 0{,}414^2}) = 2, \text{ i. e.}$$

$$\mu = \frac{2}{1 + \sqrt{1{,}1714}} = 0{,}96 \text{ and } \nu = 0{,}414 \,.\, 0{,}96 = 0{,}40.$$

We may, therefore, put approximately $\sqrt{1 + x^2} = 0{,}96 + 0{,}40\,.\,x$, and in like manner the resultant $R = 0{,}96\,P + 0{,}40\,Q$, knowing that we thereby commit, at most, the error $\pm y = 1 - \mu = 1 - 0{,}96 = 0{,}04$ = four per cent. of the true value.

This determination supposes that we know which is the greater of the forces; but if this be unknown, we may assume $\sqrt{1 + x^2} = \mu\,(1 + x)$, and so obtain $y = 1 - \frac{\mu\,(1 + x)}{\sqrt{1 + x^2}}$. Here not only the limit $x = 0$ gives the error $= 1 - \mu$, but also the limit $x = \infty$, the same error $= 1 - \frac{\mu x}{x} = 1 - \mu$; but if we put $x = \frac{\nu}{\mu} = 1$, we then obtain the greatest negative error $= -\left(\frac{2\mu}{\sqrt{2}} - 1\right) = -(\mu\sqrt{2} - 1,)$ and by making these errors equal: $1 - \mu = \mu\sqrt{2} - 1$, therefore,

$$\mu = \frac{2}{1 + \sqrt{2}} = \frac{2}{2{,}414} = \frac{1}{1{,}212} = 0{,}825, \text{ for which } 0{,}83 \text{ may be put.}$$

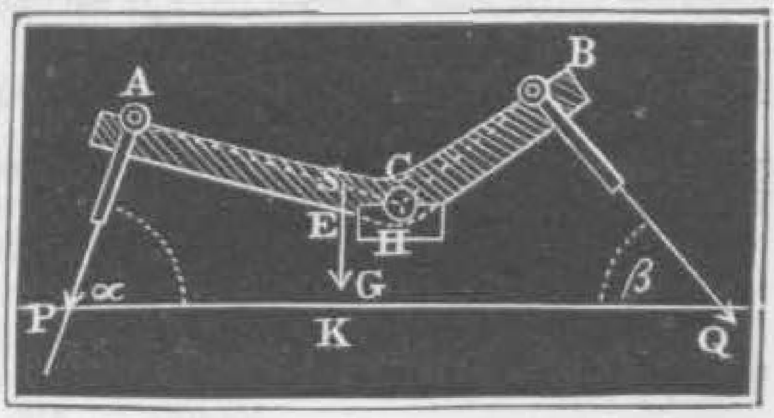
In the case where we do not know which is the greater of the forces, R may be put $= 0{,}83\,(P + Q)$, and we know that the greatest error committed will be $\pm\, y = 1 - 0{,}83 = 0{,}17$ per cent. $= \frac{1}{6}$ of the true value.

If, lastly, we know that x does not exceed 0,2, we may disregard it altogether, and write $\sqrt{P^2 + Q^2} = P$, but if x exceeds 0,2, then

$\sqrt{P^2+Q^2}$ is more accurately $= 0{,}888 \cdot P + 0{,}490 \cdot Q$; in both cases the greatest error is about two per cent.*

§ 170. *Lever.*—The theory of friction above developed finds its application in the material lever, the wheel and axle, and in other machines. Let us, in the first place, treat of the lever, and take the general case, viz., that of the bent lever ACB, Fig. 176. Let us represent as before (§ 127) the arm CA of the power P by a, the arm CB of the weight Q by b, and the radius of the axle CH by r, let us put the weight of the lever $= G$, its arm $CE = s$, and the angles APK and BQK, by which the directions of the forces deviate from the horizon $= \alpha$ and β. The power P gives the vertical pressure $P\, sin.\, \alpha$, and the weight $= Q\, sin.\, \beta$; the whole vertical pressure is, therefore, $V = G + P\, sin.\, \alpha + Q\, sin.\, \beta$. The power P gives further the horizontal pressure $P\, cos.\, \alpha$, and the weight Q an opposite pressure $Q\, cos.\, \beta$: since there remains for the horizontal pressure, $H = P\, cos.\, \alpha - Q\, cos.\, \beta$, we may put the whole pressure on the axle: $R = \mu\, V + \nu\, H = \mu\,(G + P\, sin.\, \alpha + Q\, sin.\, \beta) + \nu\,(P\, cos.\, \alpha - Q\, cos.\, \beta)$, of which the second part $\nu\,(P\, cos.\, \alpha - Q\, cos.\, \beta)$ must never be taken negative, and, therefore, in the case where $Q\, cos.\, \beta$ is $> P\, cos.\, \alpha$, the sign must be changed, or rather $P\, cos.\, \alpha$ must be subtracted from $Q\, cos.\, \beta$. In order to find that value of the power which corresponds to unstable equilibrium, so that the smallest addition of force produces motion, we must put the moment of power equal to the moment of weight, plus or minus the moment of weight of the machine (§ 127) plus the moment of friction, therefore,

Fig. 176.



$$Pa = Qb \pm Gs + fRr$$
$$= Qb \pm Gs + f(\mu\, V + \nu\, H)\, r, \text{ from which follows}$$
$$P = \frac{Qb \pm Gs + f[\mu\,(G + Q\, sin.\, \beta) \mp Q\, cos.\, \beta]\, r}{a - \mu\, fr\, sin.\, \alpha \mp \nu\, fr\, cos.\, \alpha}.$$

If P and Q act vertically, R is simply $= P + Q + G$, therefore, $Pa = Qb \pm Gs + f\,(P + Q + G)\, r$. If the lever is one-armed, P and Q act opposite to each other, then $R = P - Q + G$, and consequently the friction is less. Besides R must be put constantly positive in the calculation, because the friction fR only impedes, but does not produce motion. From this we see that a one-armed, is mechanically more perfect than a two-armed lever.

Example. If the arms of a bent lever, Fig. 176, are: $a = 6$ ft., $b = 4$ ft., $s = \frac{1}{3}$ ft. and $r = 1\frac{1}{2}$ inches, the angle of inclination $\alpha = 70^0$, $\beta = 50^0$, and further the weight $Q = 5600$ lbs., and that of the lever $G = 900$ lbs., the power required to restore the unstable equilibrium is the following. Without regard to friction $Pa + Gs = Qb$, therefore,

* Polytechnische Mittheilungen, Heft 1, 1844.

$P = \frac{Qb - Gs}{a} = \frac{5600 . 4 - 900 . \frac{1}{2}}{6} = 3658$ lbs. If we put $\mu = 0{,}96$ and $\nu = 0{,}40$, we obtain $\mu (G + Q \sin. \beta) = 0{,}96 (900 + 5600 \sin. 50^0) = 4982$ lbs., $\nu Q \cos. \beta = 0{,}40 . 5600 . \cos. 50^0 = 1440$ lbs.; $\mu \sin. \alpha = 0{,}96 . \sin. 70^0 = 0{,}902$, $\nu \cos. \alpha = 0{,}40 . \cos. 70^0 = 0{,}137$. It is easy to see, that here $P \cos. \alpha$ is less than $Q \cos. \beta$, for since approximately $P = 3658$, we have $P \cos. \alpha = 1251$ lbs., and $Q \cos. \beta = 3600$ lbs., let us therefore take for $\nu Q \cos. \beta$, and $\nu \cos. \alpha$ the lower sign and put $P = \frac{5600e\, 4 - 900 . \frac{1}{2} + fr\, (4982 + 1440)}{6 - fr\, (0{,}902 - 0{,}173)}$. Let us further take the co-efficient of friction $f = 0{,}075$, and we shall have $fr = 0{,}075 . \frac{3}{24} e = 0{,}009375$, and the power sought,

$P = \frac{22400 - 450 + 60}{6 - 0{,}00683} = \frac{22010}{5{,}9932} = 3673$ lbs. Here the vertical pressure, if we substitute the value $P = 3658$ lbs., and neglect friction, is $V = 3658e \sin. 70^0 + 5600 \sin. 50^0 + 900 = 3437 + 4290 + 900 = 8627$ lbs., on the other hand, the horizontal pressure:

$$H = 5600 \cos. 50 - 3658 \cos. 70^0 = 3600 - 1251 = 2349 \text{ lbs.}$$

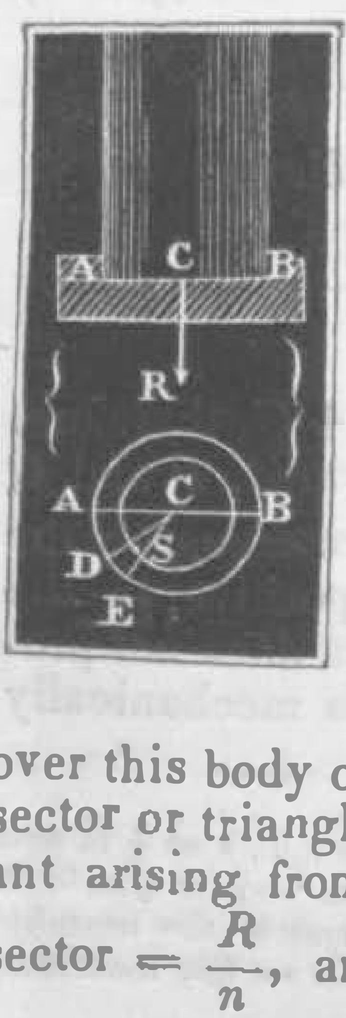
Here H is $> 0{,}2\, V$, therefore, more correctly:

$$R = 0{,}888 . H + 0{,}490\, V = 0{,}888 . 8627 + 0{,}490e\, 2349 = 8811,$$

and it follows that the moment of friction $= f r R = 0{,}009375 . 8811 = 82{,}6$ ft. lbs., and lastly, the power $P = \frac{22400 - 450 + 82{,}6}{6} = 3672$ lbs., which value varies little from the above.

§ 171. *Pivot Friction.*—When in the wheel and axle a pressure takes place in the direction of the axis, as in the case, for example, of upright axles, in consequence of their weight, there is a friction on the base of the one axle. Because pressure is there exerted on points between the pivot and its step, this friction approximates to the simple sliding friction, and to the axle friction which we have hitherto considered, and we must put for it the co-efficients of friction given in Table II. To find the mechanical effect absorbed by this friction, we must know the mean space which the base AB, Fig. 177, of such an upright axle describes in a revolution. Let us assume that the pressure R is equally distributed over the whole surface, let us also suppose that on equal parts of the bases the frictions are equal. Let us further divide the base by radii CD, CE, &c., into equal sectors or triangles DCE; to these will correspond not only equal amounts of friction, but also equal moments, therefore, it will be necessary only to find the moment of friction of one of these triangles. The frictions of such a triangle may be regarded as parallel forces, for they all act tangentially, *i. e.* at right angles to the radius CD, and since the centre of gravity of a body or a surface is nothing more than the point of application of the resultant of the parallel forces equally distributed over this body or surface, accordingly the centre of gravity S of the sector or triangle DCE is here the point of application of the resultant arising from its different frictions. If now the pressure on this sector $= \frac{R}{n}$, and the radius $CD = CE$, the base $= r$, it follows that (from § 104) the moment of friction of this sector $= CSB \frac{f R}{n}$

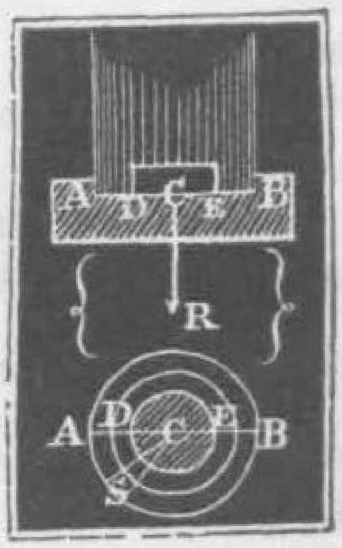
Fig. 177.



$= \frac{2}{3} r \cdot \frac{fR}{n}$, and lastly, the moment of the entire friction of the axle $= n \cdot \frac{2}{3} r \frac{fR}{n} = \frac{2}{3} f R r$.

Sometimes the rubbing surface is a ring $ABED$, Fig. 178. If its radii are $CA = r_1$ and $CD = r_2$, we have then to determine the centre of gravity S of a portion of a ring, and from § 109, obtain the arm $CS = \frac{2 r_1^3 - r_2^3}{3 r_1^2 - r_2^2}$, therefore, the moment of friction $= \frac{2}{3} f R \left(\frac{r_1^3 - r_2^3}{r_1^2 - r_2^2}\right)$. If we introduce the mean radius $\frac{r_1 + r_2}{2} = r$, and the breadth of the ring $r_1 - r_2 = b$, we obtain this moment of friction also $= f R \left(r + \frac{b^2}{12 r}\right)$.

Fig. 178.

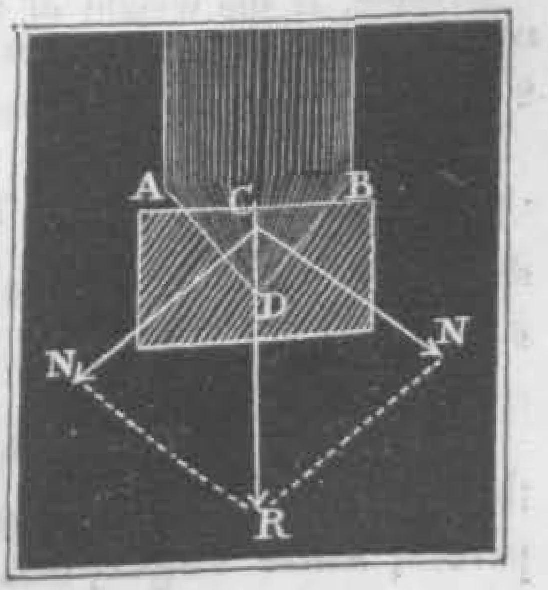


The mechanical effect of friction for a revolution of the axle is in the second case $= 2 \pi \cdot \frac{2}{3} f R r = \frac{4}{3} \pi f R r$, and in the first $\frac{4}{3} \pi f R \left(\frac{r_1^3 - r_2^3}{r_1^2 - r_2^2}\right)$. Here we easily see that to diminish this loss of mechanical effect, the upright axle or shaft must be made as light as possible, and that a greater loss of mechanical effect would arise if, under otherwise similar circumstances, the friction were to take place in a ring instead of a complete circle.

Example. In a turbine making 100 revolutions a minute, and 1800 lbs. weight, the size of the pivot at the base, is $\frac{1}{2}$ inch; how much mechanical effect does the friction of this pivot consume in one second? The co-efficient of friction being taken $= 0{,}1$ we have the friction $f R = 0{,}1 \cdot 1800 = 180$ lbs., the space per revolution $= \frac{4}{3} \pi r = \frac{4}{3} \cdot 3{,}14 \cdot \frac{1}{24} = 0{,}1745$ ft. lbs., hence the mechanical effect per revolution $= 180 \cdot 0{,}1745 = 31{,}41$ ft. lbs. But now this machine makes in a second $\frac{100}{60} = \frac{5}{3}$ of a revolution, hence it follows that the loss of mechanical effect sought $= \frac{314{,}1}{6} = 52{,}3$ ft. lbs.

§ 172. *Pointed Axles.*—If the axle ABD, Fig. 179, has conical ends, the friction comes out greater than if it has plane ends, because the pressure of the axle R is resolved into the normal forces N, N_1, which produce the friction, and which together are greater than R alone. If the half of the convergent angle $ADC = BDC = \alpha$, we have $2 N = \frac{R}{\sin. \alpha}$, and consequently the friction of the pointed axle $= f \frac{R}{\sin. \alpha}$. Let the radius of the axle $CA = CB$ at the entrance into the step be re-

Fig. 179.



presented by r_1, we shall then have as before the moment of friction $= \frac{2}{3} f \frac{Rr_1}{sin.\ a}$. Let this axle dip a little only into the step, the mechanical effect of this axle will be less than that of an axle with a plane base, and on this account the application of the pointed axle is of service. When, for example, $\frac{r_1}{sin.\ a} = \frac{r}{2}$, therefore, $r_1 = \frac{1}{2} r\ sin.\ a$, the pointed axle of the radius r_1 causes only half the loss of mechanical effect through friction which the truncated axis of the radius r does.

If the pivot forms a truncated cone, Fig. 180, friction takes place as well at the envelop as the truncated surface, and the moment of friction comes out $= \left(r_1^3 + \frac{r^3 - r_1^3}{sin.\ a}\right) \frac{2}{3} \frac{f R}{r^2}$, if r be the radius of the place of entrance into the step, and $r_1 = DE$ that of the base, and a° the half of the convergent angle.

Fig. 180.

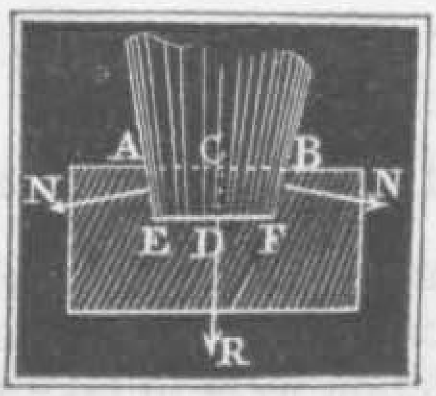


Fig. 181.

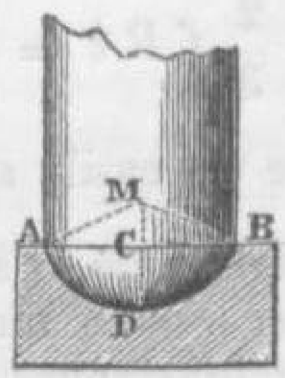
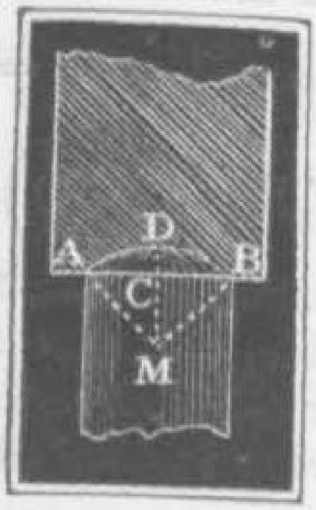


Fig. 182.



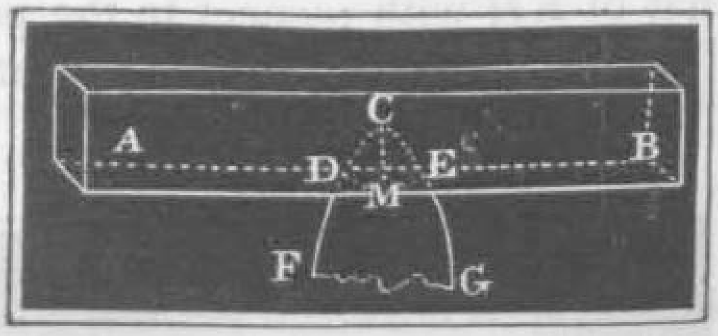
Lastly, the pivot or upright axles (Figs. 181, 182) are very often rounded. Although by this rounding, the friction itself is by no means diminished, there arises nevertheless a diminution of the moment of friction, from the extremity not dipping far into the step. If we suppose a spherical rounding, we obtain by the aid of the higher calculus for a semi-spherical step, the moment of friction $= \frac{f\pi}{2} . R r$; but for that of a step having a less segment $= \frac{2}{3}\left[1 + 0{,}3\left(\frac{r_1}{r}\right)^2\right] Rr_1$, r being the radius of the sphere $MA = MB$, r_1 the radius of the step $CA = CB$.

Example. If the weight of the armed axle of a horse capstan $R = 6000$ lbs., the radius of the conical pivot $= r = 1$ inch, and the angle of convergence of the cone $2a = 90^\circ$, then the moment of friction of this pivot $= \frac{2}{3} . f . \frac{Rr}{sin.\ a} = \frac{2}{3} . 0{,}1 . \frac{6000}{sin.\ 45^\circ} . \frac{1}{12} = \frac{100}{3\sqrt{\frac{1}{2}}} = 47{,}1$ ft. lbs. This axle makes during the lifting up of a ton from a shaft or mine $= u = 24$ revolutions, then the mechanical effect which is expended at the pivot in this time by friction $= 2\pi u . \frac{2}{3} f \frac{R r}{sin.\ a} = 2\pi . 24 . 47{,}1 = 7103$ ft. lbs.

§ 173. *Points and Knife Edges.*—To avoid as much as possible the friction of the axle, rotatory bodies are supported on pointed pivots, knife edges, &c. If we had only to do with perfectly rigid and inelastic bodies, no loss of labor would arise through friction by this method of support or suspension, because no measurable space here is described by the friction; but since every body possesses a certain degree of elasticity, by the resting of such a body on a point or knife edge, a slight penetration takes place, and a rubbing surface is thereby caused, upon which a space is described by the friction which gives rise to a certain loss of labor, although very small. In rotations and vibrations long sustained, bodies supported in this manner, present similar surfaces of friction arising from the abrasion of their points or knife edges, and the friction must then be estimated by what has been already mentioned. For these reasons this mode of support is applicable only to such instruments as the compass, the balance, &c., where it is of importance to diminish the amount of friction, and where motion is only allowed from time to time.

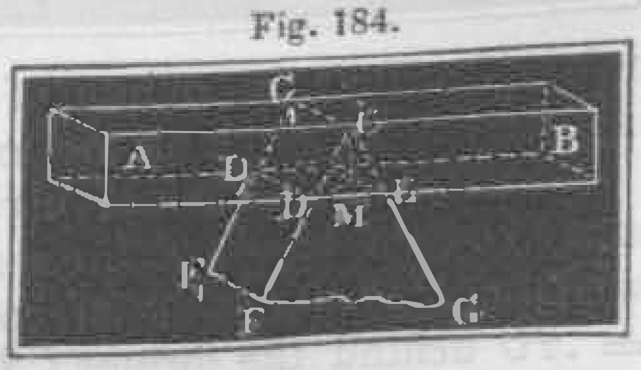
Experiments on the friction of a body resting upon a hard steel point, and revolving about it, have been made by *Coulomb.* From these, it results that the friction increases somewhat more than the pressure, and varies with the thickness of the supporting pivot. It is least for a granite surface, greater for one of agate and of rock crystal, greater still for a glass surface, and greatest of all for a steel one. For a very small pressure, as in the magnetic needle, the pivot may be pointed to 10° or 12° of convergence. But if the pressure is great, we must then apply a far greater angle of convergence, viz. 30° to 45°. The friction is less when the body having a plane surface reposes upon a point than when it lies in a conical or spherical step. Similar relations take place in the knife edge as applied to the balance, and the beams of balances, that are intended for heavy loads, require sharp axes of 90° convergence, while an edge of 30° is sufficient for the lighter ones.

Fig. 183.



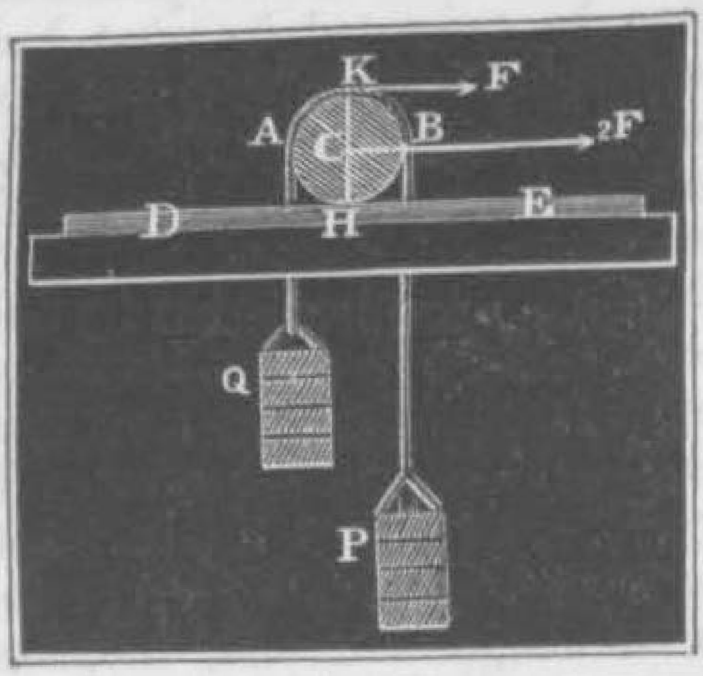
Remark. If we assume that the needle AB, Fig. 183, rests on the point DCE of the pivot FCG, of the height $CM = h$, and radius $DM = r$, and suppose that the volume $\frac{1}{3} \pi r^2 h$ is proportionate to the pressure R, the amount of friction may be found in the following manner. If we put $\frac{1}{3} \pi r^2 h = \mu R$, where μ is a number resulting from experience, and introduce the angle of convergence $DCE = 2\alpha$, and, therefore, put $h = r$ cotg. α, we obtain the radius of the base $r = \sqrt[3]{\frac{3 \mu R \, tang.\, \alpha}{\pi}}$, and $f r R = \sqrt[3]{\frac{3 \mu R^4 \, tang.\, \alpha}{\pi}} = \sqrt[3]{\frac{3\mu}{\pi}} . R^{\frac{4}{3}} (tang.\, \alpha)^{\frac{1}{3}}$. We must, therefore, assume that the friction on the pivot increases equally with the cube root of the fourth power of the pressure, and the cube root of the tangent of half the angle of convergence. The amount of friction of a beam AB, Fig. 184, which oscillates on a sharp edge CC_1, may be found in like manner. If α be half the angle of convergence DCM, l the length CC_1 of the edge, and R the pressure, this amount is given $= \sqrt{\frac{(P \, tang.\, \alpha)^3}{l}}$.

Fig. 184.



§ 174. *Rolling Friction.*—The theory of rolling friction is by no means firmly grounded; we know that it increases with the pressure, and that it is greater for a smaller than for a larger diameter of the rolling body; but in what algebraical dependence this friction stands to the pressure and diameter of the rolling body, cannot as yet be considered as determined. *Coulomb* made only a few experiments with rollers, from two to twelve inches thick, of lignum vitæ and elm, which were rolled along a surface of oak, by means of a thin thread passing over the roller AB, whose extremities were stretched by unequal weights P and Q, Fig. 185. From the resultshof these experiments, rolling friction appears to increase directly with the pressure, and inversely with the diameter of the roller, so that the force necessary to overcome this friction may be expressed by $F = f \cdot \frac{R}{r}$, if R be the pressure, r the radius of the roller, and f the co-efficient of friction derived from experiment. If r be given in inches, then from these experiments

Fig. 185.

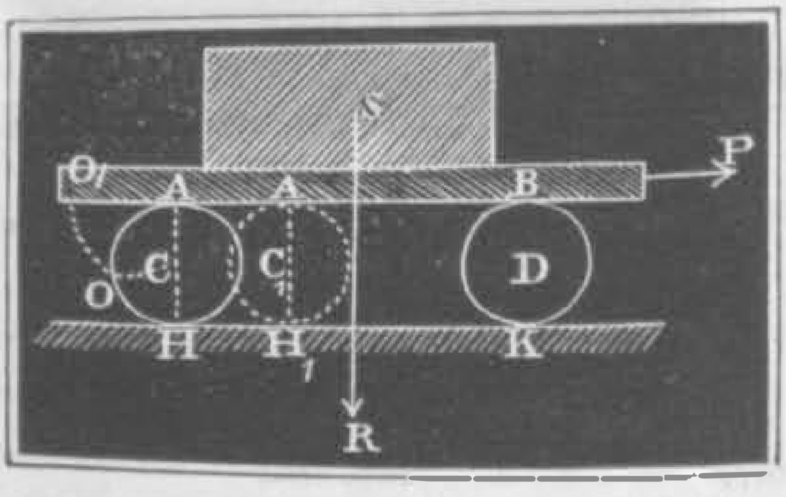


For rolling upon compressed wood $f = 0{,}0189$,
" " " " $f = 0{,}0310$.

These formulas suppose that the force F acts at the circumference of the roller, but if the force be applied to the axis C of the rolling bodies, by which, as in every description of carriage, axle friction ensues, the required force is $2\ F$, because here the arm CH is only half that of KH with respect to the point of application H.

A body ABS is moved forwards, Fig. 186, lying on the rollers C and D, the required force P here comes out very small, because two rolling frictions only, viz., that between AB and the rollers, and that between the rollers and the way HR are to be overcome. The progressive space of the rollers is only half that of the load Q, and on this account for farther progression, the rollers must be replaced under it from before, because the points of contact A and B, by virtue of the rolling, recede as much as the axis of the rollers advances.

Fig. 186.



If the roller AH has revolved about an arc AO, the roller has then moved over a space AA_1 equal to this arc, and O comes into contact with O_1, the new point of contact O_1 bas, therefore, receded by $AO_1 = AO$ behind the former (A). If the co-efficients of friction are f

and f_1, the power necessary to draw the load R forward is $P = (f + f_1)\frac{R}{r}$.

Remark. The extended experiments of *Morin* on the resistance of carriages upon roads, accord with the law by which this resistance increases equally with the pressure, and inversely with the thickness of the roller. Another French engineer, *Dupuit*, on the contrary, deduces from his experiments, that rolling friction increases indeed directly with the pressure, but for the rest, only inversely proportional to the square root of the radius of the roller. Particular theoretical views upon rolling friction may be found in *Gerstner's* Mechanics, vol. i. § 537, and developed in *Brix's* Treatise upon Friction, art. 6.*

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* The following demonstrations are applicable to wheel carriages in general, and especially to railway cars and locomotives. They bring into view the relation between rolling and dragging friction, as well as the resistance of fixed obstacles to rolling bodies.

These two kinds of friction may be illustrated by the motion of a cylinder, (Fig. 186$_1$), moved over any plane surface by a force applied, first, opposite to its centre of gravity, and at *right angles* to its axis, and secondly, opposite to the same centre, but in the *same direction* with its axis. The former force will, if both the cylinder and the plane be *perfectly smooth*, unyielding, and free from foreign matter, produce a progressive motion only in the cylinder. But in every practical case such an application of force produces likewise a rotation, and, in proportion as the roughness of the surfaces prevents or resists the sliding of one over the other, in the same proportion will the *rotary*, sooner or later, correspond to the *progressive* motion. We may easily conceive, however, that while a body is moving forward with accelerated velocity, that is, while its centre of gravity advances with increasing rapidity, the revolution on its axis shall be uniform, and the motion of any point on the periphery may, at any given moment, be either greater or less than that of the centre of gravity. Should a cylinder, revolving under such circumstances, come to apply its periphery suddenly to a part of the plane, where, from the roughness of surface, it should be compelled to coincide in its revolving velocity with the motion of progression, while the two motions were coming to an equality, a portion of rubbing must take place, and the extent of surface rubbed must be equal to the difference of motion between the centre of gravity of the cylinder and the assumed point of the periphery. Thus, if while the centre traverses a straight line of four feet, the circumference revolved through five, *in the same direction*, then the extent actually rubbed over would be one foot. If it revolved in the opposite direction to that which friction would of itself produce, then we may conceive that each point of the plane passes over some one point of the cylinder, and, therefore, that there is from this cause a friction through four feet of space due to the progressive motion; and again, that each point of the cylinder rubs against some point of the plane, and produces a friction through five feet due to the rotation, and consequently, that the united effects of these opposite motions would be to change the existing rotary motion into one in the opposite direction, by a quantity equal to a direct friction through *nine* feet; that is, through the sum of the two motions. To illustrate the preceding remarks, we may easily suppose a wheel or cylinder to receive a sudden percussion, which shall cause a rapid progressive motion commencing from a state of rest; this motion may generate a degree of rotary motion, which may or may not be equal to the progressive velocity, according to the nature of the surface over which it moves. If, after a short time, the surface of the rolling body ceases to touch the plane surface and traverses free space, the rotary motion will continue nearly uniform, while the progressive motion may be greatly retarded, or may entirely cease. If in this case the body comes again in contact with the plane, the uniform rotary motion will generate a friction, which will increase the rectilinear motion, by communicating to the centre

Fig. 186$_1$.

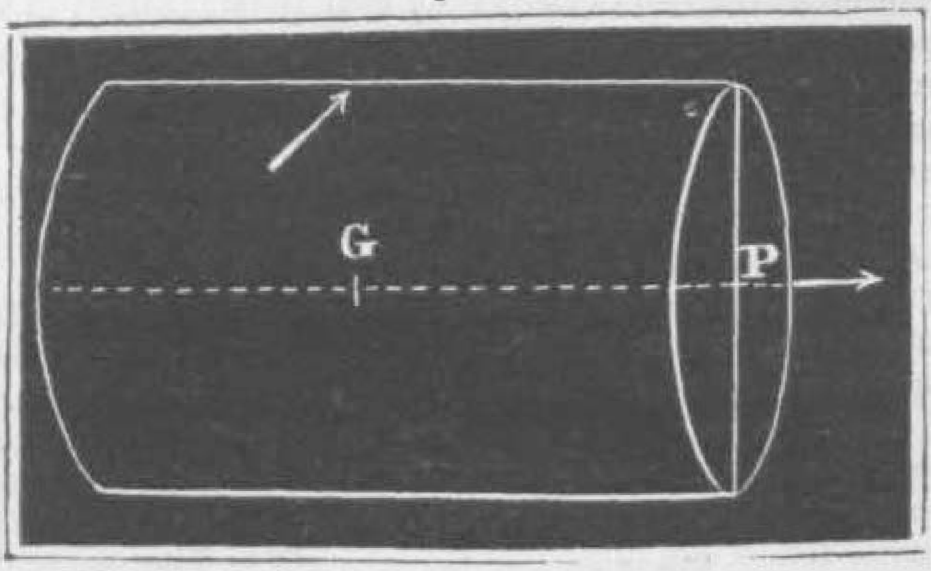
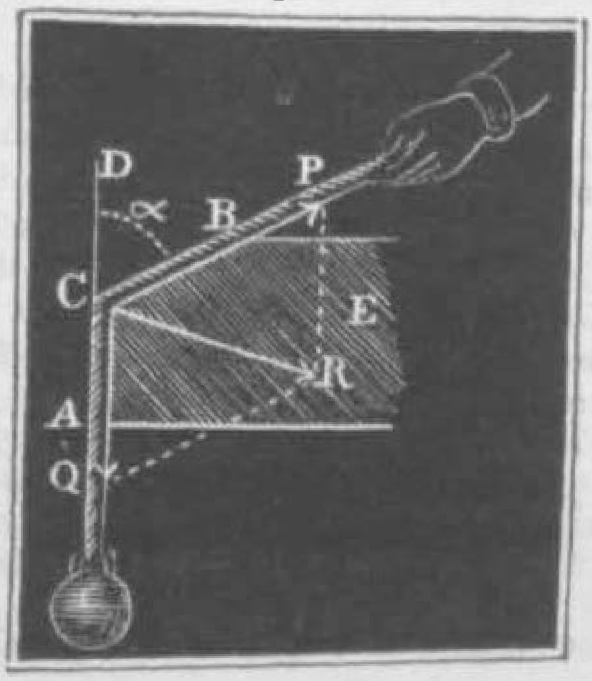


Fig. 187.



§ 175. ***Friction of Cords.***—We have now to investigate the friction of a flexible body. When a perfectly flexible cord is stretched by a weight Q over the edge C of a rigid body ABE, Fig. 187, and thereby deviates from its original direction by an angle $DCBP = \alpha°$, there arises at the edge a pressure R from which a friction F takes place, and requires for the restoration of unstable equilibrium that the force P should be greater or less than Q. The pressure is (§ 74):

$$R = \sqrt{P^2 + Q^2 - 2\,PQ \cos.\ \alpha},$$

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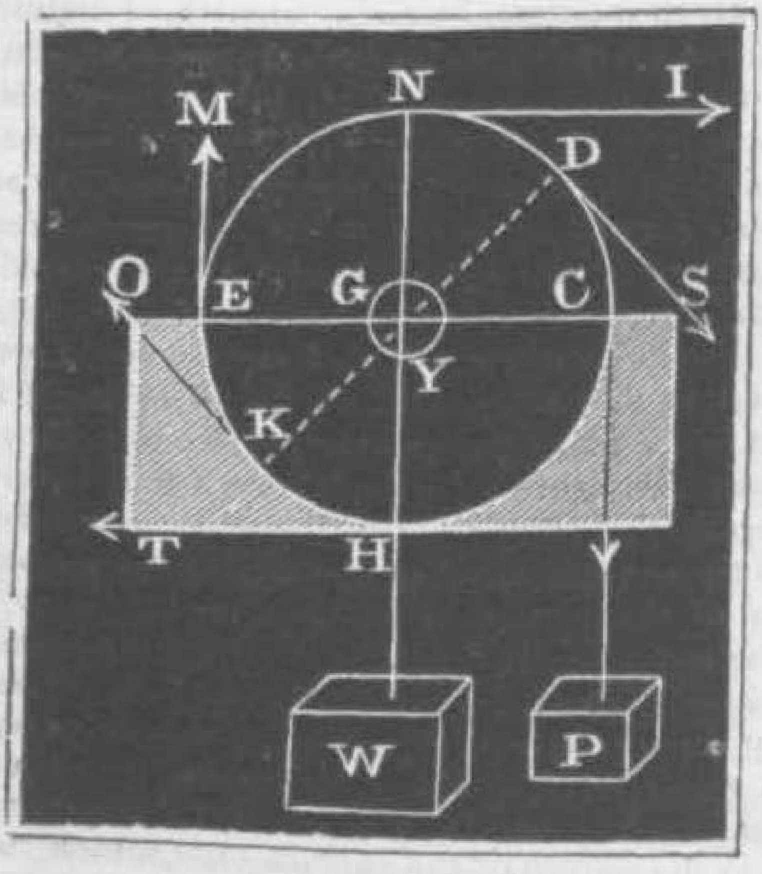
of gravity a part of that force which had been employed in producing rotation. The case now supposed is precisely that which is seen in balistics, when a cannon ball, after having received, by traversing the gun, a rotary motion, and subsequently nearly expended its force in overcoming the resistance of the air, is seen to acquire an additional onward velocity by coming in contact with the ground.† A similar transfer of motion from the rotary to the rectilinear direction, through the interference of friction, is seen, when a billiard ball is caused to retrograde on the table, by giving it an oblique stroke downwards in a direction which passes below the centre of gravity. Another example to the same effect is that of a ball falling from a tower at the same time that it revolves on some axis of its own. When it comes to the ground the rotary is converted into a rectilinear motion by the agency of friction, and the ball rolls off horizontally from the spot where it first struck.

A still more remarkable illustration of rolling friction is presented in the common rolling mill, for converting metals into plates, where it puts in motion not only the bar of metal rolled, but often one of the rollers also, notwithstanding all the friction of its axle.

If the cylinder before supposed were to be moved along a plane by a force applied in the direction GP of its length, (Fig. 186,,) and passing through the centre of gravity G, it would generate a real *dragging friction*. The same thing would be true if the solid were laid in a semi-cylindrical groove or hollow, (Fig. 186,,) and drawn along that groove endwise, or caused to revolve about its axis, its surface being in contact with the concave part of its bed, and pressing it with a force due to the weight of the cylinder. The force, then, which would be necessary to cause this body to revolve, would be equal to that which would be required to drag the cylinder lengthwise along the plane. Every revolution of the cylinder must, therefore, produce the same amount of friction as if its surface were reduced to a parallelogram, and the body were dragged without revolving through its breadth over a plane surface. Such would be the case, if the forces applied to the cylinder to produce rotation were directed in such a manner as neither to increase nor diminish the effect of gravity; as, for example, the two equal and opposite forces acting simultaneously in the directions CP and EM, or NI and HT, or DS and KO.

Fig. 186,.



But if a single weight were applied as represented at P, Fig. 186,, it is obvious that its own gravity would increase the pressure at H, and consequently, augment the friction caused by the weight of the cylinder, so that, after allowing for the friction caused by W, we must make an additional allowance for P itself, according to the nature of the materials consequently the friction:

$$F = f\sqrt{P^2 + Q^2 - 2\,PQ\cos.\,\alpha}.$$

If further we put $P = Q + F$ and P^2 approximately $= Q^2 + 2\,QF$, we then obtain:

$$F = f\sqrt{Q^2 + 2\,QF + Q^2 - 2\,Q^2\cos.\,\alpha - 2\,FQ\cos.\,\alpha}$$

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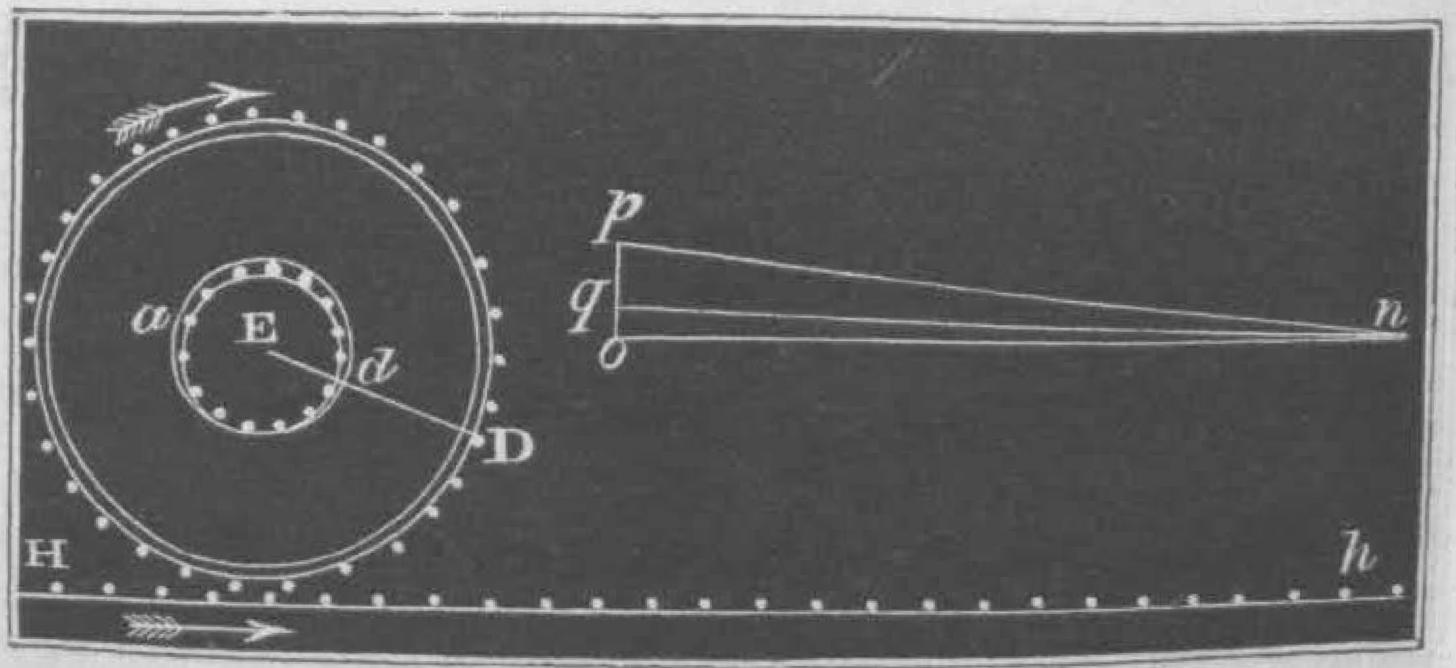
of which the cylinder and the bed, *EHC*, are composed; this allowance would again cause a new pressure and friction, and thus a decreasing geometrical series of weights must be added at the point *C*, having for the first term such a part of *W* as is expressed by the relation of pressure to friction, in the case of the given materials, and for a common ratio of the progression, the fraction expressing the same relation. The sum of all the terms, continued to *zero*, will be the actual amount of *P* at the moment when motion commences. The sum of all the terms *following the first*, will be found by *multiplying together the first and second terms of the series*, and *dividing the product by their difference. The quotient added to the first term gives the sum of the series required.* The applicability of a similar method of computation to the friction on the gudgeons of water wheels, moved by the gravity of water, is too obvious to require demonstration.

If instead of applying a weight at *P* only, we should apply, as above supposed, two equal forces, one in the direction of *CP*, and the other in that of *EM*, the amount of friction caused by the former would be relieved by the latter, and consequently, there would remain only the friction of the cylinder. The same would be true if the forces were to take either the directions *NI* and *HT*, or *KO* and *DS* respectively. Supposing the cylinder to be placed on an axis smaller in any given proportion than its own diameter, as *GV*, then the whole effect of gravity would be transferred to this axis, and if this were to be caused to revolve by a force applied tangentially to the axis itself, it must be of the same magnitude as that which had before been applied to the cylinder when placed in the groove. But if applied to the exterior of the cylinder, it must be as much less than before as the diameter of the axis is less than that of the cylinder. In other words, the difficulty of overcoming friction at the axle, is to that of overcoming the same at the outer periphery, when, confined in a bearing, as the diameter of the axle is to that of the cylinder. If *D* be the diameter of the cylinder, *d* of the axle, and *F* the relation of weight to friction, we shall have the proportion $D : d = F : \frac{dF}{D} =$, the force required to overcome the friction on the axle.

This subject may be still further illustrated by Fig. 186_3, where the horizontal plane *Hh*, is represented as furnished at equal distances, with small balls or prominences so attached to its surface as to present equal obstructions to the dragging of heavy bodies along that surface.

Fig. 186_3.



The exterior of the wheel *ED* is likewise represented as furnished with equal prominences at equal distances. When, therefore, the wheel is compelled to make one revolution without advancing, as many prominences would be broken from its periphery as would be dislodged from the plane surface while it advanced, without revolving through a space equal to its circumference. This applies to a locomotive slipping its

$= f \sqrt{2(1-\cos. \alpha)(Q^2+QF)} = 2 f \sin. \frac{\alpha}{2} \sqrt{Q^2+QF}$, which again may be put $= 2 f \sin. \frac{\alpha}{2} (Q + \frac{1}{2} F)$, if we have regard to the two

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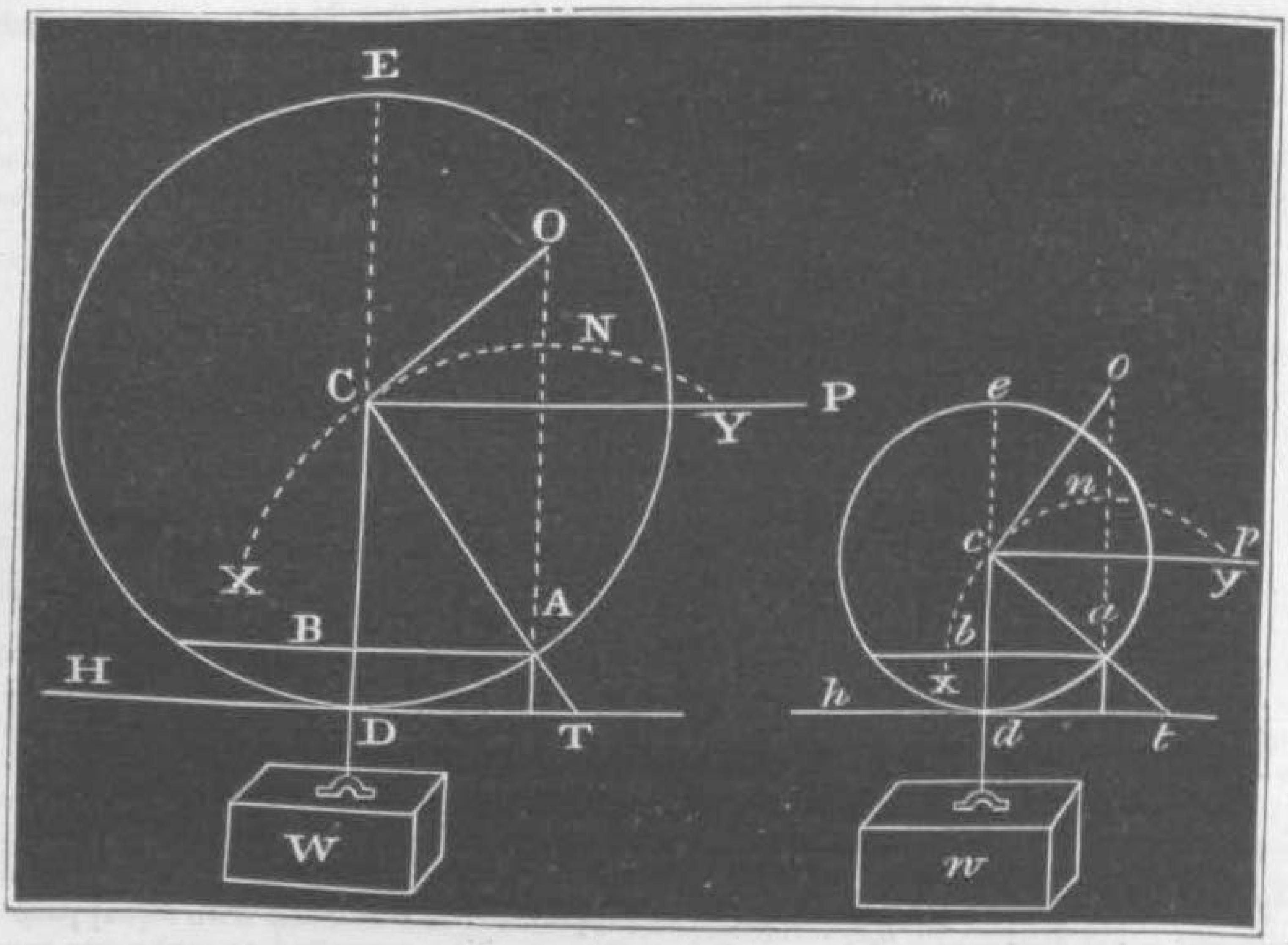
wheels on the rails in the one case, and sliding with wheels cloggedor engine reversed in the other. But when the wheel is allowed both to revolve and to advunce in such a manner as to apply its periphery to a length of plane just equal to the space traversed by the centre, the prominences will be geared together like the teeth of a rack and pinion. But in the latter case the prominences may all remain unbroken. But when the wheel rests with its whole weight on an axle, as *ad*, the number of prominences which can be disposed at the same distances as before, on the circumference of the axle, will be diminished, in proportion as the radius *Ed* of the axle is smaller than *ED*, that of the wheel.

When a wheel or cylinder rolls on a surface as nearly *plane* as it is possible for art to produce, the amount of friction, being no more than is due to the moment of inertia, is extremely small compared with that of dragging, but the observations already made, and the examples cited, will be sufficient to show that the actual advancement tends, by a force equal to that which produces the rotation, to break down the prominences of the surface, for if we consider the cylinder rolled forward by a fine thread unrolled from its upper side, we may consider also the plane beneath to oppose a force tending to draw it in the opposite direction, and this force is friction.

In experimenting with wheel carriages, or cars descending by their own gravity along inclined planes, to ascertain the ratio of *friction* to *weight*, we have to determine separately the rolling friction of wheels and axles with various weights and diameters, and then their influence combined with that of the insistent weight of cars and loads, which latter can alone produce sliding friction at the axle. The weight of the wheels resisted only by the slight amount of rolling friction at the periphery, tends to accelerate the velocity of the car and load. If we suppose the wheels and axle only to be placed on a plane *n q o*, Fig. 186$_3$, so little inclined as just to continue their rolling motion, and afterwards on another *n p o* so much inclined as to allow a car to descend with all the friction at its axles, we shall readily conceive that over the latter plane the wheels would, by themselves, have descended with a constantly accelerated motion, and consequently, that they

Fig. 186$_3$.



first members of the square root only. Now if $F - f\,F\,sin.\frac{a}{2}$ is given $= 2f\,Q\,sin.\frac{a}{2}$, then the friction sought is

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would, to the extent of their accelerating force, overcome a portion of the resistance which friction opposes to the motion of the car.

Thus, in every case where we would compute the effect of friction by comparing the actual distance passed over by a carriage, with the theoretical descent as caused by the inclination of the plane, we must consider the weight of the car and load as the cause of friction on the axle, and the gravitating power of the wheels (and that of the axles when they revolve with the wheels), as aiding to overcome the friction occasioned by the load.

To compute the effect of any obstacle of given height which a rolling body is compelled to surmount, as dependent on the diameter of the wheel, we may take two wheels EAD and ead, Fig. 186_4, of different heights, intended to surmount the equal obstacles TA and ta. Let the weights W and w be the same for both wheels, and the powers P and p be such as to produce an equilibrium in the wheels DAE and dae respectively. Then since (§ 75 and 139) three forces are in equilibrium, where each is represented by the *sine of the angle comprehended between the directions of the other two*, the direction of gravity and that of the horizontal line of traction being at right angles to each other, the sine of the angle comprehended between their directions is equal to *radius*, and is, therefore, represented by CA or ca. Again, the horizontal force is represented by the sine of the angle BCA or bca, which is the line BA or ba, while the vertical force or gravitating power of W is represented in the two cases by the sines of BAC and bac respectively, which are the lines BC and bc. These two forces multiplied respectively by the distance at which they act, in a perpendicular direction from the point A or a, about which the wheel must revolve in order to surmount the obstacle, ought to give equal moments to those of W multiplied by BA and ba. Hence $BA \times W = BC \times P$, or $P = \frac{BA \times W}{BC}$ and $ba \times W = bc \times p$, or $p = \frac{ba \times W}{bc}$. In order, therefore, to know the absolute values of P and p, we must determine the actual lengths of BA and BC, of ba and bc. BC is easily found by subtracting the perpendicular height of the obstacle from the radius, and BA is found by subtracting the square of BC from that of AC, and extracting the square root of the remainder. Thus $\overline{CA}^2 - \overline{BC}^2 = \overline{BA}^2$. But $BC = CA - BD$, therefore $\overline{BC}^2 = \overline{CA}^2 - 2\,CA \times BD + \overline{BD}^2$. Substituting this value of $\overline{BC}^2$ in the equation $\overline{CA}^2 - \overline{BC}^2 = \overline{BA}^2$, we obtain $2\,CA \times BD - \overline{BD}^2 = \overline{BA}^2$; hence $BA = \sqrt{2\,CA \times BD - \overline{BD}^2}$. The value of P, therefore, must be $\frac{W \times \sqrt{2\,CA \times BD - \overline{BD}^2}}{BC, \text{or } (CA - BD)}$, which is now expressed in terms of the radius and versed sine. Substituting R for the radius of the larger wheel, and r for that of the smaller, as also h for the perpendicular height of the obstacle in both cases, the above expression of the value of P becomes $\frac{W \times \sqrt{2\,Rh - h^2}}{R - h}$, and by a course of reasoning precisely similar, we obtain $p = \frac{W \times \sqrt{2\,rh - h^2}}{r - h}$. Hence

$$P : p = \frac{W \times \sqrt{2\,Rh - h^2}}{R - h} : \frac{W \times \sqrt{2\,rh - h^2}}{r - h}, \text{ or } = \frac{\sqrt{2\,Rh - h^2}}{R - h} : \frac{\sqrt{2\,rh - h^2}}{r - h},$$

or, as the sines of the angles DCA, dca, divided by the cosines of the same angles. And since the relation of sine to cosine is the same as that of tangent to radius, we may substitute the proportion $P : p = \frac{tang.\,DCA}{R} : \frac{tang.\,dca}{r}$. We may be assured that this statement is true, when we consider that the centre of gravity of the wheel, where we suppose the force P or p, to be applied, must describe the curve CNY or cny, and that it must commence its motion in the direction CO or co, Fig. 186_4, and that the greatest effort will be required when the centre is at C and c respectively. But the tangent CO, which expresses the length of the inclined plane which the weight begins to surmount, is of the same magnitude as the tangent DT of the angle DCA, and the successive planes will be more nearly coincident with the horizontal line. Having, on the foregoing principles,

$$F = \frac{2 f\, Q\, sin.\, \frac{a}{2}}{1 - f\, sin.\, \frac{a}{2}}, \text{ which generally is} = 2 f\, Q\, sin.\, \frac{a}{2}\left(1 + f\, sin.\, \frac{a}{2}\right)$$

and indeed very often $= 2 f\, Q\, sin. \frac{a}{2}$. In order to draw the cord

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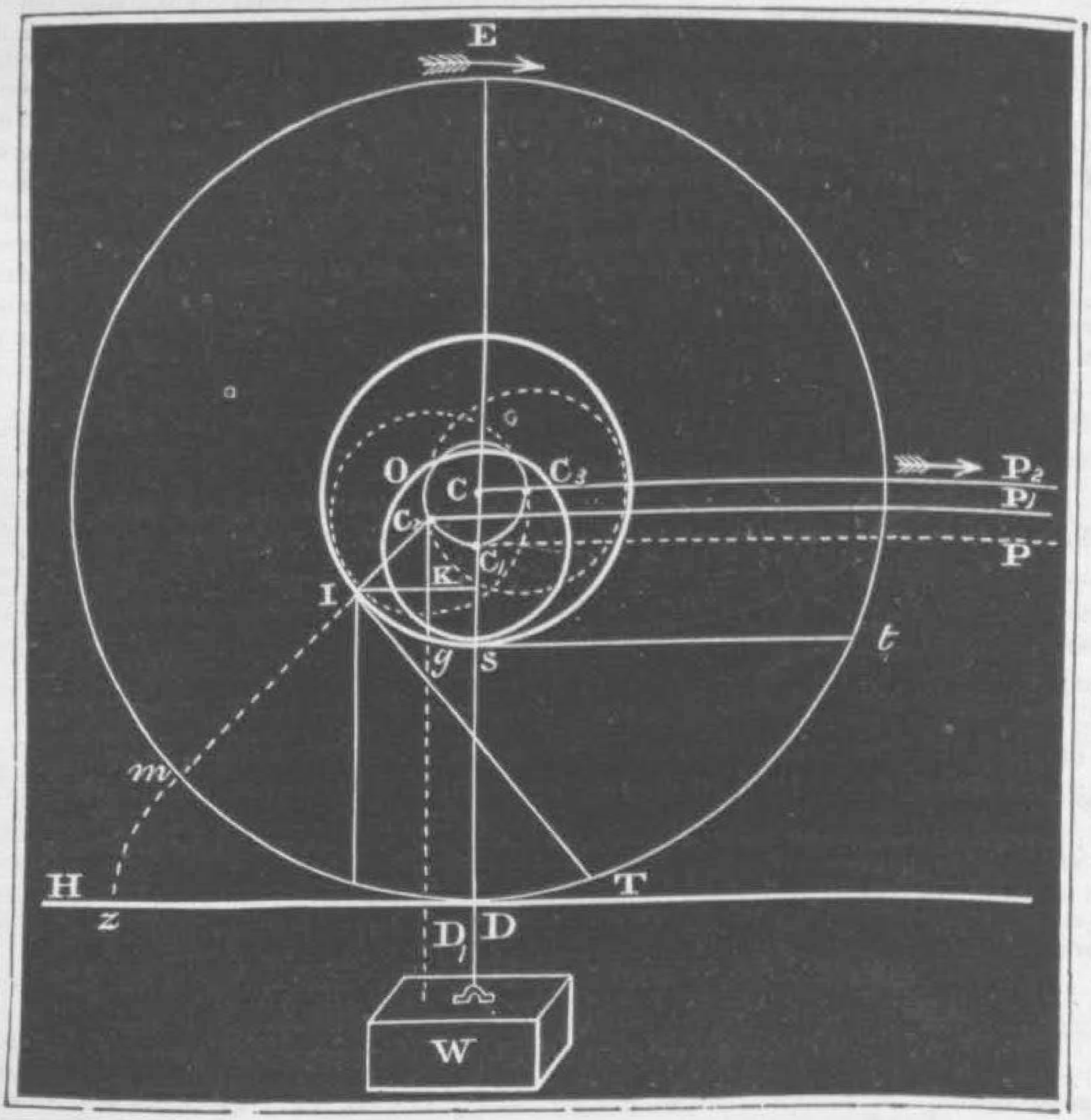
determined the effect of obstacles which oppose the rolling of a wheel, we may proceed to consider the influence of that resistance on the quantity of friction at the axle.

As carriage wheels are ordinarily constructed, and as roads, and especially railroads, are commonly made, the resistance of obstacles at the circumference is much more easily overcome than that of friction at the nave. Thus, in Fig. 186_3, where the wheel

Fig. 186_1.



$mDTE$ turns on its axle, sO, the line or spoke sD becomes the proper representative of a suspended lever, impelled at the upper end in the direction of st or C_1P by the resistance of friction at s, and at the lower by the resistance of opposing obstacles, (the amount of which resistance has just been stated,) in the direction DH. It generally happens, however, that, except on very rough roads, the resistance from the latter cause is less than that from the former. Hence, the wheel commences its forward motion sooner than the axle begins to slide in the box; so that its sliding motion is not (except in cases of great resistance at the periphery), in the direction of the tangent st, but in that of some other over the edge, a force $P = Q + F = \left(1 + \dfrac{2f \sin. \frac{a}{2}}{1 - f \sin. \frac{a}{2}}\right) Q$ is necessary;

and inversely, to prevent the descent of the weight Q by the cord, a

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line, as IT. The axle, then, ought to be found bearing not on the bottom, but on a part *back of the lowest line of the cylinder*. Having been led to this conclusion from the theory now developed, the writer was induced to inquire of several wheelwrights, coachmakers, carriage-smiths, and keepers of livery stables, whether they had ever noticed the fact, or whether they supposed it to be true, that the axle *did* rest in its box elsewhere than on its lowest part; all, after a moment's reflection, answered, that as a force was applied to draw it forward, it must press and be most worn against its front and lower side; but upon examining the old axles in their possession, they have uniformly found the above views to be confirmed by evidence which they could not doubt. In moving a carriage, then, the animal exerts his strength to bring the axle into such a position that it will descend by the gravity of the load along an inclined plane as if to follow some direction, as IT. If the axle be smaller than the box, so as to leave considerable space between them, the centre may retreat from the vertical C_1D, ascending at the same time from C_1 to C_2, where it exercises a gravitating force due to the weight, and in the direction $C_2 D_1$, acting of course on the arm of the lever equal in length to D_1 and D, and which would, if the force P were relaxed, cause the wheel to retreat and again depress the point m towards z, describing the portion mz of a cycloidal curve. This effect is often observed to take place. This position of the axle likewise accounts for the *retrograde rotation* of a wheel which is sometimes observed to take place through a portion of a revolution, when a heavily loaded car first passes from rough ground to smooth ice. The gravitating force, when the centre takes the position C_2, may be resolved into $C_1 I$, perpendicular to the side of the box, and the inclined tangent Ig; then the force which presses the surface causes the friction and opposes motion, is less than when it lies on the horizontal plane st, and the friction is diminished in proportion as $C_2 I$ is less than $C_2 g$. Again, as the force P *now* acts in the direction $C_2 P_1$, it tends to relieve even the remaining portion $C_2 I$, in the ratio of that line to $C_2 K$. When the force necessary to surmount the obstacle becomes infinite, the centre of the axle will take the position C_3, but this can happen only when the height of the obstacle is equal to the radius of the wheel. The tangent of the angle formed at the centre will then be infinite also, and the expression before given, viz. $P = \dfrac{W \, tang. \, DCA}{R}$, will be as applicable to this extreme case as to any other where the height of obstacle is less.

The conclusions drawn from the foregoing remarks, are, that the friction of a roller, moving over a horizontal surface, depends on the relation between the velocity of the periphery and that of the centre of gravity; also, that this relation between the tangential velocity and that of transportation, will depend on *the moment of inertia of the cylinder*.

Again, the advantage of a wheel over a sledge, where the same materials are employed to slide over each other as those which compose the box and axle, is *as the radius of the wheel to that of the axle*.

If friction wheels be employed, the ratio just stated must be multiplied by the relation between their diameter and that of their axles.

The amount of friction at the axle to be overcome by the moving force will be proportional to the weight of the load, but will depend also on the obstacles which oppose the progressive rotation, and will attain its *maximum* when the height of the obstacle is equal to the radius of the wheel, at which moment the advantage of the wheel to surmount the obstacle is a *minimum* or *zero*.

The advantage of a wheel to overcome any obstacle of a given height when the plane over which the wheel moves, and the line of draught, are both horizontal, will be *as the tangent of the angle formed by a vertical drawn from the centre of the wheel ~~downwards~~, and another line drawn from the same point to the top of the obstacle, divided by ~~the radius of~~ the wheel.*

For a more full exposition of the views of the writer of this note on the friction of carriage wheels, see *Journal of the Franklin Institute*, vol. v. p. 57.—AM. ED.

force $P_1 = Q \div \left(1 + \dfrac{2 f \sin. \frac{\alpha}{2}}{1 - f \sin. \frac{\alpha}{2}}\right)$ is requisite; therefore approximately

$P = \left[1 + 2 f \sin. e \frac{\alpha}{2} \left(1 + f \sin. e \frac{\alpha}{2}\right)\right] Q$, or more simply, we may put

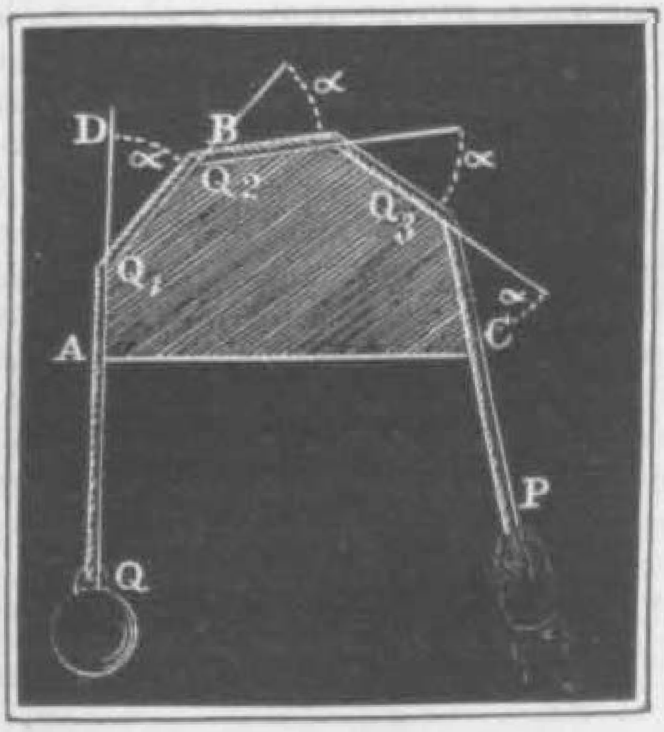
$P = \left(1 + 2 f \sin. \frac{\alpha}{2}\right) Q$, and

$P_1 = \dfrac{Q}{1 + 2 f \sin. e = \frac{\alpha}{2}\left(1 + f \sin. \frac{\alpha}{2}\right)}$, or more simply

$P_1 = \dfrac{Q}{1 + 2 f \sin. \frac{\alpha}{2}} = \left(1 - 2 f \sin. \frac{\alpha}{2}\right) Q.$

If the cord passes over several edges, the forces P and P_1 at the other extremity of the cord may be in like manner calculated by the repeated application of these formulæ. Let us take the simple case of a cord ABC, Fig. 188, passing over a body of n edges, and at each edge making the same small angle α. The tension of the first portion of the cord will be $Q_1 = \left(1 + 2 f \sin. \frac{\alpha}{2}\right) Q$, that of the extremity be $= Q$, that of the second

Fig. 188.



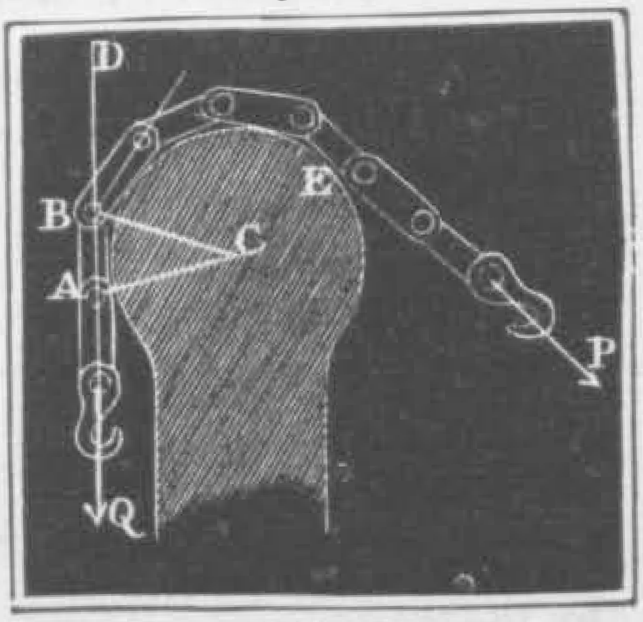
$$Q_2 = \left(1 + 2 f \sin. \frac{\alpha}{2}\right) Q_1$$

$$= \left(1 + 2 f \sin. e \frac{\alpha}{2}\right)^2 Q;$$ that of the

third $Q_3 = \left(1 + 2 f \sin. \frac{\alpha}{2}\right) Q_2 = \left(1 + 2 f \sin. \frac{\alpha}{2}\right)^3 Q$; therefore, the force at the remaining extremity $P = \left(1 + 2 f \sin. \frac{\alpha}{2}\right)^n Q$, in so far as motion takes place in the direction of the force P. If we change P into Q, and Q into P, we obtain $P_1 = \dfrac{Q}{\left(1 + 2 f \sin. \frac{\alpha}{2}\right)^n}$, provided only a motion in the direction of Q is to be prevented.

The friction $F = P - Q$ is in the first case $= \left[\left(1 + 2 f \sin. \frac{\alpha}{2}\right)^n - 1\right] Q$, and in the second $= Q - P_1 = \left[\left(1 + 2 f \sin. \frac{\alpha}{2}\right)^n - 1\right] P_1 = \left[1 - \left(1 + 2 f \sin. \frac{\alpha}{2}\right)^{-n}\right] Q.$

Fig. 189.



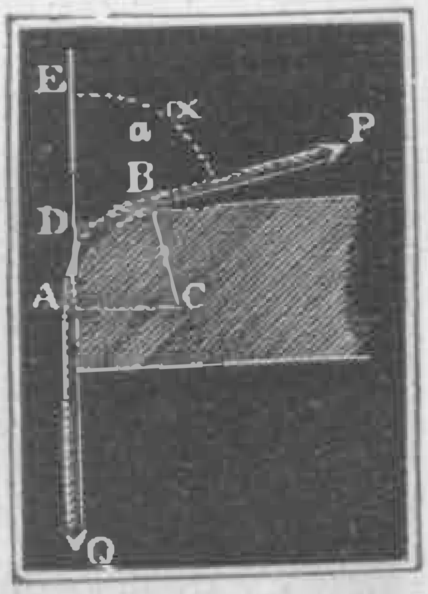
The same formulæ are applicable to a body winding round a cylinder, and consisting of members, as, for instance, a chain ABE, Fig. 189, where n is the number of links in contact, the length AB of a link $= l$, and the distance CA of the axis A of a link from the centre of the arc covered $= r$, we then have $sin. \frac{a}{2} = \frac{l}{2r}$.

Example. What is the amount of friction at the circumference of a wheel 4 feet in diameter, if twenty links of a chain, five inches long and one inch thick, pass over it, one end of which is fixed, and the other stretched by a force of 50 lbs.? Here $P_1 = 50$ lbs. $n = 20$, $sin. \frac{a}{2} = \frac{5}{48+1} = \frac{5}{49}$, let us now put for f the mean value 0,35, we then obtain the friction with which the chain acts against the wheel in its revolution:

$$F = \left[\left(1 + 2 \,.\, 035 \,.\, \frac{5}{49}\right)^{20} - 1 \right] . \, 50 = \left[\left(1 + \frac{35}{490}\right)^{20} - 1 \right] . \, 50$$

$$= \left[\left(\frac{15}{14}\right)^{20} - 1 \right] . \, 50 = 2{,}974.50 = 149 \text{ lbs.}$$

Fig. 190.



§ 176. A stretched cord AB, Fig. 190, lies about a fixed and cylindrically rounded body ACB, the friction may be likewise found from the rule of the former paragraph. Here the angle of deviation $EDB = a^{\circ} =$ the angle ACB at the centre subtended by the arc of the cord AB; if we divide this into equal parts, and consider the arc AB as consisting of n straight lines, we have then n corners, each with a deviation of $\frac{a^{\circ}}{n}$, and consequently the equation between the power and weight, as in the former §:

$$P = \left(1 + 2 f sin. \frac{a}{2n}\right)^n Q.$$

From the smallness of $\frac{a}{2n}$ we may put the $sin. \frac{a}{2n} = \frac{a}{2n}$, whence $P = \left(1 + \frac{f a}{n}\right)^n Q$. If further we make use of the binomial series, we obtain:

$$P = \left(1 + n\frac{f a}{n} + \frac{n\,(n-1)}{1\,.\,2}\,\frac{(f a)^2}{n^2} + \frac{n\,(n-1)\,(n-2)}{1\,.\,2\,.\,3}\,\frac{(f a)^3}{n^3} + \ldots\right) Q,$$

but as n is very great, therefore $n-1 = n-2 = n-3 \ldots = n$ it may be put:

$$P = \left(1 + f a + \frac{1}{1\,.\,2}\,(f a)^2 + \frac{1}{1\,.\,2\,.\,3}\,.\,(f a)^3 + \ldots\right) Q.$$

But now $1 + x + \frac{x^2}{1\,.\,2} + \frac{x^3}{1\,.\,2\,.\,3} + \ldots = e^x$, where e denotes the base 2,71828 . . e of the hyperbolic system of logarithms, therefore, it may also be put:

$$P = e^{f\alpha} \,.\, Q, \text{ as also } Q = Pe^{-f\alpha}, \text{ lastly } \alpha = \frac{1}{f}\, hyp.\, Log.\, \frac{P}{Q}$$

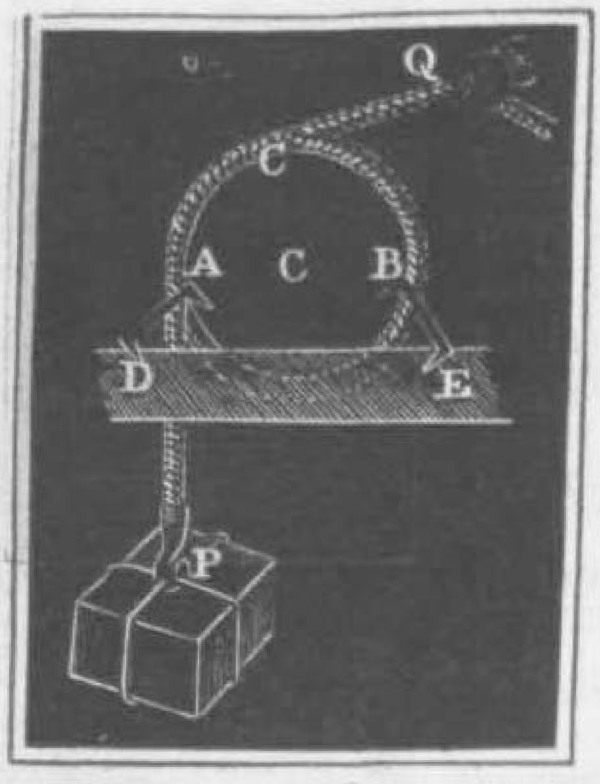
$$= \frac{2{,}3026}{f}(Log.\, P - Log.\, Q).$$

If the arc of the cord is not given in parts of π, but in degrees, we have then to substitute $\alpha = \frac{a^{0}}{180^{0}} \,.\, \pi$; if lastly, it be expressed by the number of coils u, we have then to put $\alpha = 2\,\pi\,u$.

The formula $P = e^{f\alpha} \,.\, Q$ expresses that the friction of the cord $F = P - Q$ upon a fixed cylinder is not dependent on the diameter of the same, but on the number of coils of the cord, and moreover shows that it may very easily be increased, almost to infinity. If we put $f = \frac{1}{3}$, we have:

For $\frac{1}{4}$ of a winding		$P = 1{,}69\ Q$
" $\frac{1}{2}$	"	$P = 2{,}85\ Q$
" 1	"	$P = 8{,}12\ Q$
" 2	"	$P = 65{,}94\ Q$
" 4	"	$P = 4348{,}56\ Q$, &c.

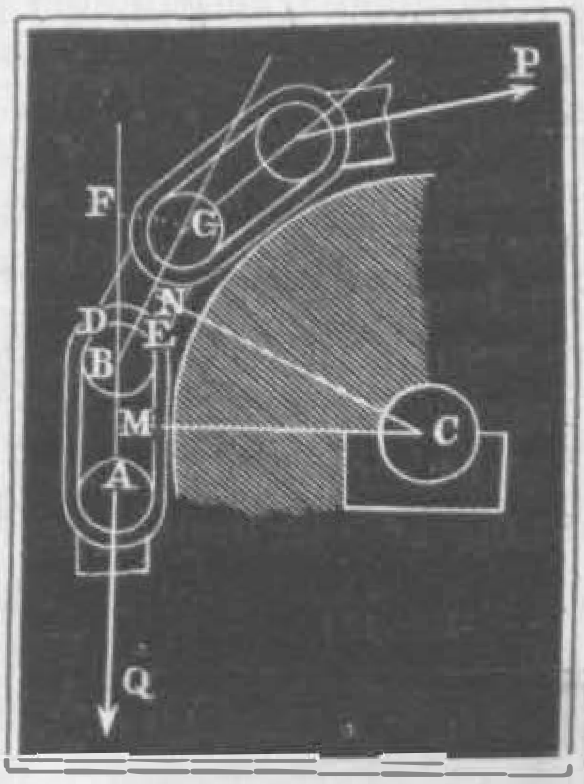
Fig. 191.



Example. To let down a shaft a load P of 1200 lbs. from a certain height, the rope to which this weight is attached is wrapped $1\frac{3}{8}$ times about a round firmly clamped holder AB, Fig. 191, and the other extremity of the rope is held by the hand. With what force must this extremity be stretched, that the load may slowly and uniformly descend? If we put $f = 0{,}3$ we obtain this power $Q = Pe^{-f\alpha}$ $= 1200 \,.\, e^{-0{,}3 \,.\, \frac{11}{8} \,.\, 2\pi} = 1200 \,.\, e^{-\frac{33}{40}\pi}$, therefore, $hyp.\, Log.\, Q = hyp.\, Log.\, 1200 - \frac{33}{40}\pi = 7{,}0901$ $- 2{,}5918 = 4{,}4983 \,.\, Log.\, Q = 1{,}9536$, $Q = 89{,}9$ lbs.

§ 177. *Rigidity of Chains.*—If ropes, or other similar bodies, &c., are placed over a pulley, or on the circumference of other cylinders revolving about an axis, the cord or chain friction considered in the foregoing paragraph ceases, because the circumference of the wheel has the same velocity as the rope; but now the force of bending by the winding of the rope about the pulley, and also that of unbending by the unwinding, becomes perceptible. If it is a chain which winds round a drum, there arises the resistance of the winding and unwinding manifested in a friction of the chain pins, while these last are revolving through a certain angle. If AB, Fig. 192, is one link, and BG the one lying next, if, further, C is the axis of revolution of the wheel on which the chain stretched by the weight Q winds itself, if, lastly, CM and CN are let fall perpendicularly to the longer axes of the links AB and BG, $MCN = \alpha^{0}$ is the angle through which

Fig. 192.



the wheel revolves whilst a fresh link is laid on, $FBG = 180^0 - ABE$ is the angle by which the link BG with its bolt BD revolves about the link AB. If now $BD = BE = r_1$ is the radius of the bolt, the point of friction or pressure D describes an arc $DE = r_1 a$, and the mechanical effect of friction $f_1 Q$ hereby produced at the point B is $f_1 Q \,.\, r_1 a$. The force P_1 expended in overcoming this friction, acting in the direction of the longer axis BG, describes the simultaneous space $s = CN$ times the arc of the angle $MCN = CN \,.\, a$, and the mechanical effect $= P_1 \,.\, CN \,.\, a$; by equating both labors we have $P_1 \,.\, CN \,.\, a = f_1 \,.\, Q r_1 a$, and the required force, if a represent the radius of the drum CN increased by half the thickness of the chain: $P_1 = f_1 Q \,.\, \frac{r_1}{a}$.

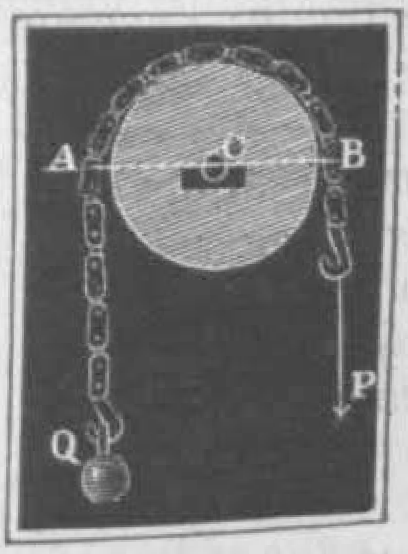
Without regard to friction, the force for a revolution of the wheel would be $P = Q$, having regard to the friction in the winding up of the chain $P = Q + P_1 = \left(1 + f_1 \frac{r_1}{a}\right) Q$. If the chain unwinds itself from the drum, an equal resistance takes place; if, therefore, a winding on one side, and an unwinding on the other take place, the force $P = \left(1 + f_1 \frac{r_1}{a}\right)^2 Q$, or approximately:

$$= \left(1 + 2 f_1 \frac{r_1}{a}\right) Q.$$

Lastly, if the pressure on the axle $= R$, and its radius $= r$, it follows that the force, taking into account all resistances, is:

$$P = \left(1 + 2 f_1 \frac{r_1}{a}\right) Q + f \frac{r}{a} R.$$

Fig. 193.



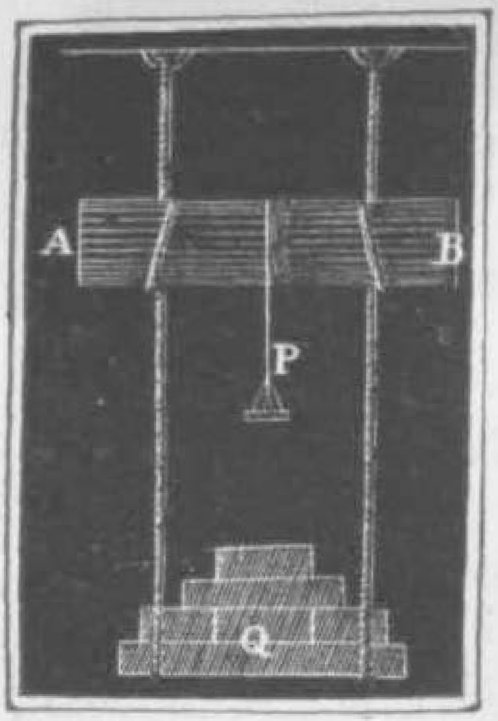
Example. What is the magnitude of a force P at the extremity of a chain passing over a pulley ACB, Fig. 193, if the weight Q drawing vertically downwards $=$ 110 lbs., the weight of the pulley with the chain 50 lbs., the radius of the pulley measured to the middle of the chain $=$ 7 in., that of the axle C $\frac{5}{8}$ inch, and that the chain bolts $= \frac{3}{8}$ in.? The co-efficients of friction $f = 0{,}075$ and $f_1 = 0{,}15$, therefore from the last formula we obtain the force:

$$P = \left(1 + 2 \,.\, 0{,}15 \,.\, \frac{3}{8 \,.\, 7}\right) \,.\, 110 + 0{,}075 \,.\, \frac{5}{8 \,.\, 7} (110 + 50 + P),$$

or, if we assume P on the right hand nearly $=$ 110 lbs. $P = 1{,}016 \,.\, 110 + 0{,}0067 \,.\, 270 = 111{,}76 + 1{,}81 + 113{,}6$ lbs.

§ 178. *Rigidity of Cords.*—In bending a cord over a pulley or wheel, rigidity comes in as a resistance opposed to motion. The same takes place, but in a far less degree, in the unrolling from cylinders. Amontons and Coulomb set about measuring the amount of this resistance by experiment. The results obtained by them are by no means satisfactory; partly because they are not in sufficient accordance with each other, and partly because they have not that extension so desirable for practical application. The experiments of Coulomb, which are those only of which we shall speak, were mostly made with hempen cords, of ¼ to ¾ inch thick, and with pulleys of from 1 to 4 inches diameter. Other experiments must be made before we can know what is the resistance of rigidity of a hempen rope of from 2 to 3 inches thick, when wrapped round a drum of from 1 to 6 feet in height; and also what is the amount of this resistance in the case of the wire-ropes, now come generally into use.*

Fig. 194.



Coulomb made hish experiments in two ways; at one time with the apparatus of Amontons, Fig.hl 94, where AB is a roller, with two cords winding round it, the tension is effected by a weight Q, and the rolling down of the cylinder by a second one P, which pulls, by means of a thin string at this roller; at another time, with a cylinder, which was allowed to roll upon a horizontal line, and round which a cord was wound, and from the difference of the weights suspended at both extremities, which effected a slow rolling forward, and after abstraction of the rolling friction, the resistance of the rigidity was deduced.

It results from the experiments of Coulomb, that the rigidity increases equally with the tension of the winding cord; that it consists, moreover, of a constant part K, which is no more than might be expected, because a certain force is necessary to bend an unstretched cord. It also appears that this resistance increases inversely as the diameter of the pulley; that it is, therefore, with twice the diameter of the pulley, only half as great; with three times the diameter, one-third, &c. The relation between the thickness and the rigidity of the cord is only approximately given from thesehexperiments, since the rigidity depends upon the quality of the materials, the twisting of the strings, &c. For new ropes, the rigidity was found proportional to the power $d^{1.7}$, for old $d^{1.4}$, d being the diameter of the rope. It is, therefore, only an approximation, when some assume that this resistance increases proportionally with the thickness, others with the square of the thickness of the rope.

§ 179. The rigidity of cords may be therefore expressed by the formulæ:

$$S = \frac{d^n}{a}(K + \nu Q),$$ where d is the thickness of the cord, a the radius of the pulley measured to the axis of the cord, n, K and ν,

* See Appendix.

numbers from experiment. *Prony* deduced from *Coulomb's* experiments that for new cords

$$S = \frac{d^{1,7}}{a}(2{,}45 + 0{,}053\ Q), \text{ and for old}$$

$$S_1 = \frac{d^{1,4}}{a}(2{,}45 + 0{,}053\ Q),$$ a and d being expressed in lines, Q and S in pounds. These expressions refer to the Paris measure; expressed in Prussian inches and pounds, they become,

$$S = \frac{d^{1,7}}{a}(14{,}23 + 0{,}295\ Q) \text{ and } S_1 = \frac{d^{1,4}}{a}(6{,}83 + 0{,}141\ Q).$$

As these complicated formulæ do not always give the results in accordance with experiment, we may, until other experiments supersede them, put with *Eytelwein*

$$S = \nu \,.\, \frac{d^2}{a} Q = \frac{d^2\ Q}{3500\ ah},$$ provided that a be expressed in Prussian feet, and d in Prussian lines, Q and S in the same weight, which, however, may be arbitrary. For the metrical standard $S = 18{,}6 \,.\, \frac{d^2\ Q}{a}$. This formula, as might be expected, will give satisfactory approximative results only for great tensions, as they generally occur in practice.

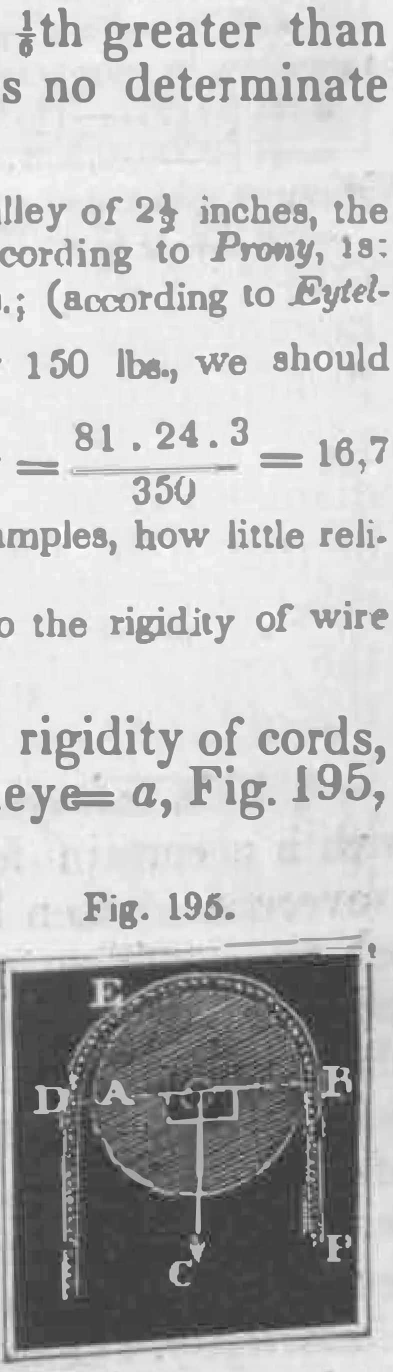
The rigidity of tarred ropes is found to be about $\frac{1}{6}$th greater than that of untarred; for wetted ropes, however, there is no determinate relation of this kind.

Example. With a tension of 350 lbs., and a radius of the pulley of $2\frac{1}{2}$ inches, the rigidity of a new rope of 9 lines $= 0{,}78$ (English) inches, according to *Prony*, is: $S = \frac{2}{5} \,.\, (\frac{3}{4})^{1,7} \,.\, 14{,}23 + 0{,}295 \,.\, 350) = 0{,}613 \,.\, 47{,}0 = 28{,}8$ lbs.; (according to *Eytelwein*) $S = \frac{9^2 \,.\, 350 \,.\, 24}{3500 \,.\, 5} = 38{,}9$ lbs. Were the tension Q only 150 lbs., we should have from *Prony*, $S = 0{,}613 \,.\, 23{,}4 = 14{,}34$ lbs.; from *Eytelwein* : $= \frac{81 \,.\, 24 \,.\, 3}{350} = 16{,}7$ lbs., therefore, here a better accordance. We see from these examples, how little reliance is to be placed on the formula.

Remark. A farther extension of this subject, viz. in respect to the rigidity of wire ropes, will be given under the article, windlass and capstan.

§ 180. Let us now apply the formula given for the rigidity of cords, to the theory of pulleys. The radius CA of a fixed pulley $= a$, Fig. 195, the radius of the axle $= r$, the thickness of rope $= d$, the weight Q at one extremity of the cord, (whose weight $= G$,) and the power which must be applied to the other extremity to draw it slowly up $= P$. Without friction on the axle, and without rigidity, P would be $= Q$, but because the axle exerts a pressure $P + Q + G$ against its bearing, there arises a friction $f\ (P + Q + G)$ which, since it acts at the radius r, makes an increase of power $\frac{f\,r}{a}(P + Q + G)$ necessary; since the rigidity of the rope must be added to this, which manifests

Fig. 195.



itself in this, that the cord does not at once take the curvature of the circumference of the pulley, but lays itself upon the pulley with an increasing curvature, and in this manner causes an extension of the arm of Q; the arm, therefore, of the weight Q is not CA but CD, and the force at the arm CB

$$= CA = a, P = \frac{CD}{CA}. Q = \left(1 + \frac{AD}{CA}\right) Q = Q + S = Q + \frac{d^n}{a}(K + \nu Q).$$

The complete equation between the power and the weight is now

$$P = Q + \frac{d^n}{a}(K + \nu Q) + \frac{fr}{a}(P + Q + G).$$

In the wheel and axle the power P acts at a different arm a to that of the weight, whose arm $= b$, therefore,

$$Pa = Qb + d^n (K + \nu Q) + fr (P + Q + G), \text{ and}$$

$$P = \frac{b}{a}Q + \frac{d^n}{n}(K + \nu Q) + \frac{fr}{a}(P + Q + G).$$

Hence the force

$$P = \frac{(b + \nu d^n + fr) Q + d^n . K + fr G}{a - fr}.$$

Example.—A weight $Q = 200$ lbs. is to be raised with the wheel and axle by a power $P = 50$ lbs.; suppose the wheel to be 1½ feet, and the pivot ½ inch radius, and the rope applied ½ an inch thick, and the weight of the whole machine 70 lbs., what radius must we give to the axle? It must be:

$$b = [Pa - d^n (K + \nu Q) - fr (P + Q + G)] \div Q,$$

therefore, in numbers if we put $f = 0{,}075$,

$$b = [50 . 18 - (\tfrac{1}{2})^{1,7} . (14{,}23 + 0{,}295 . 200) - 0{,}075 . \tfrac{1}{2} . 320] \div 200$$

$$= [900 - 0{,}308 . 73{,}23 - 12] \div 200 = 865{,}4 \div 200 = 4{,}327 \text{ inches}.$$

Without additional resistances b would be

$$= Pa \div Q = 75 \div 200 = 0{,}375 \text{ feet} = 4\tfrac{1}{2} \text{ inches}.$$

CHAPTER VI.

ELASTICITY AND RIGIDITY.

§ 181. *Elasticity.*—The parts of a rigid body adhere to each other with a certain force, which is called *cohesion*, and which must be overcome when bodies are changed in their figure and extension, or broken. The first effect which forces produce in a body, is a change in the position of their parts relatively to each other, and a resulting change of form or volume of the body. If the forces acting upon a body exceed certain limits, a separation of the parts, and a breaking of the whole body ultimately take place. The capability of bodies, which suffer a change of form by the action of forces, to resume perfectly their former state after the withdrawal of the forces, is called *elasticity*. The elasticity of every body has a certain limit. If the change of form or volume exceeds a certain amount, the body retains an alteration of its volume, even when the forces which have effected it cease to act. The limit of elasticity is different for different bodies.

Bodies which suffer a considerable change of form before this limit is attained, are called *perfectly elastic*. Those, on the other hand, in which there is scarcely any appreciable change of form preceding the limit, are called *inelastic*, although in reality there exist no bodies of this kind.

It is an important rule in building and in machinery never to load the materials to such an extent, that any alteration of their form should attain, much less exceed, the limits of elasticity.

§ 182. *Elasticity and Strength.*—Different bodies present different phenomena when their form is changed beyond the limits of elasticity. If a body be brittle, it flies into pieces. If it be ductile, as many of the metals, it will admit of alterations of form beyond the limits of elasticity, without suffering a separation of its parts. Many bodies are hard, others soft; the one opposes a great resistance to a separation of their parts, whilst the others easily allow of this to be brought about.*

In the restricted sense of the word, we understand by *elasticity*, the resistances which a body opposes to a change of form; on the other hand, by *strength*, the resistance which a body opposes to a separation of its parts. We will accordingly, in what follows, consider each of these separately.

According to the way in which external forces act upon a body, and change their form and dimensions, we distinguish the elasticity and strength of bodies, into:

1. The *absolute resistance*,
2. The *relative resistance*,
3. The *resistance to compression*, and
4. The *resistance to torsion*.

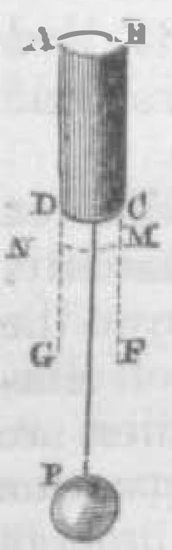
If two external forces act by tension in the direction of the axis of a body, it resists by its *absolute elasticity* and *strength* any extension or rupture. If, on the other hand, these forces act at right angles to the axis of a body, the body will resist by its relative elasticity and strength any bending or fracture. If, further, two forces act in the direction of the axis of a body by compression, so that the body becomes either compressed or crushed, then there is the *elasticity* and *strength of compression* to be overcome. If, lastly, forces strive to turn a body in opposite directions about an axis, or which do not act in the same plane normal to the axis, then there is the *elasticity* and *strength of torsion* to be overcome.

§ 183. *Modulus of Elasticity.*—The change of volume within the limits of elasticity, *i.e.* the extension or compression of a body, is pretty nearly proportional to the force exerted, but if this change exceeds that limit, this proportionality ceases, and the change goes on rapidly to that of rupture or crushing. As a measure of the elasticity, the *modulus of elasticity* E, is that which expresses the force which is necessary to elongate a prismatic body of a transverse section, unity = *i.e.* a *square foot*, to double, or to compress it to one-half of its original length. A different modulus corresponds to different mate-

* See Appendix.

rials: for each substance it must be determined by experiment. For the rest, we must bear in mind that the modulus of elasticity only holds good for extensions and compressions within the limit of elasticity, and its measure is one, not of observation, but of hypothesis and calculation, because it is not easy to find a body which, without exceeding the limit of elasticity, allows so great a change of form as the modulus of elasticity supposes.*

Fig. 196.



A body AC, Fig. 196, which has the initial length $AD = BC = l$, and the transverse section 1, requires for its extension $DG = l$, the force E, if, however, its transverse section is F, that is, if it consists of F contiguous prisms, this force is then $F \cdot E$. If, on the other hand, this body is to be extended a length $DN = CM = \lambda$, then for the force P

$P : F \cdot E = \lambda : l$, it therefore follows

1. That $P = \frac{\lambda}{l} F \cdot E$, and inversely, 2. $\lambda = \frac{P}{F \cdot E} \cdot l$.

The same formulæ are also applicable to a body AC, Fig. 197, of the length $AD = l$, and the transverse section $AB = F$, if it become shortened a length λ by the compression of a force P.

Fig. 197.

By the aid of these formulæ we may calculate from the change of volume (λ) the corresponding force P, or from the force P the quantity of the extension or compression.

Example. If the modulus of elasticity of brass wire amounts to 14625000 lbs., what force is necessary to stretch $\frac{1}{12}$ inch a wire 5 feet in length and $\frac{1}{6}$ inch in thickness? $l = 5 \cdot 12 = 60$ inches, $\lambda = \frac{1}{12}$ inch consequently $\frac{\lambda}{l} = \frac{1}{720}$; further $F = \frac{\pi d^2}{4} = 0{,}7854 \cdot \left(\frac{1}{6}\right)^2 = 0{,}0218$ square inches, the required force accordingly is $P = \frac{1}{720} \cdot 0{,}0218 \cdot 14625000 = 442$ lbs. — 2. The modulus of elasticity of iron wire is 263250000 lbs.; if an iron chain, 60 feet long and 0,2 inches thick, be stretched by a force of 150 lbs., the same will be increased by a length $\lambda = \frac{150 .}{0{,}7854 \cdot (0{,}2)^2} \cdot \frac{60 \cdot 12}{263250000} = \frac{108000}{31416 \cdot 263{,}25}$ $= 0{,}013$ inches0$= 0{,}156$ lines.

§ 184. *Modulus of Working Load and Strength.*—The force T, which a body of the transverse section unity accumulates when its extension attains the limit of elasticity, is easily determined from the modulus of elasticity E and the elongation λ corresponding to this limit, for $T : E = \lambda : 1$, therefore, $T = \lambda E$. This is the strain beyond which materials used in construction and machinery must not be loaded if they are to maintain sufficient safety together with durability. If the transverse section of a body, which has to sustain a tensile strain P be $= F$, we have then

1. $P = FT$, and 2. $F = \frac{P}{T}$.

The force T by which we judge of the working load of bodies, may be introduced into calculations under the name of *modulus of working load*.

* See Appendix.

The *modulus of strength K*, which expresses the force by whichha body of the transverse section unity becomes ruptured, is entirely different from this modulus. If the transverse section of a prismatic body, or its least section = F, it follows that the force, for the rupture of this body, is:

$$1.\ P_1 = FK, \text{ and inversely, } 2.\ F = \frac{P_1}{K}.$$

Generally the strength of materials of constrnction and parts of machines are calculated by the co-efficient K, which is divided for security's sake, by one of the numbers 3, 4 to 10. This makes little difference in the result, as we may see froin a comparison of the values found in the succeeding table, but the supposition is incorrect, or to be justified only in so far as the modulus of strength is from 3, 4 to 10 times that of the modulus of tenacity, or generally bears a constant relation to it.

If the section of the body be a circle of the diameter d, we have therefore,

$$\frac{\pi d^2}{4} = F, \text{ so that } d = \sqrt{\frac{4\ F}{\pi}} = 1{,}128\sqrt{F} = 1{,}128\sqrt{\frac{P}{T}},$$

and hence, from the load or strain P on a body, and the modulus of tenacity T of its material, the strength may be found, for which the body will not be strained beyond the limit of elasticity.

Example. 1 What load will a column of fir sustain, if it be 5 inohes in breadth and 4 inches in thickness? The modulus of tenacity being taken at 3000 lbs. and the section F being = 5 . 4 = 20 square inohes, we obtain P = 20 . 3000 = 6000 lbs. for the power of tenacity of this column. But if we take the modulus of strength K = 12000 lbs., and assume a triple security, we obtain $P = 20 \cdot \frac{12000}{3} = 80000$ lbs.; but to maintain security for a long period, we must only take one-tenth of K, and we shall then have P = 20 . 1200 = 24000 lbs.—2. A round and wroughtiron pump-rod is to sustain a weight of 4500 lbs.; what diameter ought it to have? Here T = 20000 lbs., therefore, $d = 1{,}128\sqrt{\frac{4500}{20000}} = 1{,}128 \cdot \sqrt{\frac{9}{40}} = 0{,}535$ feet. The modulus of strength for wrought iron of the medium kindt= 58000 lbs., and if we take one-sixth for the security, we then obtain K = 10000 lbs., and $d = 1{,}128\sqrt{\frac{450}{10000}} = 0.756$ inch, the requisite thickness of the rod.

§ 185. *Strongest Form of Body.*—If a vertically suspended prismatic body, for example, a pole or cord, is very long, its weight G must be added to the force of rupture, and, therefore, $P + G$ must be put $= FT$. If now l be the length of the body, and γ the weight of a cubic inch of its mass, we have then $G = Fl\gamma$, and, therefore, $P = F(T - l\gamma)$, as inversely $F = \frac{P}{T - l\gamma}$. If a body ABC . . G, Fig. 198, consists of equal portions, each of the length l, its successive transverse sections are as follows. The section of the first portion is as before $F_1 = \frac{P}{T - l\gamma}$. For the second portion, whose section is F_2

and weight $l\gamma$, $P + F_1 l\gamma + F_2 l\gamma = F_2 T$, hence $F_2 = \frac{P + F_1 l\gamma}{T - l\gamma} = F_1 + \frac{F_1 l\gamma}{T - l\gamma} = F_1\left(1 + \frac{l\gamma}{T - l\gamma}\right)$. For the third portion it follows that $F_3 = F_2\left(1 + \frac{l\gamma}{T - l\gamma}\right) = F_1\left(1 + \frac{l\gamma}{T - l\gamma}\right)^2$, for the fourth $F_4 = F_3 \left(1 + \frac{l\gamma}{T - l\gamma}\right) = F_1\left(1 + \frac{l\gamma}{T - l\gamma}\right)^3$, and generally for the nth portion; $F_n = F_1 \cdot \left(1 + \frac{l\gamma}{T - l\gamma}\right)^{n-1}$ or $F_n = \frac{P}{T - l\gamma}\left(1 + \frac{l\gamma}{T - l\gamma}\right)^{n-1}$, the corresponding section.

If l is very small, the portions therefore very short, we may then put:

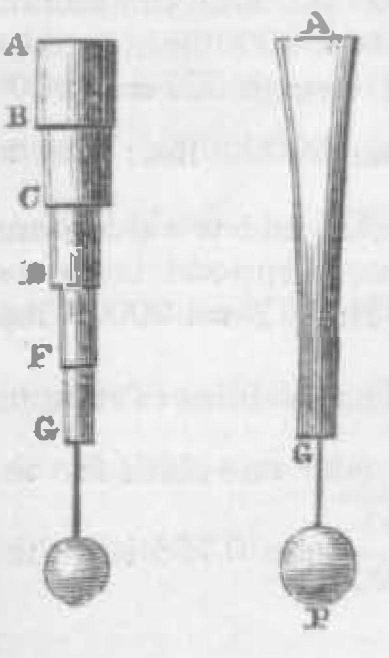
$$F_n = \frac{P}{T}\left(1 + \frac{l\gamma}{T}\right)^{n-1}.$$

If the number of portions is very great, or if the thickness of the body AG, Fig. 199, increases uniformly from below upwards, we may then (from the reasons in § 175,) put the cross section

$$F_n = \frac{P}{T} \cdot e^{(n-1)\frac{l\gamma}{T}} = \frac{P}{T} \cdot e^{\frac{nl\gamma}{T}} = \frac{P}{T} \cdot e^{\frac{L\gamma}{T}}$$

where e represents the base 2,71828 . . of the Naperien logarithms, and L the entire length of the body.

Fig. 198. Fig. 199.



A body of uniform thickness to have the same tenacity throughout, must have a transverse section $F = \frac{P}{T - L\gamma}$. If $L\gamma$ is small as compared with T, $\frac{L\gamma}{T}$ is a small fraction, so that we may put:

$$F_n = \frac{P}{T}\left[1 + \frac{L\gamma}{T} + \frac{1}{2}\left(\frac{L\gamma}{T}\right)^2\right] \text{ and}$$

$$F = \frac{P}{T}\left[1 + \frac{L\gamma}{T} + \left(\frac{L\gamma}{T}\right)^2\right],$$

further, the weight of the first body is

$$= \frac{F_1 + F_n}{2} \cdot L\gamma = \left[1 + \frac{1}{2}\frac{L\gamma}{T} + \frac{1}{4}\left(\frac{L\gamma}{T}\right)^2\right]\frac{P}{T} L\gamma;$$

and that of the second $= F \cdot L\gamma$

$$= \left[1 + \frac{L\gamma}{T} + \left(\frac{L\gamma}{T}\right)\right]\frac{P}{T} L\gamma;$$

hence the prismatic body is heavier, and on that account more costly than one having at each point in its length a cross section corresponding to the load it has to bear, and which may therefore be called a body of uniform resistance, or *a body of the strongest form.*

Examples.—1. What cross section ought a wrought iron shaft 100 feet long to have, when besides its own weight it has to sustain a load $P = 75000$ lbs.? The modulus of tenacity or strain is taken at $T = \frac{1}{8} K = 10311$ lbs., and the weight of a cubic inch of wrought iron $\gamma = \frac{7{,}60 \cdot 62{,}4}{12 \cdot 12 \cdot 12} = 0{,}27444$ lbs. The section sought is $F = \frac{P}{T - L\gamma} =$

$\frac{75000}{10311 - 1200 . 0,27444} = 7.51$ square inches, and the weight of the shaft $G = F . L\gamma = 7,51 . 1200e\ 0,27444 = 2473$ lbs.—2. If we were to give to this shaft the form of a body of uniform resistance, we should then obtain for the least section $F = \frac{P}{T} = \frac{75000}{10311} = 7,28$ square inches; for the greatest section $F_n = 7,28 . e^{0,27444 . 0,116} = 7,28 . e^{0,03183} = 7.513$ square inches, and the weight $= \left(\frac{7,28 + 7,513}{2}\right) . 329,3 = 2435,5$ lbs. (approximately).

§ 186. *Numerical Values.*—In the following table are given the mean values of the different moduli, of elasticity, tenacity, and strength of the materials most commonly occurring in construction.

TABLE I.

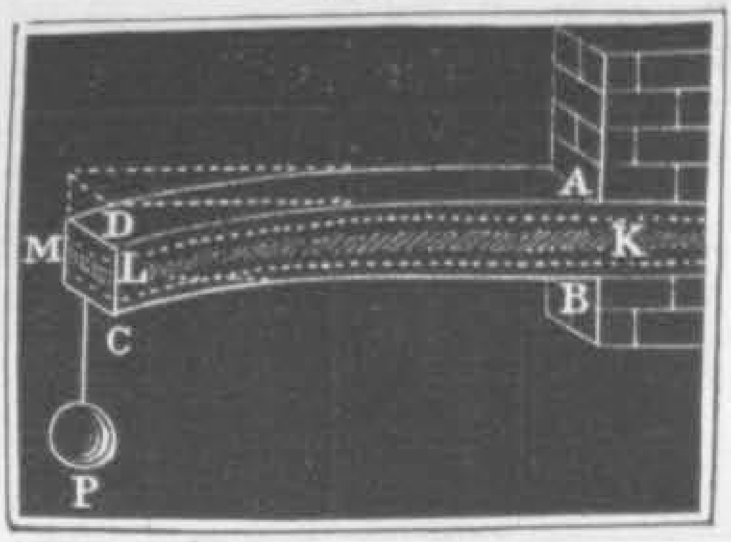
THE MODULI OF ELASTICITY AND STRENGTH.

NAMES OF THE SUBSTANCES.	Extension at the limits of the elasticity. $\frac{\lambda}{l}$	Modulus of elasticity. E.	Modulus of working load. T.	Modulus of strength. K.	Modulus of safety. K_1.
Box, oak, fir, firm Scotch fir	$\frac{1}{600}$	1856005	3094	12373	1237
Iron in wires	$\frac{1}{1250}$	26808964	21650	87645	14436
Iron in bars	$\frac{1}{1520}$	29902306	20622	59805	10311
Iron in plates		26808969		56712	9280
Cast iron	$\frac{1}{1200}$	17528938	14436	19592	3094
Steel	$\frac{1}{835}$	30933420	37120	123700	20622
Hard cast steel	$\frac{1}{4500}$	45369016	98987	150543	24740
Copper				38151	6187
Copper wire				75271	12370
Brass	$\frac{1}{1320}$	97955830	7218	18560	3093
Brass wire	$\frac{1}{742}$	149511530	20622	75271	12370
Bell metal	$\frac{1}{1590}$	48462358	3093	35058	5774
Lead	$\frac{1}{477}$	721779	1547	928	329
Leaden wire	$\frac{1}{1500}$	1031114	722	2062	351
Marble		2680896		2062	206
Ropes under 1 inch				9280	3093
" 1—3 inches				7218	2371
" above 3 "				5156	1753
Straps					299

The values contained in the second vertical column of this table, of the relative extension $\left(\frac{\lambda}{l}\right)$ at the limits of elasticity, give likewise the relation $\frac{T}{E}$ of the values of the fourth and third columns. The sixth column is derived from the fifth, if we divide the woods by 10, the metals by 6, and the cords by 3. The strength of wires is always greater than that of rods, because the enveloping crust of wires is stronger than their nucleus.

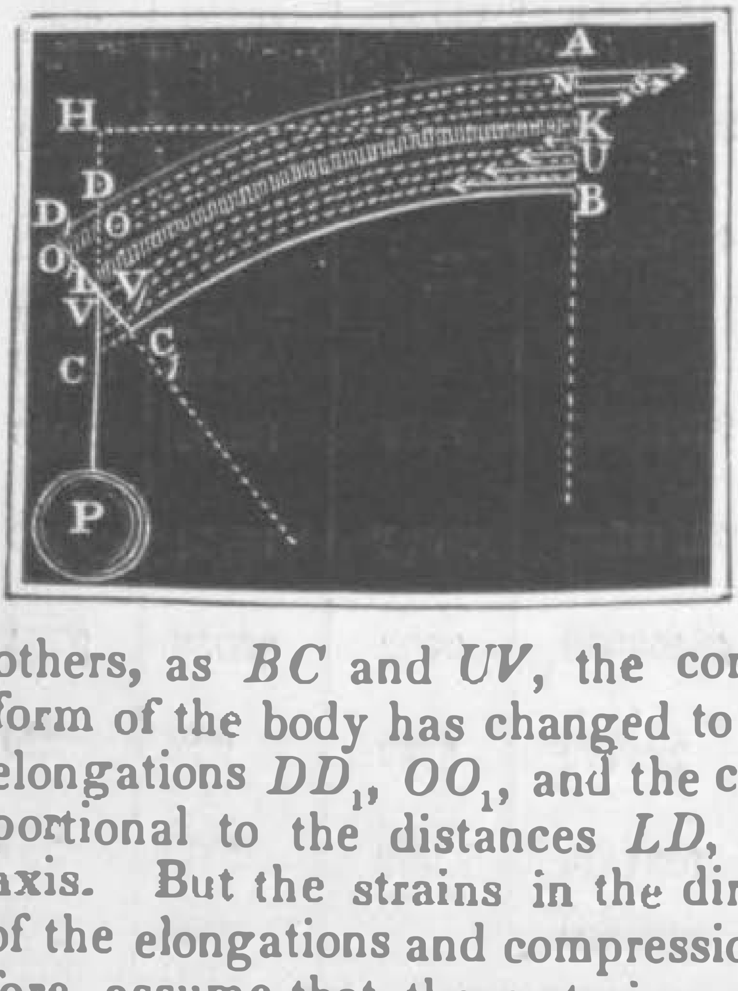
§ 187. *Flexure of Bodies.*—A prismatic body $ABCD$, Fig. 200, is fixed at one extremity, for instance, imbedded in a wall, and at the other extremity acted upon by a force P; strains then take place in this body, in consequence of which, one part is extended, and the other compressed, and the whole becomes deflected. If we imagine the whole body to be decomposed into thin laminæ by planes parallel to the axis, and at right angles to the direction of force, we may then assume that there is a certain mean lamina KLM, which is called the *neutral surface* or the *neutral axis* of the laminæ, which is not strained by this flexure, and remains unaltered in length, while the laminæ on the convex side undergo an extension, and those on the concave side a compression.

Fig. 200.



Let ABC_1D_1, Fig. 201, be the longitudinal section of the body, KL its neutral axis, NO_1 an extended and UV_1 a shortened or compressed lamina. If the flexure had taken place without any change of volume, KL would be $= AD = NO$, &c.; *i. e.* the length of all the lamina would be one and the same; the body also would have the form $ABCD$, but because the body has sustained extensions and compressions, certain laminæ, such as AD, NO, &c., have undergone the elongations DD_1, OO_1, &c., and others, as BC and UV, the compressions CC_1, VV_1, &c., and the form of the body has changed to that of ABC_1D_1. In every case the elongations DD_1, OO_1, and the compressions CC_1, VV_1, &c., are proportional to the distances LD, LO, LC, LV, &c., from the neutral axis. But the strains in the direction of the laminæ are in the ratio of the elongations and compressions effected by them; we must, therefore, assume that these strains are proportional to the distances from

Fig. 201.



* See Appendix.

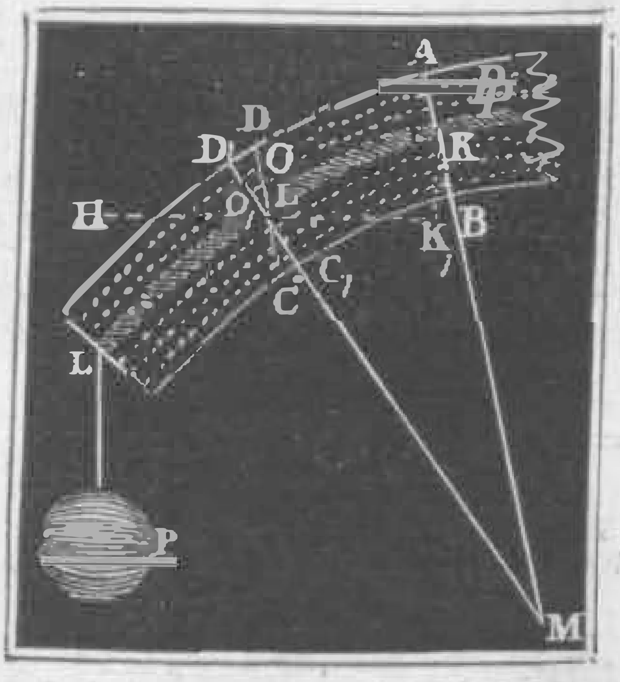
the neutral axis. If, then, we put the strain on a fibre, or layer of fibres, of a transverse section equal to unity (a square inch), and at a unit of distance (one inch) from the neutral axis $= S$; the strain for the distance $KN = z$ is Sz, and for the section F, it is FSz. If now the experimental number S represents both the extension and compression, we know the sum of all the strains $= (F_1 z_1 + F_2 z_2 + \ldots) S$, where F_1, F_2, &c., are the sections and z_1, z_2, &c., the distances from the neutral axis. In order that the tensions may produce no pressure, and therefore no alteration in the length, at the extremity K of the neutral axis, which we may regard as the fulcrum of a lever, the sum of the tensions $(F_1 z_1 + F_2 z_2 + \ldots) S$, and therefore also $F_1 z_1 + F_2 z_2 + \ldots$ must be $= 0$; *i. e. the neutral axis or the neutral lamina must pass through the centre of gravity of the cross section of the body.*

We may now compare the condition of the body with the equilibrium of a bent lever. The force P acts at the arm $KH = l$, the moment is, therefore, $M = Pl$, and balances the collective forces of extension and compression, whose moments are $z_1 \,.\, F_1 S z$, $z_2 \,.\, F_2 S z_2$, &c., or $F_1 z_1^2 \,.\, S$, $F_2 z_2^2 \,.\, S$, &c.; consequently we must put

$$M = Pl = (F_1 z_1^2 + F_2 z_2^2 + \ldots) \,.\, S.$$

This formula holds good for each cross section of the body, only for l we must substitute its distance each time from the point of application L of the force P. The factor $F_1 z_1^2 + F_2 z_2^2 + \ldots$ is dependent only on the cross section of the deflected body, and may be represented by the letter W. Hence we may put $M = Pl = WS$, and assert that the tension or strain of a transverse section is proportional to its distance l from the point of application of the force.

Fig. 202.



§ 188. From the modulus of elasticity E, the length of a fibre l at a unit of distance (an inch) from the neutral axis, and the elongation λ which it undergoes, the corresponding tension $S = \frac{\lambda}{l} E$ is known. If now ABC_1D_1, Fig. 202, is a short portion of the deflected body, $KL = l$ its length, and $MK = ML = \rho$ its radius of curvature, we have then $DD_1 : KL = LD : ML$, and also $OO_1 : KL = LO : ML$; *i.e.* $OO_1 : l = LO : \rho$. If we now assume $LO = 1$ and $OO_1 = \lambda$, we obtain $\lambda : l = 1 : \rho$, and hence $S = \frac{\lambda}{l} E = \frac{E}{\rho}$. If, finally, we substitute this value of S in the formula $M = WS$, we have the moment $M = \frac{WE}{\rho}$, and inversely, $WE = M\rho$.

The product WE is called the moment of flexure, *and hence the* *product of the moment M and the radius of curvature ρ is equivalent to the moment of flexure for all cross sections.*

If we divide the neutral axis KL, Fig. 204, into n equal parts, as

Fig. 203.

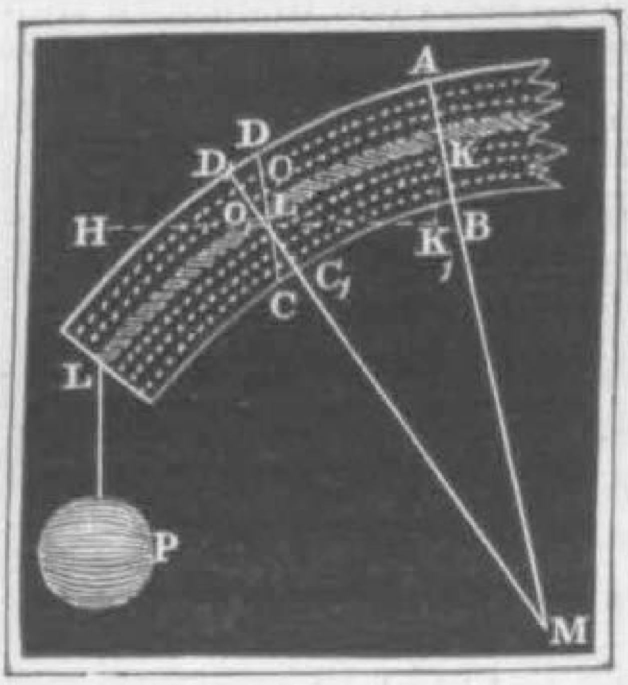
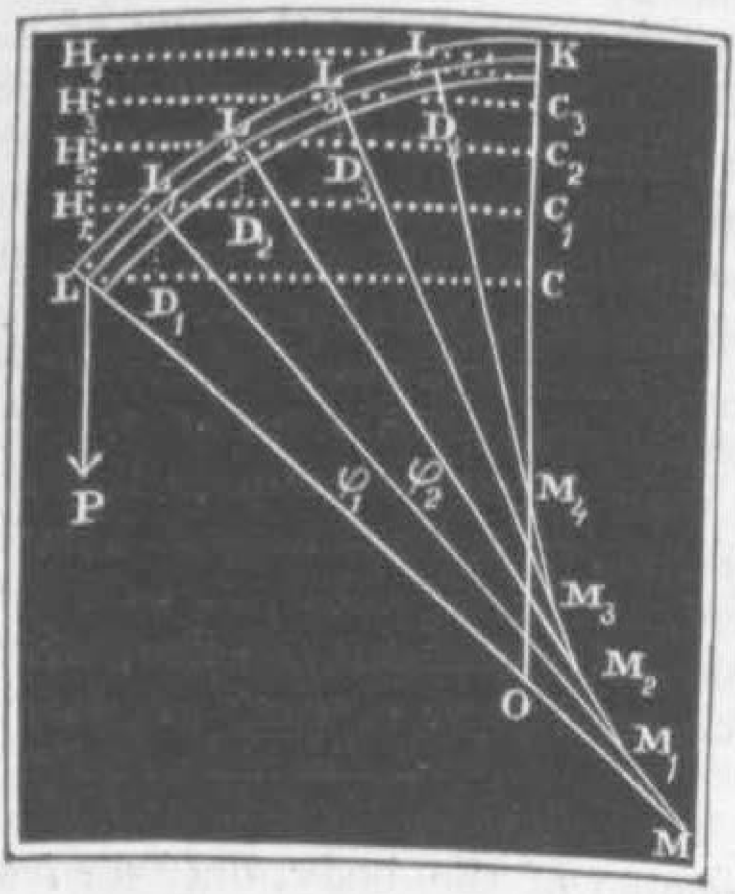


Fig. 204.



LL_1, L_1L_2, L_2L_3, &c. $= \frac{l}{n}$, and determine the radii of curvature $\overline{ML_1} = \rho_1$, $\overline{M_1L_2} = \rho_2$, &c., corresponding to these parts, the angle of curvature $\overline{LML_1} = \phi_1{}^0$, $\overline{L_1M_1L_2} = \phi_2{}^0$, &c., which every two radii of curvature include, are known, viz. $LL_1 = \frac{l}{n} = \rho_1\,\phi_1$, $L_1L_2 = \frac{l}{n} = \rho_2\phi_2$, &c., and therefore $\phi_1 = \frac{l}{n\,\rho_1}$, $\phi_2 = \frac{l}{n\,\rho_2}$, &c. If, further, we substitute $\rho_1 = \frac{WE}{M_1}$, $\rho_2 = \frac{WE}{M_2}$, &c., we then obtain $\phi_1 = \frac{M_1 l}{n\,WE}$, $\phi_2 = \frac{M_2 l}{n\,WE}$, &c.; and by the summation of all these angles we find the angle $LOK = \alpha^0$, by which a greater portion, or the whole neutral axis, is deflected.

§ 189. *Elastic Curve.*—If we suppose a small flexure, we may take the projection $CL = KH_4$, parallel to the initial direction of the undeflected beam, and equal to the length of the beam itself, and likewise the projections LD_1, L_1D_2, &c., equal to the parts LL_1, L_1L_2, &c., of the neutral axis, *i. e.* $= \frac{l}{n}$, and we obtain the moments

$M_1 = \frac{Pl}{n}$, $M_2 = \frac{2\,Pl}{n}$, $M_3 = \frac{3\,Pl}{n}$, &c. If we substitute these values in the formulæ for ϕ_1, ϕ_2, &c., then the measures of the angles of curvature are given:

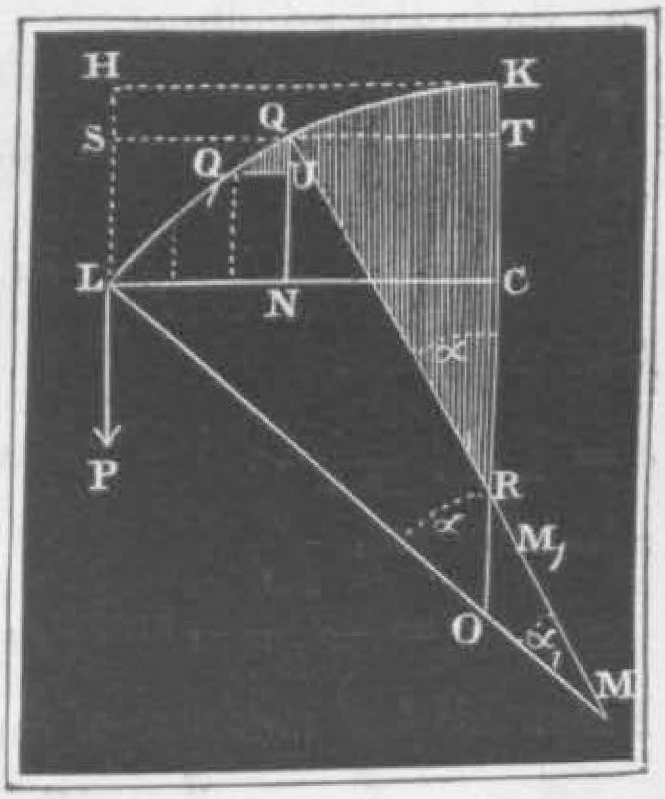
$$\phi_1 = \frac{Pl^2}{n^2\,WE},\ \phi_2 = \frac{2\,Pl^2}{n^2\,WE},\ \phi_3 = \frac{3\,Pl^2}{n^2\,WE},\ \text{\&c.};$$

and by addition, the measure of the whole angle of curvature KOL $= \alpha$ of the neutral axis:

$$\alpha = \frac{Pl^2}{n^2 WE}(1 + 2 + 3 + \ldots + n) = \frac{Pl^2}{n^2 WE} \cdot \frac{n^2}{2} = \frac{Pl^2}{2 WE}.$$

With the assistance of the last formula, we may now find the equation to the curve formed by the neutral axis, KL, Fig. 205. Let us divide the absciss $LN = x$, commencing at the point L, into m equal parts, and find the parts of the ordinate $NQ = y$ corresponding to them. Since the radius of curvature QR is perpendicular to the part of the arc QQ_1, the angle $QQ_1U = QRK = \alpha_2$, and therefore the part QU of the ordinate $y_1 = Q_1U \cdot tang. \alpha_2$,

Fig. 205.



or Q_1U being put $= \frac{x}{m}$ and $tang. \alpha_2 = \alpha_2$, $QU \frac{x\alpha^2}{m}$. Now $\alpha_2 = LOK - LMQ = \alpha - \alpha_1 = \frac{Pl^2}{2WE} - \frac{Px^2}{2WE} = \frac{P}{2WE}(l^2 - x^2)$; it follows, therefore, that $QU = \frac{x}{m} = \frac{P}{2WE}(l^2 - x^2)$. If for x^2 we substitute successively $\left(\frac{x}{m}\right)^2, \left(\frac{2x}{m}\right)^2, \left(\frac{3x}{m}\right)^2$, &c., we then obtain by the last formula all the parts of y, and by the addition of these, the whole ordinate:

$$NQ = y = \frac{x}{m} \cdot \frac{P}{2WE}\left[l^2 - \left(\frac{x}{m}\right)^2 + l^2 - \left(\frac{2x}{m}\right)^2 + l^2 - \left(\frac{3x}{m}\right)^2 + \ldots\right] = \frac{x}{m} \cdot \frac{P}{2WE}\left[ml^2 t - \left(\frac{x}{m}\right)^2 (1^2 + 2^2 + 3^2 + \ldots + m^2)\right], \text{i. e. } y = \frac{Px}{2WE}\left(l^2 - \frac{x^2}{3}\right).$$

By this formula we may calculate for every absciss x the corresponding ordinate y, and likewise for the whole length $CL = l$, the height of the arc $CK = a$. This last is:

$$ae = \frac{Pl}{2WE}\left(l^2 - \frac{l^2}{3}\right) = \frac{Pl^3}{3WE}.$$

Therefore, *the height of the arc increases as the force and the cube of the length.*

If we have a by measurement, we may find from this formula the modulus of elasticity, $E = \frac{Pl^3}{3Wa}$.

§ 190. If the whole load is uniformly distributed over the beam, and if each unit of length sustains a portion $= q$, therefore, for the whole length l, $Q = lq$, we must substitute for the moments $\frac{1}{n}\ Pl$, $\frac{2}{n}\ Pl$, $\frac{3}{n}\ Pl$, &c., the moments $\frac{1}{2}q\left(\frac{l}{n}\right)^2$, $\frac{1}{2}q\left(\frac{2\,l}{n}\right)^2$, $\frac{1}{2}q\left(\frac{3\,l}{ne}\right)^2$, &c., because the centres of gravity of the loads $q\,.\,\frac{l}{n}$, $q\,.\,\frac{2\,l}{n}$, $q\,.\,\frac{3\,l}{n}$, &c., lie in the middle of $\frac{l}{n}$, $\frac{2\,l}{n}$, $\frac{3\,l}{n}$, the arms are, therefore, $\frac{1}{2}\,.\,\frac{l}{ne}$, $\frac{1}{2}\,.\,\frac{2\,l}{n}$, $\frac{1}{2}\,.\,\frac{3\,l}{n}$. Hence we obtain

$$\phi_1 = \tfrac{1}{2}\,.\,\frac{ql^3}{n^3\,WE},\ \phi_2 = \tfrac{1}{2}\,.\,\frac{2^2\,.\,ql^3}{n^3\,WE},\ \phi_3 = \tfrac{1}{2}\,.\,\frac{3^2\,.\,ql^3}{n^3\,WE},\ \&c.$$

And therefore,

$$a = \tfrac{1}{2}\,.\,\frac{ql^3}{n^3\,WE}\,(1^2+2^2+3^2+\ldots+n^2) = \frac{ql^3}{2n^3\,WE}\,.\,\frac{n^3}{3} = \frac{ql^3}{6\,WE},$$

and likewise,

$$a_1 = \frac{qx^3}{6\,WE},\ \text{and}\ a_2 = \frac{q}{6\,WE}\,(l^3 - x^3).$$

From this last measure of the angles an element of the ordinate $= \frac{x}{m}\,a_2 = \frac{x}{m}\,.\,\frac{q}{6\,WE}\,(l^3 - x^3)$, and now for x^3 putting successively $\left(\frac{x}{m}\right)^3$, $\left(\frac{2x}{m}\right)^3$, $\left(\frac{3x}{m}\right)^3$, we have $y = \frac{x}{m}\,.\,\frac{q}{6\,WE}$

$$.\left[ml^3 - \left(\frac{x}{m}\right)^3 (1^3 + 2^3 + \ldots + m^3)\right] = \frac{x}{m}\,.\,\frac{q}{6\,WE}$$

$$.\left[ml^3 - \left(\frac{x}{m}\right)^3 . \frac{m^4}{4}\right],\ i.\,e.\ y = \frac{qx}{6\,WE}\left(l^3 - \frac{x^3}{4}\right),$$

the equation of the curve sought.

If again we take $x = l$, we obtain the height of the arc

$$a = \frac{ql}{6\,WE}\,.\,\tfrac{3}{4}\,l^3 = \frac{ql^3}{8\,WE} = \frac{Ql^3}{8\,WE} = \tfrac{3}{8}\,.\,\frac{Ql^3}{3\,WE},\ i.\,e.$$

$\frac{3}{8}$ths as great as if the load Q were suspended at the extremity of the beam.

If the beam is loaded by a weight Q, uniformly distributed, and by a force P at the extremity, the height of the arc is then

$$a = \frac{Pl^3}{3\,WE} + \frac{Ql^3}{8\,WE} = \left(\frac{P}{3} + \frac{Q}{8}\right)\frac{l^3}{WE}.$$

If a beam AMB, Fig. 206, is supported at both extremities, and loaded in its middle by a weight P, both the extremities are deflected upwards by the reactions $\frac{1}{2}\,P$ and $\frac{1}{2}\,P$, as was in the former case (§ 189), the one extremity downwards, the formula then found here holds good, if instead of P, we put $\frac{P}{2}$, and instead of the whole length $LL = l$, half the length $KL = \frac{l}{2}$. Hence the height of the arc is:

$a = \frac{\frac{1}{2}P\,.\,(\frac{1}{2}l)^3}{3\,WE} = \frac{1}{16}\,.\,\frac{Pl^3}{3\,WE}$ = a sixteenth of the height of the arc of the beam, which is loaded at its extremity.

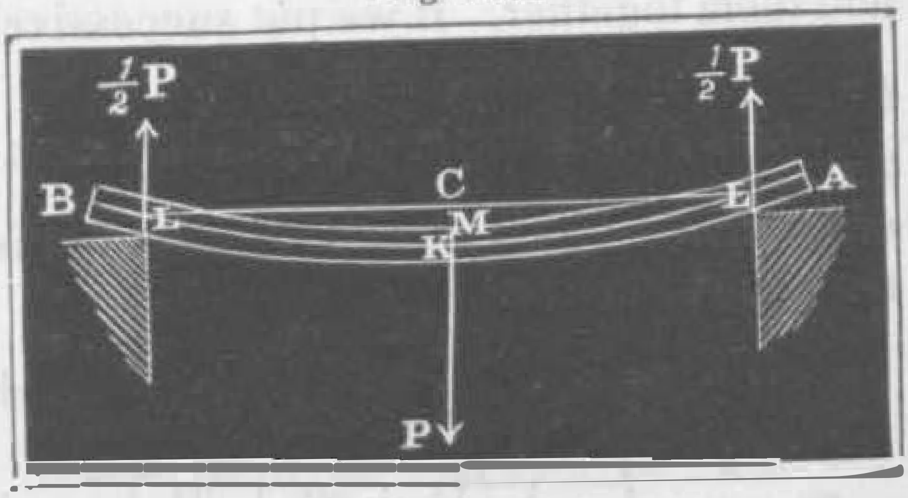


Fig. 206.

If, lastly, the load $Q = ql$ is uniformly distributed over the body AB, Fig. 207, supported at both extremities, we must put in the formula $a = \left(\frac{P}{3} + \frac{Q}{8}\right)\frac{l^3}{WE}$ in place of l, $\frac{l}{2}$, in place of P, $\frac{P+Q}{2}$ and for Q, $-\frac{Q}{2}$, because with respect to K, the weight $\frac{Q}{2}$ at the arm $\frac{l}{4}$ is opposed to the reaction $\frac{P+Q}{2}$ at the arm $\frac{l}{2}$. Consequently

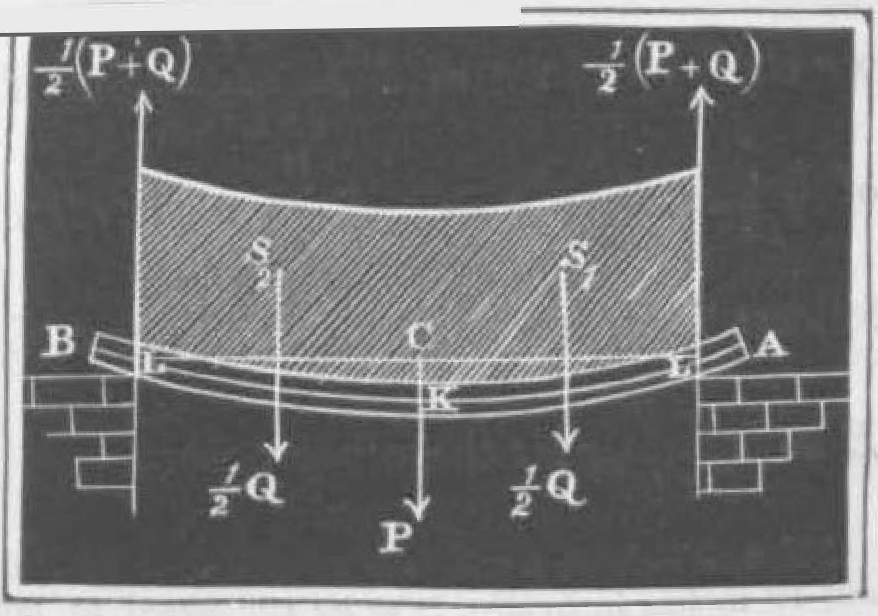


Fig. 207.

$$a = \left(\frac{P+Q}{6} - \frac{Q}{16}\right)\frac{l^3}{8\,WE} = \left(P + \frac{5}{8}\,Q\right)\frac{l^3}{48\,WE}.$$

For $P = 0$, $a = \frac{5}{8}\,.\,\frac{Ql^3}{48\,WE}$; the load is, therefore, uniformly distributed over the whole arc, and the height of the arc is $\frac{5}{8}$ times as great as if the weight acted at the middle of the beam.

§ 191. *Rectangular Beams.*—In order to give the relations of flexure of a beam or other prismatic body, and the elastic curve formed by its neutral axis, the transverse section of the body must be known, and the moment of flexure WE, calculated from it.

If the section of the beam be a rectangle $ABCD$, Fig. 208, of the width $AB = CD = b$, the height $AD = BC = h$, the moment of flexure $WE = (F_1 z_1^2 + F_1 z_2^2 + \ldots)\,E$ will be known if we decompose this cross section by lines parallel to the neutral axis NO into $2\,n$ equal laminæ, each having the area $b\,.\,\frac{h}{2n} = \frac{bh}{2n}$; and determine the moments of these laminæ, and

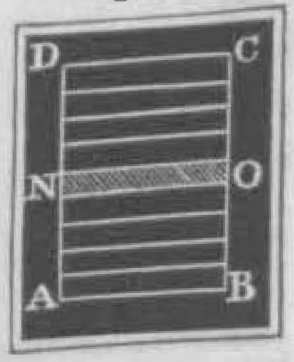


Fig. 208.

add them together. If we put successively $\frac{1}{n} \cdot \frac{h}{2}, \frac{2}{n} \cdot \frac{h}{2}, \frac{3}{n} \cdot \frac{h}{2}$ for z in $\frac{bh}{2n} \cdot z^2 E$, we shall then obtain the moments of the laminæ on one side of the neutral axis; but if we double their sum, we have the complete moment of flexure

$$WE = 2 \cdot \frac{bh}{2n}\left[\left(\frac{h^2}{2n}\right) + \left(\frac{2h}{2n}\right)^2 + \left(\frac{3h}{2n}\right)^2 + \ldots\right] E$$

$$= \frac{bh}{n} \cdot \left(\frac{h}{2n}\right)^2 (1^2 + 2^2 + 3^2 + \ldots + n^2) E = \frac{bh^3}{4 \cdot 3} E = \frac{bh^3}{12} \cdot E.$$

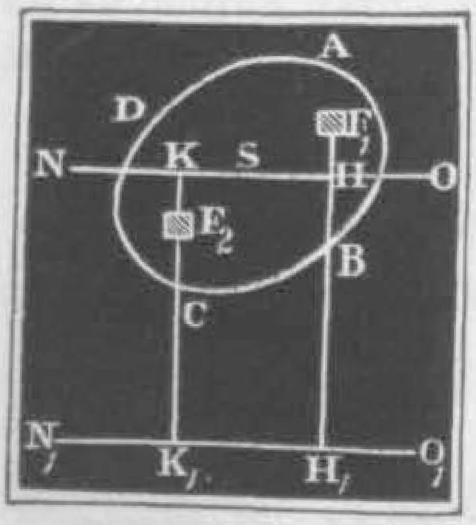
The moment of flexure, therefore, of a rectangular beam *increases as the width and the cube of the depth of the beam.*

If we put this value of WE into the formula $a = \frac{Pl^3}{3\,WE}$ of § 189, we shall obtain $a = 4 \cdot \frac{P\,l^3}{b\,h^3\,E}$, but if into the formula $a = \frac{1}{48} \frac{P\,l^3}{WE}$ of § 190, then $a = \frac{P\,l^3}{4\,b\,h^3\,E}$. Inversely, the modulus of elasticity follows from the height of the arc a $E = \frac{4\,P\,l^3}{a\,b\,h^3}$ for the one, and $E = \frac{P\,l^3}{4\,a\,b\,h^3}$, for the other case.

Example.—1. A wooden beam, 10 feet $=$ 120 inches in length, 8 inches in width, and 10 inches in height, is to be supported at both its ends, and bear a uniform load Q $= 10000$ lbs., what flexure will it undergo? The height of the arc is $a = \frac{5}{8} \cdot \frac{Q\,l^3}{4\,b\,h^3\,E}$ $= \frac{5}{8} \cdot \frac{10000 \cdot 120^3}{8 \cdot 10^3 \cdot E} = \frac{50000 \cdot 12^3}{32 \cdot 8\,E} = \frac{1350000}{4 \cdot E}$. Now E being put $= 1800000$ lbs. it follows that $a = \frac{135}{4 \cdot 180} = 0{,}1875$ inches.—2. If a rectangular cast iron bar, 2 inches wide and $\frac{1}{2}$ inch thick, has been deflected $\frac{1}{4}$ inch by a weight $P = 18$ lbs. lying in the middle of it, whilst the distance of the supports amounts to 5 feet, the modulus of elasticity of cast iron will be $E = \frac{P\,l^3}{4\,a\,b\,h^3} = \frac{18 \cdot 60^3}{4 \cdot \frac{1}{4} \cdot 2 \cdot (\frac{1}{2})^3} = \frac{18 \cdot 60^3}{\frac{1}{4}} = 72 \cdot 216000 =$ 15552000 lbs.

§ 192. *Reduction of the Moment of Flexure.*—If we know the moment of flexure of a body, $ABCD$, Fig. 209, about an axis $N_1 O_1$, lying without the centre of gravity, the moment about another axis NO, passing through the centre of gravity S, and running parallel with the former, may be found. If the distance $HH_1 = KK_1$ of both axes $= d$, and the distances of the elementary surfaces $F_1\,F_2$, &c., from the neutral axis $NO = z_1, z_2$, &c., we shall have the distances from the axis $N_1 O_1$, $= d + z_1$, $d + z_2$, &c., and the moment of flexure will be $W_1 E = [F_1\,(d + z_1)^2 + F_2\,(d + z_2)^2 + \ldots] E = [F_1\,(d^2 + 2dz_1 + z_1^2) + F_2\,(d^2 + 2dz_2 + z_2^2) + \ldots] E = [d^2(F_1 + F_2 + \ldots) + 2d\,(F_1 z_1 +$

Fig. 209.



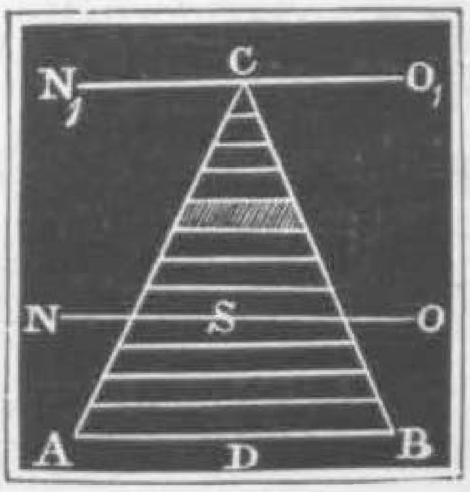
$F_2 z_2 + \ldots) + (F_1 z_1^2 + F_2 z_2^2 + \ldots)]$. But $F_1 + F_2 + \ldots$ as the sum of all the elements = the transverse section F of the whole body; further, $F_1 z_1 + F_2 z_2 + \ldots$ as the sum of the moments about an axis passing through the centre of gravity of the body = 0, and $F_1 z_1^2 + F_2 z_2^2 + \ldots$ is the moment of flexure WE about the neutral axis NO; it follows, therefore, that $W_1 E = (F d^2 + W) E$, or $W_1 = F d^2 + W$; and inversely, $W = W_1 - F d^2$.

The measure W of the moment of flexure about the neutral axis is equal to the measure W_1 of the moment of flexure about a second parallel axis, less the product of the transverse section F and the square (d^2) of the distance of both axes. Hence it follows, that of all the moments of flexure, that about the neutral axis is the least.

The moments of flexure of many bodies about any axis may be easily found; we may therefore avail ourselves of these to determine, by means of the formulæ found, the moments about the neutral axis.

§ 193. To find the moment of flexure of a prism having a triangular transverse section ABC, we must decompose this section by lines parallel to the base AB into n thin laminæ, and determine the moments of these about the axis $N_1 O_1$ passing through the point C parallel to AB. If h is the height CD, and b the breadth AB of the triangular section ABC, we have the height of these laminæ = $\frac{h}{n}$, their lengths = $\frac{b}{n}, \frac{2b}{n}, \frac{3b}{n}$, &c., to $\frac{nb}{n}$, and their distances from $N_1 O_1 = \frac{h}{n}, \frac{2h}{n}, \frac{3h}{n}$, &c., to $\frac{nh}{n}$. From these the areas of the laminæ are $F_1 = \frac{bh}{n^2}$, $F_2 = \frac{2bh}{n^2}$, $F_3 = \frac{3bh}{n^2}$, and their moments $F_1 z_1^2 = \frac{bh^3}{n^4}$, $F_2 z_2^2 = 2^3 \cdot \frac{bh^3}{n^4}$, $F_3 z_3^2 = 3^3 \cdot \frac{bh^3}{n^4}$, &c., and the moment of flexure about the axis $N_1 O_1$:

Fig. 210.



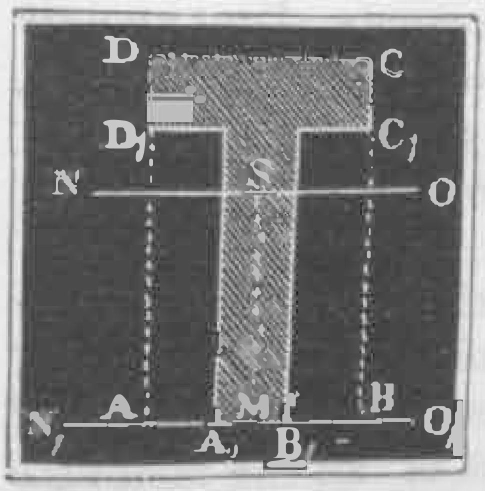
$$W_1 = \frac{bh^3}{n^4}(1^3 + 2^3 + 3^3 + \ldots + n^3) = \frac{bh^3}{n^4} \cdot \frac{n^4}{4} = \frac{bh^3}{4}.$$

The distance of the centre of gravity S from the point C is $d = \frac{2}{3} h$, and the area of the whole triangle $F = \frac{bh}{2}$; therefore $Fd^2 = \frac{bh}{2} \cdot \frac{4}{9} h^2 = \frac{2bh^3}{9}$, and the moment of flexure about the neutral axis NO sought is:

$$WE = (W_1 - Fd^2) E = \left(\frac{bh^3}{4} - \frac{2bh^3}{9}\right) E = \frac{1}{3} \cdot \frac{bh^3}{12} E =$$

a third of the moment of flexure of the rectangular beam, which has the same depth and width as the triangular one. But since this beam has double the volume, it then follows, that under otherwise similar circumstances, the triangular beam has $\frac{2}{3}$ of the moment of flexure of the rectangular.

Fig. 211.



We may find in the same manner the moments of flexure of many other bodies used in construction. For the transverse section of a T-shaped body A_1B_1CD, Fig. 211, whose dimensions are $AB = b$, $AB - A_1B_1 = AA_1 + BB_1 = b_1$, $AD = BC = h$ and $AD_1 = BC_1 = BC - CC_1 = h_1$, the moment of flexure about the lower edge A_1B_1 = the moment of the rectangular figure $ABCD$, less the moments of the rectangles A_1D_1 and B_1C_1, *i. e.*

$$W_1 = \frac{1}{2}\cdot\frac{b(2h)^3}{12} - \frac{1}{2}\cdot\frac{b_1(2h_1)^3}{12} = \frac{bh^3 - b_1h_1^3}{3},$$

as follows, if we consider each of these rectangles as the half of rectangles having double the height with the neutral axis N_1O_1. Now the area $A_1C_1D = bh - b_1h_1$, and its moment $Fd = bh\,.\,\frac{h}{2} - b_1h_1\,.\,\frac{h_1}{2} = \frac{1}{2}(bh^2 - b_1h_1^2)$; hence it follows that the arm MS

$$= d = \frac{bh^2 - b_1h_1^2}{2(bh - b_1h_1)},$$ the moment $Fd^2 = \frac{1}{4}(bh^2 - b_1h_1^2)^2 \div (bh - b_1h_1)$, and the moment of flexure about the neutral axis passing through the centre of gravity S:

$$W = W_1 - Fd^2 = \frac{bh^3 - b_1h_1^3}{3} - \frac{1}{4}(bh^2 - b_1h_1^2)^2 \div (bh - b_1h_1)$$

$$= \frac{4(bh^3 - b_1h_1^3)(bh - b_1h_1) - 3(bh^2 - b_1h_1^2)^2}{12(bh - b_1h_1)}$$

$$= \frac{(bh^2 - b_1h_1^2)^2 - 4\,bh\,b_1h_1\,(h - h_1)^2}{12(bh - b_1h_1)}.$$

§ 194. *Hollow Beams.*—The moment of flexure of a hollow rectangular beam $ABCD$, Fig. 212, is determined, if we deduct from the moment of the complete beam that of the hollow part. $AB = b$ is the external breadth, and $BC = h$ the height, and $A_1B_1 = b_1$ the internal breadth, and $B_1C_1 = h_1$ the height, we then have the moments of flexure of both $= \frac{bh^3}{12}$ and $\frac{b_1h_1^3}{12}$, and by subtraction we get the moment of flexure of the hollow beam

$$W = \frac{bh^3 - b_1h_1^3}{12}.$$

Fig. 212.

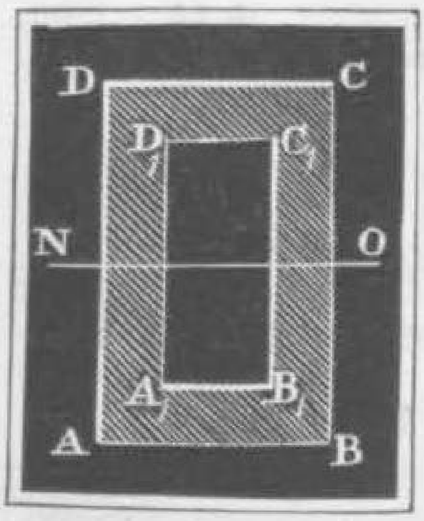
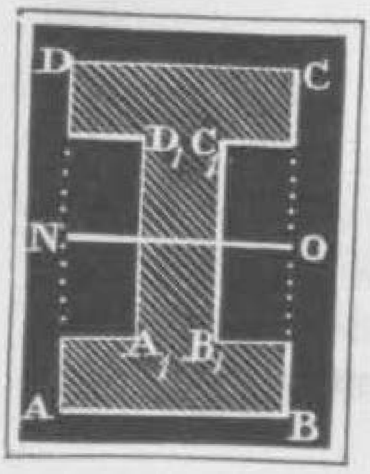


Fig. 213.

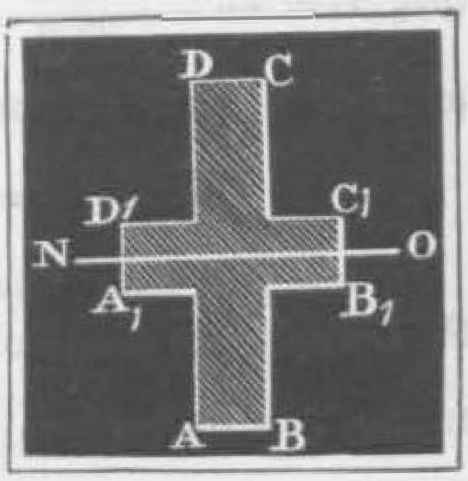


We may find in an exactly similar manner the moment of flexure of a body $ABCD$, Fig. 213, hollowed out at the sides. $AB = b$ is the outer breadth and $BC = h$ the height; and if $AB - A_1B_1 = b_1$, and $B_1C_1 = h_1$, the sum of the breadth and the heights of both hollows, by subtraction we have again:

$$W = \frac{bh^3 - b_1h_1^3}{12}.$$

The moment of flexure of a body $ABCD$, Fig. 214, of a cross-shaped section, may be obtained in the same manner. Here $AB = b$ the width, and $BC = h$ the height of the middle piece, and if $A_1B_1 - AB = b_1$ and $A_1D_1 = h_1$ are the sum of the breadths and the height of the side ribs; by addition we have the moment of flexure: $W = \frac{bh^3 + b_1h_1^3}{12}$.

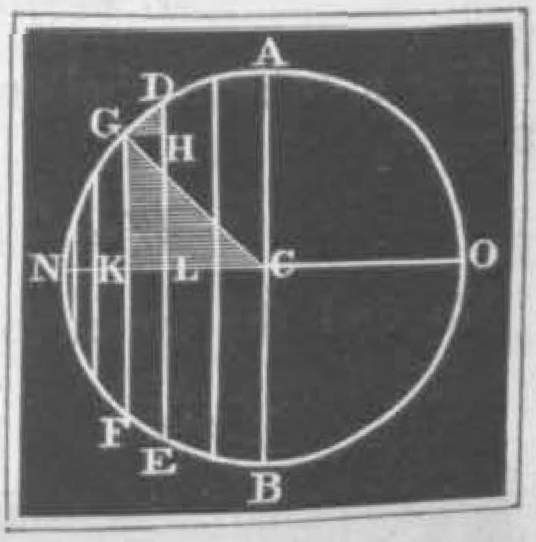
Fig. 214.



It is besides easy to see, that deep, hollow, and ribbed or flanged sections of the same area have a greater moment of flexure than square sections. Because this moment increases with the transverse section F and the square (z^2) of the distance from the neutral axis, one and the same fibre affords, therefore, a greater resistance to flexure, the further it is distant from the neutral axis. If, for example, the height h of a massive rectangular beam be equal to double its breadth b, its moment of flexure will be either $W = \frac{b\,.\,(2b)^3}{12} = \frac{2}{3}\,b^4$ or $= \frac{2b\,.\,b^3}{12} = \frac{1}{6}\,b^4$, according as we put up the beam with the lesser breadth b, or the greater $2\,b$; in the first case, therefore, the moment of flexure is four times greater than in the second. If we replace the massive beam of the cross section bh by a hollow one, whose hollow bh is equal to the massive part of the section $b_1h_1 - bh$, if, therefore, $b_1h_1 - bh = bh$, *i. e.* $b_1h_1 = 2bh$, or $b_1 = \sqrt{2}$ and $h_1 = h\sqrt{2}$, we shall obtain the moment of flexure of the last $\frac{b_1h_1^3 - bh^3}{12} = \frac{b\sqrt{2}\,(h\sqrt{2})^3 - bh^3}{12} = \frac{3}{12}\,bh^3$, *i. e.* three times as great as for the first.

§ 195. *Cylinders.*—The moment of flexure of a cylinder is determined in the following manner. Let $AOBN$, Fig. 215, be the circular transverse section, and NO the neutral axis of the cylinder. The diameter AB, divides this section into two equal parts, having equal moments of flexure, and the moment of flexure of the whole may be found by doubling the moment of the half ANB. The half may be divided by sections DE, FG, &c., parallel to AB, and at right angles to NO into thin lamina, which may be considered as rectangular. The moment of flexure of such a portion $DEFG$, $= \frac{KL\,.\,\overline{DE}^3}{12}$. Now $CA = CN = r$ the radius of the circular section, a quadrant AN has, therefore, the area $\frac{\pi r}{2}$, and if we divide this into n equal parts, any such part $DG = \frac{1}{n}\,.\,\frac{\pi r}{2} = \frac{\pi r}{2n}$. The

Fig. 215.



projection parallel to CN, $GH = KL$ corresponds to this part, and may be determined by putting, $GH : GD = GK : CG$, and, therefore, $GH = \frac{GD \,.\, GK}{CG} = \frac{\pi}{2n} \,.\, GK.$ Hence we have for the moment of flexure of the part

$$DEFG = \frac{\pi}{2n} \,.\, GK \,.\, \frac{(2\,GK)^3}{12} = \frac{\pi}{3n}\,(GK)^4.$$

If we put the variable angle corresponding to the section GF, $ACG = \phi^\circ$, we shall obtain the ordinate $GK = r\ cos.\ \phi$, and for the last moment of flexure $= \frac{\pi}{3n}\, r^4\, cos.\, \phi^4 = \frac{\pi}{3n}\, r^4 \,.\, \frac{3 + 4\ cos.\ 2\,\phi + cos.\ 4\,\phi}{8}$. The moment of flexure of the half cylinder will be now found, if for ϕ we successively put the values $\frac{1}{n} \,.\, \frac{\pi}{2}, \frac{2}{n} \,.\, \frac{\pi}{2}, \frac{3}{n} \,.\, \frac{\pi}{2}$, &c., to $\frac{n}{n} \,.\, \frac{\pi}{2}$, and add the results. But $\frac{\pi\, r^4}{3\,n} \,.\, \frac{1}{8} = \frac{\pi\, r^4}{24\,n}$ is a common factor; we have, therefore, only to consider the sum of such values, as $3 + 4\ cos.\ 2\,\phi + cos.\ 4\,\phi$. The number 3 added n times gives $3\,n$; the sum of all values of the $cos.\ 2\,\phi$ which present themselves, when ϕ is made to increase from 0 successively to $\frac{\pi}{2}$, and, therefore, $2\,\phi$ from 0 to π, equal to 0, because the cosines in the second quadrant are equal and opposite to the cosines in the first; lastly, the sum of all the cosines of all angles from 0 to $2\,\pi = 0$, hence the sum of all values of $3 + 4\ cos.\ 2\phi + cos.\ 4\,\phi$ taken between the limits $\phi = 0$ and $\phi = \frac{\pi}{2}$ is $= 3\,n$, and the measure of the moment of flexure of the half cylinder $= \frac{\pi\, r^4}{24 n} \,.\, 3\,n = \frac{\pi r^4}{8}$, and, lastly, that of the whole cylinder:

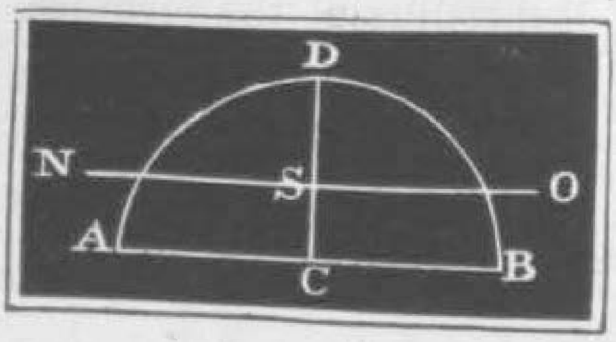
$$W = \frac{\pi}{4}\, r^4 = 0{,}7854\, r^4.$$

For a tube or hollow cylinder with the outer radius r_1 and the inner r_2

$$W = \frac{\pi}{4}\,(r_1^{\,4} - r_2^{\,4}).$$

To find the moment of flexure of a body having a semi-circular transverse section ADB, Fig. 216, we may make use of the rule found in § 192, from which the moment about the axis NO passing through the centre of gravity S is equivalent to the moment about the diameter AB, considered as a second axis, less the transverse section F ($= \frac{1}{2}\,\pi r^2$) times the square of the distance CS of both axes.

Fig. 216.



From this we obtain the moment sought $= \frac{1}{2} \cdot \frac{\pi}{4} r^4 - \frac{1}{2} \pi\, r^2 \,.\, \overline{CS}^2$
$= \frac{\pi r^4}{8} - \frac{1}{2} \pi r^2 \,.\, \left(\frac{4\, r}{3\, \pi}\right)^2$ (§ 108) $= \pi\, r^4 \left(\frac{1}{8} - \frac{8}{9\,\pi^2}\right) = 0{,}110\, r^4$.

§ 196.—*Relative Strength.*—When we know the moment of flexure of a prismatic body, we may determine from it by simple multiplication the *working load* and the *absolute* strength of the body. If a single fibre, or layer of fibres, is extended or compressed to the limits of elasticity, the body has then attained the limits of its tenacity. If we again represent by T the modulus of tenacity and the distance of the furthermost fibre from the neutral axis by e, we shall have $T = \frac{\lambda}{l} E$, and $\frac{\lambda}{l}$, or the relative elongation, $= \frac{e}{\rho}$, hence $\frac{E}{\rho} = \frac{T}{e}$. *If we substitute* $\frac{T}{e}$ *for* $\frac{E}{\rho}$ *in the formula for the moment of flexure, it will then give the statical moment of the tenacity.* We have $Px = SW = \frac{EW}{\rho}$, therefore, also, $Px = \frac{TW}{e}$. It is evident that this moment is a maximum when $x = l$, or when the arm $= l$; from this we may conclude, that at the extremity where the beam is fixed, the greatest flexure ensues, and the limit of elasticity is first attained. Accordingly, the *working load of a beam* is determined by the formula

$$P = \frac{TW}{e\, l}.$$

In like manner, the *strength*, or the resistance to rupture of the beam, may be determined. If a fibre is strained to the point of rupture, the breaking of the whole beam takes place, because the beam has now a section smaller by the section of these fibres, and therefore a greater deflexion ensues, and thus a rupture of the succeeding fibres or layer of fibres follows. If we put the modulus of strength $= K$, we have $\frac{E}{\rho} = \frac{K}{e}$, and, therefore, the force for the rupture of the beam:

$$P = \frac{KW}{e\, l}.$$

In a uniform rectangular beam, the distance of the outermost lamina of fibres from the neutral axis $= \frac{h}{2}$, hence the formula $Pl = \frac{E}{\rho} \cdot \frac{bh^3}{12}$ (§ 191) gives the resistance to rupture

$$P = \frac{2\, K}{h} \cdot \frac{bh^3}{12\, l} = \frac{bh^2}{6\, l} \cdot K.$$

If the beam is hollow, as in Fig. 212, we have $P = \frac{bh^3 - b_1 h_1^{\,3}}{6\, hl} \cdot K$, so that the formula also holds good for a body, as in Fig. 213, hollowed out at the sides.

In a prismatic body of a triangular cross section, as in Fig. 210, $e = \frac{2}{3} h$, hence $P = \frac{K}{\frac{2}{3} h} \cdot \frac{bh^3}{36\, l} = \frac{bh^2}{24\, l} \cdot K$. According to this, rectangular beams for a similar section have twice the tenacity of triangular beams.

For a cylinder of radius r, $e = r$, therefore,

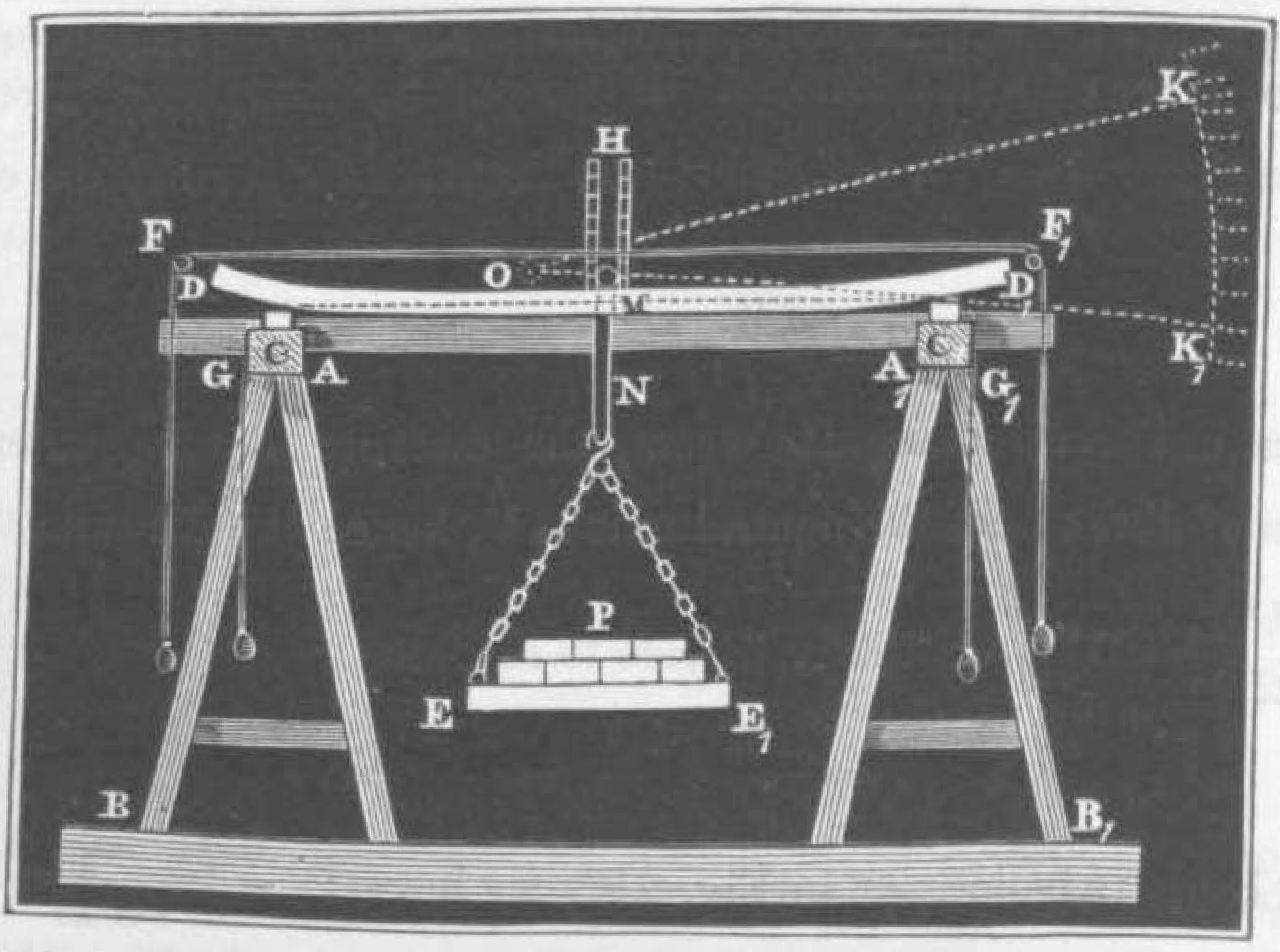
$$Pl = \frac{K}{r} \cdot \frac{\pi}{4} r^4 = \frac{\pi}{4} r^3 K.$$

If the cylinder is hollow, we have $Pl = \frac{\pi}{4}\left(\frac{r_1^4 - r_2^4}{r}\right) K.$

If we substitute the modulus of the working load T for that of the strength, or for K, an aliquot part, *i. e.* $\frac{1}{10}$th, the working load is given by the formula already found.

§ 197. *Experiments.** — To find the deflexion and tenacity of beams, we may make use of the experimental values for E and T in § 186; but as concerns the strength of beams, it is safer to replace the modulus of strength there given and derived from experiments on tensile strain, by those values of K which have been found from experiments on compression. A perfect accordance cannot exist between the moduli found by these two methods, because in rupture, not only an extension, but also a compression takes place, and both of these not only in the direction of the axis, but also in the transverse section, though here not to the same amount. Besides, many other circumstances affect the elasticity, tenacity and strength of bodies, on which account, considerable variations in the results always present themselves. Timber, for example, is stronger at the core and at the root than at the sap and the top.h Timber will also bear a greater strain when the force acts perpendicular to the annual rings,

Fig. 217.



* See Appendix.

than when parallel to them. Lastly, the soil and the situation where it has grown, temperature, dryness, age, &c., affect the resistance of woods. Besides, the deflexion of a body after it has been loaded for a long time, is always somewhat greater than on the immediate application of the load.

Experiments upon elasticity and strain were made by *Eytelwein* and *Gerstner*, with the apparatus represented in Fig. 217. AB A_1B_1 are two tressels, C and C_1 two iron supports. DD_1 the rectangular beam for experiment resting upon them. The load P for the flexure of the body lies upon a scale-pan EE_1 suspended to a stirrup MN, whose upper and rounded extremity lies in the middle M of the beam. In order to find the deflexion corresponding to a load P, *Eytelwein* applied two fine horizontal threads FF_1 and GG_1 and likewise a scale M resting upon the middle of the beam; *von Gerstner*, on the other hand, availed himself of a long one-armed delicate lever OK, whose fulcrum was at M, and whose extremity, like the hand of a watch, indicated upon a vertical scale KK_1 the deflexion of M to fifteen times its amount.

Remark. Experiments on elasticity, &c., have been made by Banks, Barlow, Buffon, Burg, Ebbels, Eytelwein, Finchan, von Gerstner, Gauthey, Muschenbroek, Rennie, Rondelet, Tredgold, &c. An ample summary of these, and besides a theory somewhat different from the above, is given by Burg in the 19th and 20th vols. of the "Jahrbücher des polytechnischen Instituts in Wien." The experiments of Eytelwein and von Gerstner are described in Eytelwein's "Handbuch der Statik fester Körper," vols. ii., and in von Gerstner's "Handbuch der Mechanik," vol. i. The Treatise printed from the transactions of the Association of Prussian Industry, "Elementare Berechnung des Widerstandes prismatischer Körper gegen Biegung," by Brix, has been used for the preparation of the foregoing article.

§ 198. *Modulus of Relative Strength.*—The following table contains the mean values of the modulus of rupture for several bodies met with in the arts. To find, with the assistance of these, the pressures which bodies can sustain with safety for a long duration, we must put for wood the tenth, for metals and stones, from the third to the fourth of K.*

TABLE II.

THE MODULUS OF FRACTURE OR MODULUS OF STRENGTH FOR THE FLEXURE OF BODIES.

Names of Substances.	Modulus of Fracture K.	Names of Substances.	Modulus of Fracture K.
Box - - - - - - -	10000 to 24000	Elm - - - - - -	6000 to 12000
Oak - - - - - - -	8000 " 24000	Cast Iron - - - -	24000 " 56000
Pine - - - - - -	8000 " 13000	Limestone - - -	700 " 1700
Scotch Fir - - -	7000 " 17000	Sandstone - - -	600 " 800
Deal - - - - - -	7000 " 14000	Brick - - - - -	180 " 340

According to this, we may assume for wood as a mean $K = 12000$ and for cast-iron $K = 40000$ pounds, and we shall then obtain for a

* See Appendix.

rectangular beam imbedded in a wall at one extremity and loaded at the other:

1. $Pl = 200 \,.\, bh^2$, if it consist of wood, and tenfold security be allowed.
2. $Pl = 1000 \,.\, bh^2$, if the beam be of cast-iron, and fourfold security be given.

If the body be cylindrical, we then have for wood

3. $Pl = 950\, r^3$, and for cast-iron
4. $Pl = 4700\, r^3$.

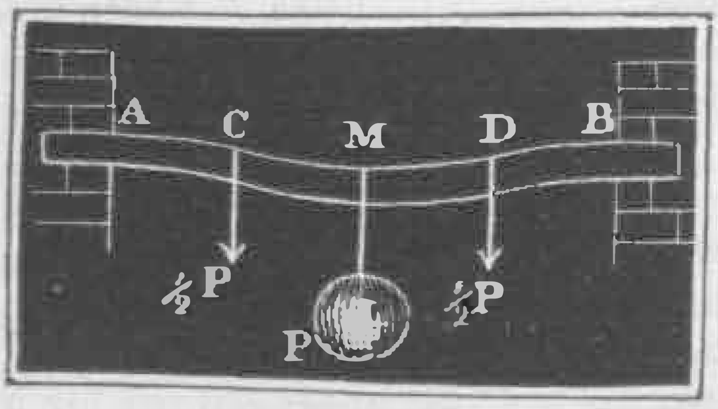
P, l, b, h, r, have the denominations hitherto used.

For wrought iron K is taken 20 per cent. less, because this bends more than cast iron; here therefore we must put

$$Pl = 800\, bh^2 = 3600\, r^3.$$

If the load Q be uniformly distributed over the beam, the beam will bear as much again, wherefore the above co-efficients must be *doubled*. If the beam rest at its extremities on points of support, whose distance is l, and if the load P act in the middle between these points, then for P we must put $\frac{P}{2}$ and for l, $\frac{l}{2}$, wherefore Pl becomes $\frac{Pl}{4}$, and the the tenacity quadrupled. But if the load between the points be uniformly distributed over the beam, we then shall have for the pressure $\frac{Q}{2}$, which acts from below upwards at a point of support, the moment $\frac{Q}{2} \,.\, \frac{l}{2}$; and for the opposite pressure $-\frac{Q}{2}$ as the half of the load pulling downwards at the centre of gravity, the moment $-\frac{Q}{2} \,.\, \frac{1}{2} \,.\, \frac{l}{2} = -\frac{Ql}{8}$; hence there will remain as the pressure for rupture at the middle, the moment $\frac{Ql}{4} - \frac{Ql}{8} = \frac{Ql}{8}$, and therefore $Ql = 8 \,.\, \frac{bh^2}{6} K$, also $= 8 \,.\, \frac{\pi}{4} br^3 K$, therefore the strength or tenacity is twice as great as if the load acted at the middle, and eight times as great as if it pulled downwards at one extremity whilst the other remained fixed.

Fig. 218.



If a beam, Fig. 218,* is imbedded in a wall at both extremities, or if its extremities are fixed, then the beam sustains as much again as if it rested freely at its extremities; for in this case the greatest flexure is not only in the middle, but likewise at the extremities; the beam, therefore, breaks at the same time in the middle and at the extremi-

* See Appendix.

ties; whilst at the intermediate points C and D, where the convexity passes into concavity, no flexure at all ensues. Consequently, for a portion AC, the pressure $= \frac{P}{2}$, the arm $= \frac{l}{4}$, and the moment $= \frac{P}{2} \cdot \frac{l}{4} = \frac{Pl}{8}$. If, finally, in this last case the load Q is uniformly distributed over the beam, the moment presents itself $= \frac{Ql}{16}$, because we may suppose, that the one half of Q is immediately sustained by the points of support, and that the other half acts in the middle of Q.

The weight G of a beam acts exactly as if the load Q were distributed uniformly over the beam; for a beam fixed at one extremity, therefore, the moment $= Pl + \frac{1}{2} Gl$; but for a beam resting on both extremities and loaded in the middle, it is $= \frac{P}{2} \cdot \frac{l}{2} + \frac{G}{2} \cdot \frac{l}{2} - \frac{G}{2} \cdot \frac{l}{4} = (P + \frac{1}{2} G) \frac{l}{4}$, &c.

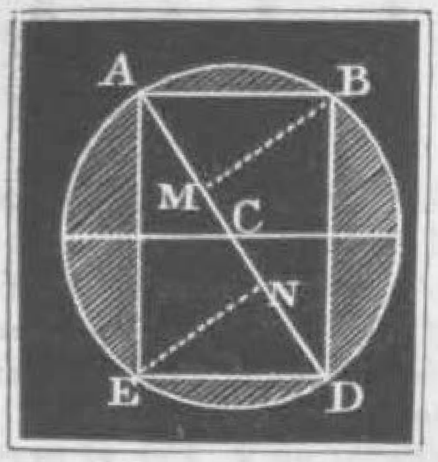
Example.—1. A rectangular beam of fir, 7 inches thick and 9 inches in depth, is to rest on both its extremities, so that the distance of the points of support may amount to 20 feet; what load, suspended from the middle, will it sustain? $b = 7$, $h = 9$, $l = 20$ feet $= 240$ inches; hence $240 . P = 4 . 200 . 7 . 9^2$; consequently this load $P = 70 . 27 = 1890$ lbs.—2. A round wooden water-wheel, and its axle, 10 feet long, is to sustain at the wheel, together with its own weight, a uniformly distributed load $Q = 10000$ lbs.; what diameter must the wheel have? $Ql = 10000 . 120 = 1200000, = 8 . 950 . r^3$, or $r^3 = \frac{1200000}{8 . 950} = 157,9$; hence the radius sought $r = \sqrt[3]{157,9} = 5,4$ inches, and the diameter of the axle $2r = 10,8$ inches, for which we may assume one foot.—3. To what height may the corn in a granary be heaped up if the bottom rest upon beams of 25 feet in length, 10 inches in breadth, and 12 in depth, the distance between the axes of any two beams $= 3$ feet, and one cubic foot of corn weighs 48,5 lbs.? If we apply the formula $Ql = 16 . 200 . bh^2$, we must put $b = 10$, $h = 12$, $l = 25 . 12 = 300$; consequently $Q = \frac{16 . 200 . 10 . 144}{300} = 15360$ lbs. A parallelopiped, 25 feet long, 3 feet broad, x feet deep, weighs $= 25 . 3 . x . 48,5$ lbs.; hence, if we put this value $= Q$, it follows that $x = \frac{15360}{75 . 48,5} = 4,22$ feet, the requisite height to which the grain may be heaped up.

§ 199. *Strongest Beams.*—Bodies of equal section very often possess different relative strengths; the formula $Pl = \frac{K}{6} . bh^2$ shows that the *strength increases, as the breadth, as the square of the depth, and inversely as the length of the beam.* The depth has consequently a greater influence upon the tenacity than the breadth; a beam of double the breadth bears twice as much, *i. e.* as much as two single beams; on the other hand, a beam of double the depth, four times that of a beam of the same depth. For this reason beams are made, namely, when they are of cast iron, much deeper than broad; they are hollowed out near the middle, and what is taken away replaced by parts at a greater distance from the neutral axis; but this rule must be particularly attended to, viz., always to lay the beam on the least side, or rather so to lay it, that the pressure may act in the direction of the greater side.

The strength of a round trunk, or of any other cylindrical body, is $P = \frac{\pi}{4} \cdot \frac{r^3}{l} K$, that of a square with equal breadths and depths $2r$, $= P_1 = \frac{2r \cdot (2r)^2}{l} \cdot \frac{K}{6} = \frac{4}{3} \cdot \frac{r^3}{l} K$; if we compare both pressures with each other, $\frac{P}{P_1} = \frac{\pi}{4} \cdot \frac{3}{4} = 0{,}588$; the cylindrical body has, therefore, only about 59 per cent. the strength of a beam having a square transverse section. Wooden beams are hewn or cut from round trunks of trees, and thereby are much weakened. But the question now is, which is the strongest form of beam that can be cut from a cylindrical trunk?

Let $ABDE$, Fig. 219, be the section of the trunk, $AD = d$ its diameter, further $AB = DE = b$ the breadth, and $AE = BD = h$ the depth of the beam. Then $b^2 + h^2 = d^2$, or $h^2 = d^2 - b^2$, and the moment of rupture.

Fig. 219.



$$Pl = \frac{K}{6} \cdot bh^2 = \frac{K}{6} b (d^2 - b^2).$$

The problem amounts to making $b(d^2 - b^2) = bd^2 - b^3$ as great as possible. If instead of b, we put $b \pm x$, where x is very small, we then obtain for the last expression

$$(b \pm x) d^2 - (b \pm x)^3 = bd^2 - b^3 \pm (d^2 - 3b^2) x - 3bx^2,$$

provided we neglect x^3, and the difference of the two $= (\mp d^2 - 3b^2) x + 3bx^2$. That the first value $bd^2 - b^3$ may in every case be greater than the last, the difference $\mp (d^2 - 3b^2) x + 3bx^2$ must be put positive, whether we take b greater or less than x. But this is only possible if $d^2 - 3b^2 = 0$, for the difference then $= 3bx^2$, therefore positive, whereas, if $d^2 - 3b^2$ is a real positive or negative value, $3bx^2$ may be neglected, and the difference may be put $= \mp (d^2 - 3b^2) x$, which if x has the same sign, is at one time positive, at another negative. But if we put $d^2 - 3b^2 = 0$, we obtain the breadth sought $b = d\sqrt{\frac{1}{3}}$, and the corresponding depth $h = \sqrt{d^2 - b^2} = d\sqrt{\frac{2}{3}}$; therefore, the ratio of the depth to the breadth: $\frac{h}{b} = \frac{\sqrt{2}}{\sqrt{1}} = 1{,}414$ or about $\frac{7}{5}$. The trunk must be so fashioned that it shall produce a beam whose depth to its breadth is as 7 to 5. To find the section corresponding to greatest strength, let us divide the diameter AD into three equal parts, raise at the points of division M and N perpendiculars MB and NE, and finally connect the points of intersection B and E by the circle with the extremities A and D of the straight line AD. $ABDE$ is the section of greatest resistance; for since $AM : AB = AB : AD$ and $AN : AE = AE : AD$, $AB = b = \sqrt{AM \cdot AD} =$

$\sqrt{\frac{1}{3} d \,.\, d} = d \sqrt{\frac{1}{3}}$ and $AE = h = \sqrt{AN \,.\, AD} = \sqrt{\frac{2}{3}} de \; d = \sqrt{\frac{2}{3}}$,

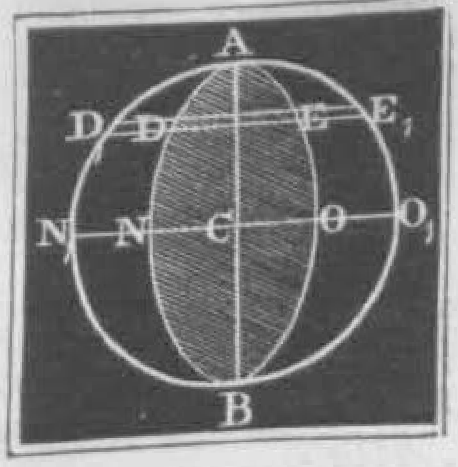
therefore $\frac{h}{b} = \frac{\sqrt{2}}{1}$, which is actually requisite.

Remark. The trunk has the moment of rupture $Pl = \frac{\pi K}{4} . r^3$, but the beam of greatest resistance formed from it $Pl = \frac{K}{6} . d \sqrt{\frac{1}{3}} . \frac{2}{3} d^2 = \frac{K}{\sqrt{243}} . d^3 = \frac{8 K}{\sqrt{243}} r^3$; the trunk, therefore, loses by squaring about $1 - \frac{8}{\sqrt{243}} . \frac{4}{\pi} = 1 - 0{,}65 = 0{,}35$, *i.e.* 35 per cent. of its strength. To spare this loss, the trunk is often hewed not quite square, but the corners rounded off. A beam with a square section formed from the same trunk, has the moment $Pl = \frac{K}{6} . d \sqrt{\frac{1}{2}} . \frac{d^2}{2}$, because here the breadth = the depth = $d \sqrt{\frac{1}{2}} = 0{,}707\, d$, hence the loss here $= 1 - \frac{8}{6 . 2 \sqrt{2}} . \frac{4}{\pi} = 1 - \frac{8}{3 \pi \sqrt{2}} = 1 - 0{,}60 = 0{,}40$ *i.e.* 40 per cent.

§ 200. *Hollow and Elliptical Beams.*—Very frequently bodies are hollowed at the inside or outside, and provided with ribs or flanges, either with a view to save material, or what comes to the same thing, to gain in strength. For a hollow rectangular beam of iron $P = 1000 . \frac{bh^3 - b_1 h_1^3}{lh}$, the hollow may be of the depth h_1 and breadth b_1, made within or without at the sides. For a hollow cylindrical body $P = 4700 . \frac{r_1^4 - r_2^4}{l r_1}$. In such cases the thickness of the solid part $r_1 - r_2$ is commonly made $= \frac{2}{5}$ of the outer radius r_1; whence it followse $P = 4700 . \frac{r_1^4 - (0{,}6 r_1)^4}{l r_1} = 4700 . \frac{0{,}8704\, r_1^3}{l} = 4090 \frac{r_1^3}{l}$. An equally heavy solid cylinder has the radius $r = \sqrt{r_1^2 - r_2^2} = \sqrt{r_1^2 - 0{,}36\, r_1^2} = 0{,}8\, r_1$; hence its moment of resistancee $= 4700 . (0{,}8\, r)^3 = 2406\, r^3$, namely, about 41 per cent. less than that of the hollow cylinder.

We gain also in strength, when, instead of a cylinder, we apply a prismatic body with an elliptical section, and place its greater axis upright or parallel to the direction of the pressure. If we suppose a circle AO_1BN_1 whose radius $CA = CB = a$ the semi-axis major, described about this elliptical section $AOBN$, Fig. 220, the strength of resistance of the body having an elliptical section may be calculated simply from that having a circular section. The length of any element DE of the elliptic elements parallel to its minor axis $NOB = 2\, b$ is always $\frac{b}{a}$ of the length of the circular element D_1E_1; but now the elasticity and strength are proportional to these dimensions singly; therefore, also the strength of the elliptic element to that of the circular element, is as b to a, and, finally, the

Fig. 220.



strength for the whole ellipse $= \frac{b}{a}$ times the strength of the whole circle, *i. e.*

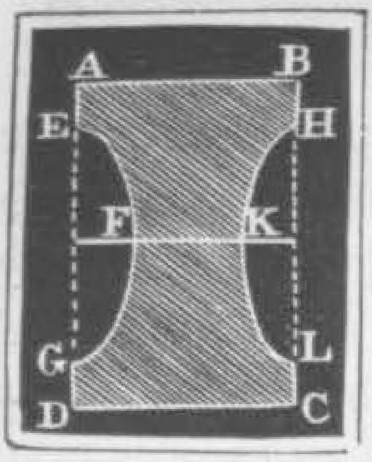
$$Pl = \frac{\pi}{4} \cdot \frac{b}{a} \cdot a^3 K = \frac{\pi}{4} b\, a^2 K, \text{ for cast iron } = 4700\, a^2\, b.$$

If now it be an elliptical hollowing whose axes are a_1 and b_1, there will remain

$$Pl = \frac{\pi}{4} \cdot \frac{ba^3 - b_1 a_1^3}{a} K, \text{ for cast iron,}$$

$$= 4700\text{t}. \frac{a^3 b - a_1^{\,3} b_1 e}{a}.$$

Fig. 221.



If, lastly, a body having a rectangular section $ABCD = bh$, Fig. 221, be hollowed at the flanks by the semi-ellipses EFG, HKL, and if the semi-axes of these are $= a_1$ and b_1, we shall have then

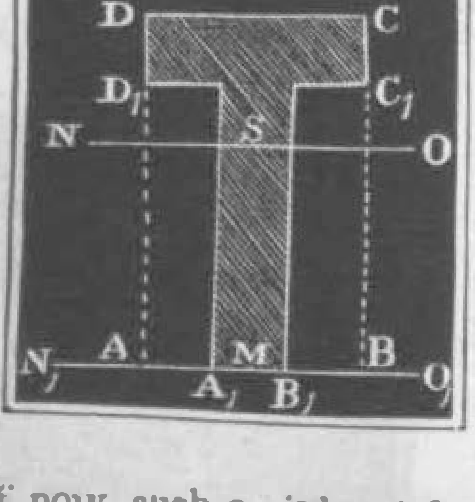
$$Pl = bh_1 \frac{K}{6} - \frac{\pi\, b_1 a_1}{4} K = \frac{2\, b\, h^3 - 3\, \pi\, b_1 a_1^{\,3}}{12\, h} \cdot K;$$

for cast iron

$$PlP = 200 \cdot \frac{bh^3 - 4{,}712\, a_1^{\,3} b_1}{h}.$$

Examples.—1. A transverse beam of oak, 9 inches broad and 11 inches deep, of known sufficient tenacity, is to be replaced by a hollow cast iron beam, of 5 inches in outer breadth and 10 in depth; of what thickness of metal must it be cast? Let this thickness $= x$, we have then for the breadth of the hollowing $= 5 - x$, and its depth $= 10 - x$; consequently, for the hollow beam $b_1 h_1^3 - b_2 h_2^3 = 5 \cdot 10^3 - (5 - x)(10 - x)^3 = 2500\, x - 450\, x^2 + 35\, x^3 - x^4$. Since the moment of resistance of the wooden beame $= 200 \cdot 9 \cdot 11^2 = 217800$, we shall have to put: $\frac{1000}{10}(2500\, x - 450\, x^2 + 35\, x^3 - x^4) = 217800$, or $2500\, x - 450\, x^2 + 35 e x^3 - x^4 = 2187$. As a first approximation $x = \frac{2178}{2500} = 0{,}9$ inches. But this value gives $450 \cdot x^2 = 450 \cdot 0{,}81 = 364{,}5$; $35\, x^3 = 25{,}5$, $x^4 = 0{,}7$; we may therefore put: $x = \frac{2178 + 364{,}5 - 25{,}5 + 0{,}7}{2500} = \frac{2517{,}7}{2500} = 1{,}01$ inch for the requisite thickness of iron.—2. If in a T-shaped girder of cast iron, the breadth $AB = CD = b$ is equal to the depth h and the thickness $A_1 B_1 = CC_1 = \frac{1}{5} b$, therefore $b_1 = \frac{4}{5} b$, and $h_1 = \frac{4}{5} b$; we shall then have for the moment of resistance (§ 193):

Fig. 222.



$$Pl = \frac{K}{12} \cdot \frac{(bh^2 - b_1 h_1^{\,2})^2 - 4\, bhb_1 h_1\, (h - h_1)^2}{(bh - b_1 h_1)\, e},$$

or by substituting $e = \frac{1}{2} \cdot \frac{bh^2 - b_1 h_1^{\,2}}{bh - b_1 h_1}$,

$$Pl = \frac{K}{6} \cdot \frac{(bh^2 - b_1 h_1^{\,2})^2 - 4\, bb_1\, hh_1\, (h - h_1)^2}{bh^2 - b_1 h_1^{\,2}}$$

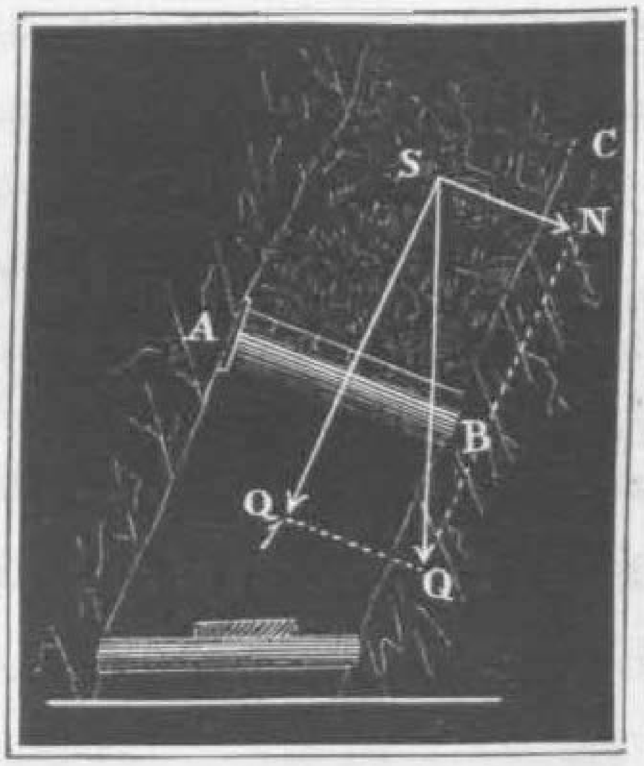
$$= 1000 \cdot \frac{(b^3 - 0{,}512\, b^3)^2 - 4 \cdot 0{,}64\, b^4 e\, (b - 0{,}8\, b)^2}{b^3 - 0{,}512\, b^3}$$

$$= 1000 \cdot \frac{0{,}2381e - 2{,}56 \cdot 0{,}04}{0{,}488} \cdot b^3 = \frac{135{,}6}{0{,}488} \cdot b^3 = 278 \cdot b^3.$$

If, now, such a girder, 4 feet in length, rest on both its extremities, and is to bear a load in its middle of 7400 lbs., Pl would then $= 7400 \cdot 4 \cdot 12 = 355200$, and therefore, $4 \cdot 278\, b^3 = 355200$; whence we should have the extreme depth and breadth $b = h = \sqrt{\frac{355200}{1112}} = 6{,}84$ in. and the thickness of iron $\frac{1}{5}\, b = 1{,}35$ inches.

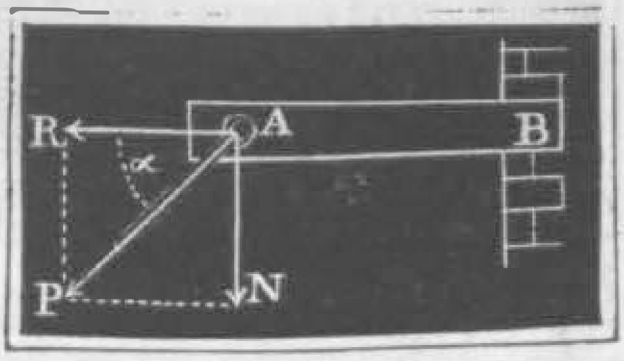
§ 201. *Oblique Pressure.*—If the pressure P act obliquely to the axis of a beam, which for example is inclined to the horizon whilst the pressure acts vertically, we have then only to take into account its components directed at right angles to the axis. If, for example, the inclined stretcher AB, Fig. 223, supports an accumulated load Q, this may be decomposed into the components Q_1 and N, and for an inclination α to the horizon of the stretcher, the pressure Q_1, counteracted by the stretcher $= Q \,.\, cos.\, \alpha$, and the pressure N counteracted by the lateral wall $BC = Q\, sin.\, \alpha$. Taking the friction into account $= Q_1 = Q \,.\, (cos.\, \alpha - f\, sin.\, \alpha)$ and hence for a round stretcher: $Q\,(cos. \quad - f\, sin.\, \alpha) = 8 \,.\, \frac{950\, r^3}{l}$, r being the radius and l the length of the stretcher.

Fig. 223.



If the pressure P be applied directly to the beam AB, Fig. 224, deviating from the axis by the angle $PAR = \alpha$, two components present themselves, $N = P\, sin.\, \alpha$ and $R = P\, cos.\, \alpha$, of which the one brings into play the relative, and the other the absolute elasticity of the beam. If F be the cross section of the beam, every unit of it is stretched by the force $\frac{P\, cos.\, \alpha}{F}$, and, therefore, the modulus of elasticity K must be taken at $\frac{P\, cos.\, \alpha}{F}$ less; therefore, we must substitute for K, $K - \frac{P\, cos.\, \alpha}{F}$, whence it follows that:

Fig. 224.



$$P\, sin.\, \alpha = \left(K - \frac{P\, cos.\, \alpha}{F}\right)\frac{W}{el};$$

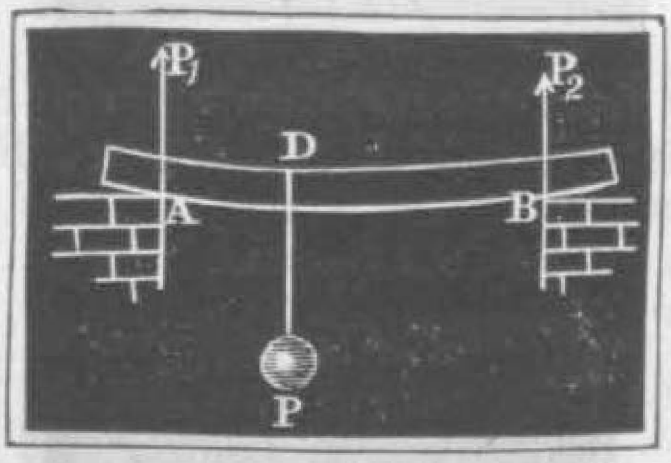
therefore, for a rectangular beam $P\, sin. B = \left(K - \frac{P\, cos.\, \alpha}{F} B\right)\frac{bh^2}{6\, l}$,

and the pressure for rupture: $P = \dfrac{K}{\frac{6\, l\, sin.\, \alpha}{bh^2} + \frac{cos.\, \alpha}{F}}$. For $\alpha^\circ = 90^\circ$, $sin.\, \alpha = 1$, $cos.\, \alpha = 0$, hence $P = \frac{K}{6} \,.\, \frac{bh^2}{l}$, for $\alpha^\circ = 0$, $sin.\, \alpha = 0$, $cos.\, \alpha = 1$, hence $P = KF$, as it should be, for we have here only to consider the absolute strength.

Example. What distance from each other must the 10-inch stretchers of ASB, Fig. 222, be laid, if it be 4½ feet wide, and run for 60 feet up a vein having a slope or inclination of 70°; the weight of a cubic foot of the ground to be supported being 65 lbs., the co-efficient of friction upon the supports is taken at $\frac{1}{8}$? Let x feet be the distance of two rafters, the weight sustained by one rafter $= 4{,}5 \,.\, 60 \,.\, 65\, x = 17550 \,.\, x$ lbs., and from the theory of the inclined plane, this rafter will have only to sustain the pressure $Q_1 = (\sin. 70° - \frac{1}{8} \cos. 70)e.\ 17550\, x = (0{,}9397 - 0{,}1140)e\ 17550\, x = 0{,}8257 \,.\, 17550\, x = 14492\, x$ lbs. But the rafter sustains $8 \,.\, 950 \,.\, \frac{r^3}{l} = 8 \,.\, \frac{950 \,.\, 5^3}{54} = 17592$; we must, therefore, put: $14492\, x = 17592$, and $x = \frac{17592}{14492} = 1{,}214$ feet $= 14{,}6$ inches. We must, therefore, only leave an interval between any two rafters of 14,6 inches.

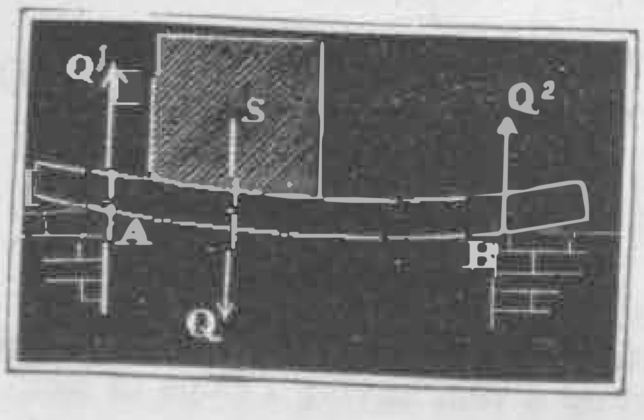
§ 202. *Loading beyond the middle.*—If a pressure P acts upon a beam, supported at both ends, not at the middle, but at a point D at distances $DA = l_1$ and $DB = l_2$ from the points of support, the beam then can bear a greater load. According to the equality of certain statical moments, the point of support A sustains the pressure $P_1 = \frac{l_2}{l_1 + l_2} P$, and the point B the pressure $P_2 = \frac{l_1}{l_1 + l_2} P$, hence the moment of rupture at the point of application $D = DA \,.\, P_1 = DB \,.\, P_2 = \frac{l_1 l_2 P}{l_1 + l_2}$. For any other point E this moment $EB \,.\, P_1$ is less, because the arm EB is less than the arm $DB = l_2$; the greatest deflexion also takes place at D, and fracture first occurs at this point. Accordingly we must put $\frac{Pl_1 l_2}{l_1 + l_2} = \frac{KW}{e}$ or the whole length $l_1 + l_2$ being represented by l, $\frac{Pl_1 l_2}{l} = \frac{K}{6} \,.\, bh^2$, if the beam is rectangular. The pressure $P = \frac{K}{6} \,.\, \frac{l}{l_1 l_2} \,.\, bh^2$ is moreover $= \infty$, when l_1 or l_2 very nearly $= 0$, and is infinitely less, the more l_1 and l_2 approach to equality. If, lastly, $l_1 = l_2$, *i. e.* if the pressure P acts in the middle of the beam, P becomes a minimum, because, if we put $l_1 = \frac{l}{2} + x$ and $l_2 = \frac{l}{2} - x$, the product forming the denominator $l_1 l_2 = \frac{l^2}{4} - x^2$ is always less than $\frac{l_2}{4}$, whether $\frac{l}{2}$ be made somewhat (x) greater or less. A beam, therefore,

Fig. 225.



supported at its extremities, sustains least when the load is applied at its middle, and one so much the greater the nearer the load approaches one of the points of support.

Fig. 226.



If a load Q be uniformly distributed over the length c, the centre of which is l_1 and l_2 distant from the points of support A and B, Fig. 226, we shall then have to take the difference $\frac{Ql_1 l_2}{l} - \frac{Q}{2} \cdot \frac{c}{4}$ for the moment of rupture, because the pressure $Q_1 = \frac{Ql_2}{l}$ at the arm l_1, and half the weight $\frac{Q}{2}$ acting at the arm $\frac{c}{4}$ is opposed to it. Therefore

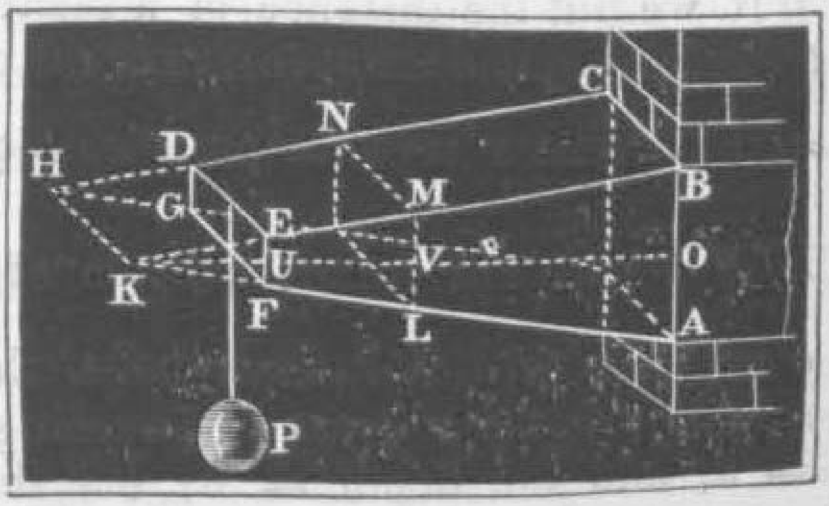
$$Q\left(\frac{l_1 l_2}{l} - \frac{c}{8}\right) = \frac{K}{6} \cdot bh^2.$$

Example. What load does a hollow cast iron beam sustain, if its outer depth and breadth amount to 8 inches and 4 inches, and inner breadth and depth 6 inches and 2 inches; and if further, the middle of the load, uniformly distributed over 3 feet in length, is distant from one point of support 4, and from the other 2 feet? It is $\frac{b_1 h_1^3 - b_2 h_2^3}{h_1} = \frac{4 \cdot 512 - 2 \cdot 216}{8} = 202$; further, $\frac{l_1 l_2}{l} - \frac{c}{8} = \left(\frac{4 \cdot 2}{6} - \frac{3}{8}\right) 12 = \frac{23}{2}$ inches; hence, $\frac{23}{2} Q = 1000 \cdot 202$; and consequently, $Q = 17565$ lbs.

§ 203. *Plane of Rupture.*—If the beams are not prismatic, if they have different transverse sections at different places, the plane of rupture, *i. e.* the plane in which rupture will ensue, will no longer be the same as for prismatic bodies, because this place is not only dependent on the arm x, but also on the transverse section. If we suppose a rectangular section of variable breadth w, and height z, and assume the beam to be fixed at one extremity, and at the other acted upon by a pressure P, and the distance of the transverse section wz from the extremity where the pressure acts $= x$, we must then put $P = \frac{K}{6} \cdot \frac{wz^2}{x}$, and find the minimum value of $\frac{wz^2}{x}$ in order to determine the weakest part, or plane of rupture of the beam.

Here many cases present themselves; let us consider only the following. Let the body $ABEG$, Fig. 227, be a truncated wedge, or have the form of a prism with a trapezoidal base, let the breadth $DE = FG$ at the extremity $= b$, the depth $EF = DG = h$, and the distance UK of the edge cut off from the terminating surface EG, $= c$. Let us now assume that the plane of rupture NL is distant $UV = x$ from the terminating surface, we shall then obtain for it the depth $ML = z = h + \frac{x}{c} h = h\left(1 + \frac{x}{c}\right)$, whilst the uniform breadth is $MN = w = b$.

Fig. 227.



The value $\frac{wz^2}{x} = \frac{bh^2}{x}\left(1 + \frac{x}{c}\right)^2 = bh^2\left(\frac{1}{x} + \frac{2}{c} + \frac{x}{c^2}\right)$ increases and diminishes simultaneously with $\frac{1}{x} + \frac{x}{c^2}$, and is, therefore, also a minimum, when this latter term is of the last value. But if in place of x, we put $c + u$, where u is a small number, we shall then obtain for it:

$$\frac{1}{c+u} + \frac{c+u}{c^2} = \frac{1}{c\left(1+\frac{u}{c}\right)} + \frac{1}{c} + \frac{u}{c^2}$$

$= \frac{1}{c}\left(2 - \frac{u}{c} + \frac{u^2}{c^2} - \ldots\right) + \frac{u}{c^2} = \frac{2}{c} + \frac{u^2}{c^3}$. As now in this last expression u appears only as a square, it follows that every other value, which is obtained when the distance x is assumed greater or less than c, gives a greater value than for $x = c$, that consequently for $x = c$, $\frac{1}{x} + \frac{x}{c^2}$, and, therefore, also $\frac{w\,z^2}{x} = bh^2 \left(\frac{1}{c} + \frac{2}{c} + \frac{1}{c}\right) = \frac{4\,bh^2}{c}$ is a minimum. From hence it follows, that the magnitude of the surface of rupture $= b \,.\, 2\,h = 2\,bh$, and is distant from the terminating surface $EG = bh$ as much again as the edge HK of the portion cut off.

In a similar manner, the distance of the plane of rupture from the terminating surface of a truncated pyramid or truncated cone is equal to half the height of the supplementary pyramid or supplementary cone.

§ 204. *Beams of the Strongest Form.*—A beam, which opposes an equal resistance to rupture throughout all its sections, of which, therefore, each may be considered as a plane of rupture, is called a *beam of the strongest form.* Of all beams of equal strength, the body of equal resistance at each point of its length has the least quantity of material, and is, therefore, the most suitable, and that which should be selected for architectural construction, and for machines, not only out of regard to economy, but also, that the weight may not be increased unnecessarily.

If we put the distance of a plane of rupture from the further extremity $= x$, and the measure of the moment of flexure for that section $= W$, we then have the pressure requisite for rupture $P = \frac{WK}{e\,x}$. As K is a constant factor, a beam of the strongest form $\frac{W}{e\,x}$ must be constant also, *i. e.* it must be of the same value for every possible section. If for a beam of a rectangular section the variable breadth $= u$, and the depth $= v$; but the breadth at the origin, or end supposed fixed $= b$, and the depth there $= h$, we have generally $W = \frac{uv^3}{12}$, and $e = \frac{v}{2}$, hence $P = \frac{uv^2}{x} \,.\, \frac{K}{6}$, and for the origin, for which x has become l, $P = \frac{bh^2}{l} \,.\, \frac{K}{6}$.

If we make these two values of P equal, we obtain the equation $\frac{uv^2}{x} = \frac{bh^2}{l}$ for the beam of the strongest form. In a beam of equal breadth $u = b$ is $\frac{v^2}{x} = \frac{h^2}{l}$, therefore, $\frac{v^2}{h^2} = \frac{x}{l}$, which is the equation to the parabola (§ 35, *Remark*), and points out that the longitudinal section *ABE*, Fig. 228, must have the form of a parabola whose vertex is the extremity or point of suspension *E* of the load. If the beam *AB*, Fig. 229, rests upon its extremities, and sustains a load in its middle *C*, or if a beam *AB*, Fig. 230, is supported in its middle *C*, and two pressures, balancing each other, are applied at the extremities *A* and *B*, then the longitudinal profile has the form of two parabolas meeting in the middle. The last

Fig. 228.

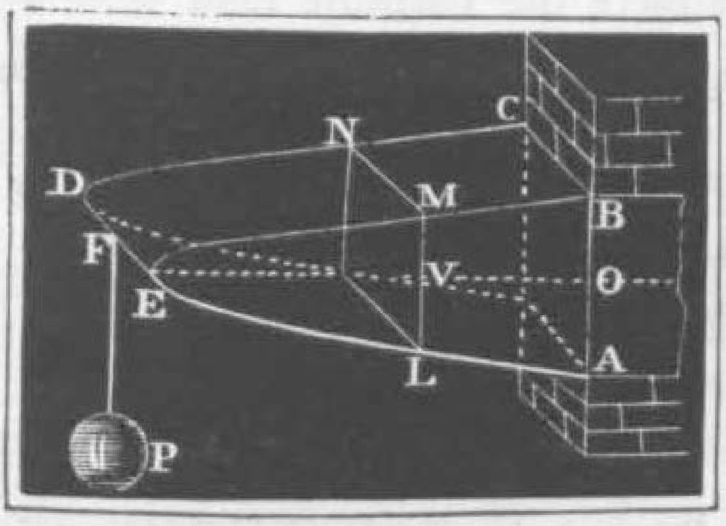


Fig. 229.

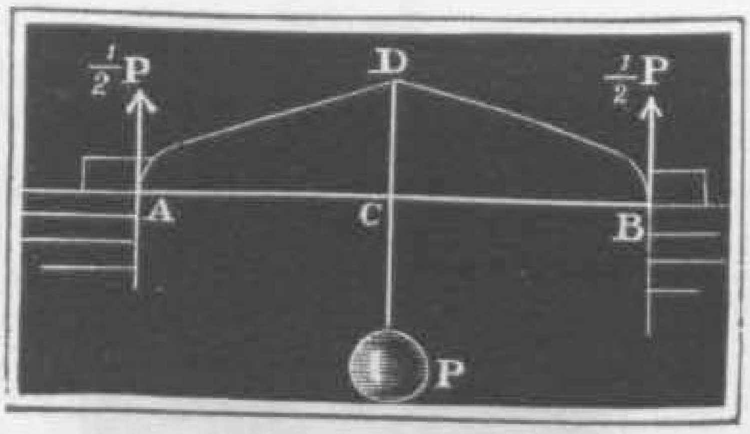
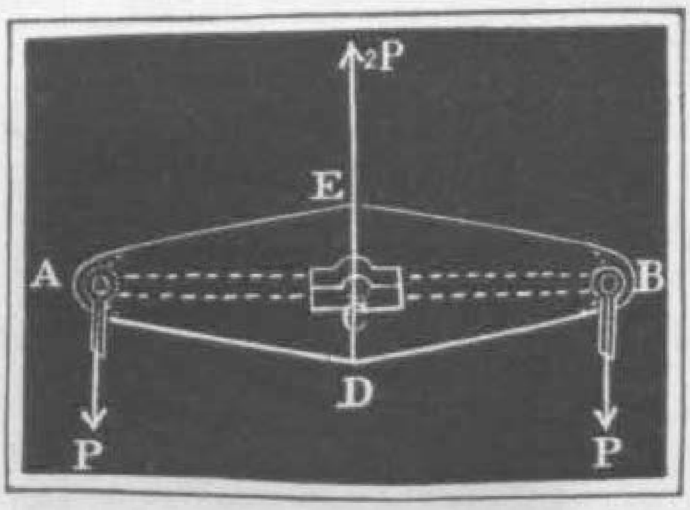


Fig. 230.



case occurs in balances, which, as they are weakened by the holes at the points *A*, *C*, *B*, are provided with ribs, or have a middle piece *AB* given to them. If the depth $v = h$ is constant $\frac{u}{x} = \frac{b}{l}$ or $\frac{u}{b} = \frac{x}{l}$, for the breadth u is proportional to its distance from the extremity, the horizontal projection, therefore, of the beam *ABD*, Fig. 231, forms a triangle *BCD*, and the whole beam a wedge with a vertical edge coinciding with the direction of force. If the body *ABD*, Fig. 232, is to have similar transverse sections, we shall then have $\frac{v}{h} = \frac{u}{b}$, hence $\frac{u \cdot u^2h^2}{b^2x} = \frac{bh^2}{l}$, *i. e.* $\frac{u^3}{b^3} = \frac{x}{l}$, therefore, the breadth and the depth in-

Fig. 231.

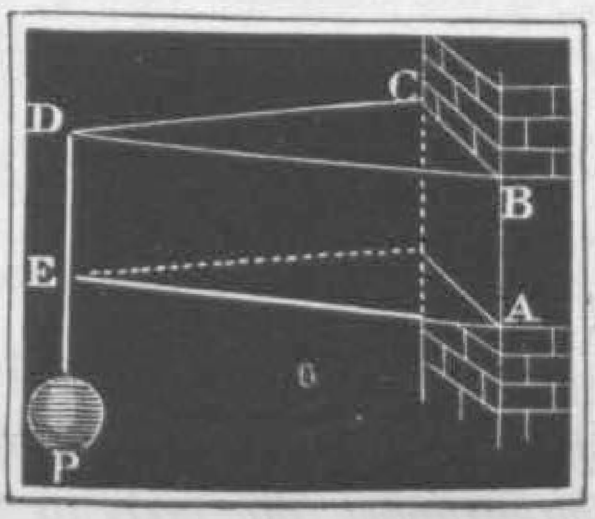
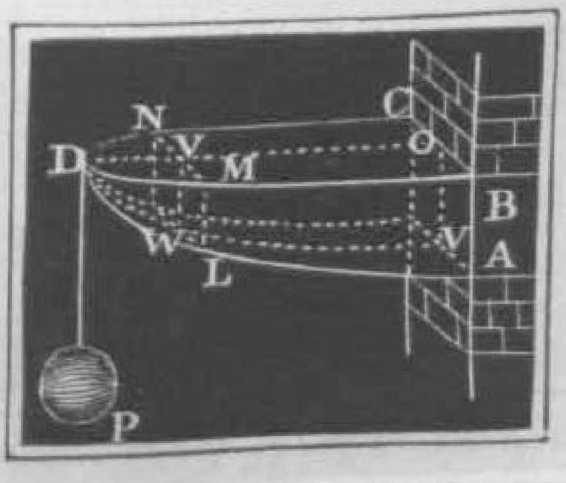


Fig. 232.



crease as the cube roots of the corresponding arms. For example, a section eight times further from the outer end than a given section, would only have double the height and breadth of that of the given section.

If a beam be uniformly loaded, we have the variable load $Q = qx$, and its arm $= \frac{xe}{2}$, hence, instead of Px, we must put $xq \cdot \frac{x}{2} = \frac{x^2q}{2}$, whence $\frac{x^2q}{2} = uv^2 \cdot \frac{K}{6}$ and also $\frac{l^2q}{2}$ must be taken $= bh^2 \frac{K}{6}$, and consequently $\frac{uv^2}{bh^2} = \frac{x^2}{l^2}$. Were the breadth invariable, that is $u = b$, we should have $\frac{v^2}{h^2} = \frac{x^2}{l^2}$, therefore, also $\frac{v}{h} = \frac{x}{l}$, and, therefore, a triangle ABE for the longitudinal section, and a wedge $ABED$, Fig. 233, for the body of the strongest form. If we take a uniform depth

Fig. 233.

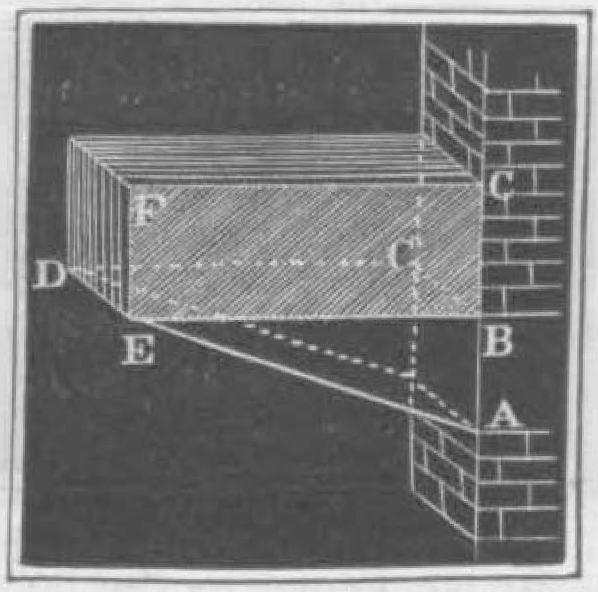
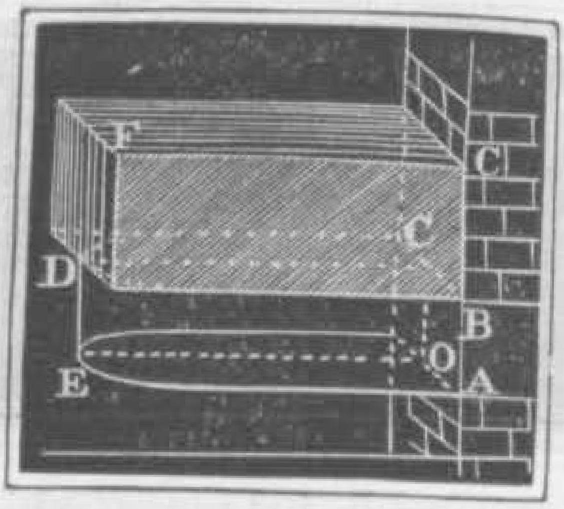


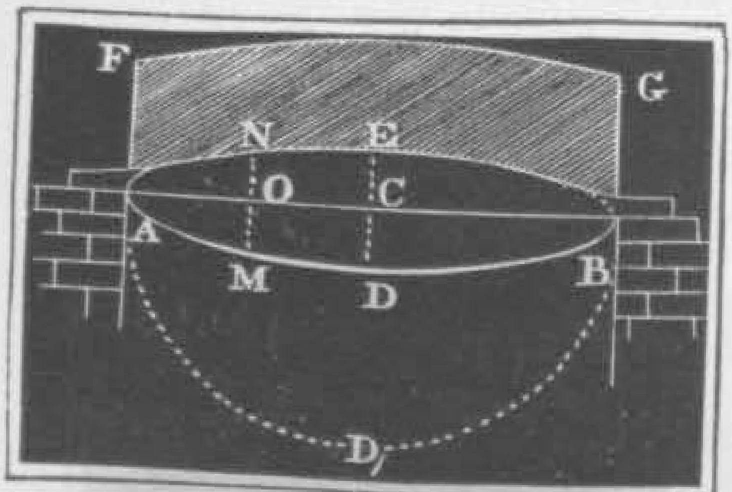
Fig. 234.



$v = h$, we then obtain $\frac{u}{b} = \frac{x^2}{l_2}$, and, therefore, for the plane a surface BDC, bounded by a parabolic arc, as in Fig. 234. If we again make similar transverse sections, then $\frac{u^3}{b^3} = \frac{x^2}{l^2}$ so that we have both in the vertical as in the horizontal profile, a cubic parabola, in which the cubes of the ordinates increase, as the squares of the abscisses.

If a body AB supported at both extremities, Fig. 235, is uniformly loaded over its whole length, we have for the moment of rupture at a distance from a point of support $AO = x$:

Fig. 235.



$$\frac{Q}{2} \cdot x - qx \cdot \frac{x}{2} = \frac{q}{2}(lx - x^2),$$

on the other hand for the middle point:

$$\frac{Q}{2} \cdot \frac{l}{2} - \frac{Q}{2} \cdot \frac{l}{4} = \frac{Ql}{8} = \frac{ql^2}{8}.$$

If we suppose the body to be of

uniform breadth, we have to put $\frac{q}{2}(lx - x^2) = bv^2 \cdot \frac{K}{6}$ and $\frac{ql^2}{8} = bh^2 \cdot \frac{Ke}{6}$, and by division $\frac{v^2}{h^2} = \frac{lx - x^2}{\frac{1}{4}l^2}$, or $v^2 = \left(\frac{h}{\frac{1}{2}l}\right)^2 (lx - x^2)$. Were $h = \frac{1}{2}l$, v^2 would be $= lx - x^2$, and therefore the longitudinal section would be a circle AD_1B described with l as a radius, but because $lx - x^2$ must still be multiplied by $\left(\frac{h}{\frac{1}{2}l}\right)^2$ in order to obtain the square v^2 of every ordinate OM, this circle passes into an ellipse ADB, whose semi-axes are $CA = \alpha = \frac{1}{2}l$ and $CD = \beta = h$.

The same relations exist for bodies with circular sections as for those with similar rectangular sections. In the case of a beam imbedded in a wall at one extremity, and loaded at the other $\frac{v^3}{r^3} = \frac{x}{l}$, *i. e.* the radii increase as the cubes of the distances from the point of application.

§ 205. *The Thickness of Axles.*—In the parts of machines, as the shafts, axles, &c., flexures may prejudicially affect the working of machines, by giving rise to vibrations and shocks; and it is here, therefore, often more desirable to determine the sections, not according to their strength, but according to their degree of flexure. *Gerstner* and *Tredgold* maintain that a beam of wood, supported at both extremities and loaded in the middle, may suffer a deflexion $a = \frac{1}{288} \cdot l$ without disadvantage, and that such a beam of cast or wrought iron can only undergo a deflexion or height of arc $a = \frac{1}{480} \cdot l$. But now from § 190: $a = \frac{Pl^3}{48\,WE}$, and from § 191: $W = \frac{bh^3}{12}$, hence follows the height of arc: $a = \frac{Pl^3}{4\,bh^3\,E}$ and $\frac{a}{l} = \frac{Pl^2}{4\,bh^3\,E}$. If now we put $\frac{a}{l} = \frac{1}{288}$, and $E = 1800000$, we obtain for wooden beams the tenacity or strength at the middle:

$$P = \frac{a}{l} \cdot \frac{4\,bh^3E}{l^2} = \frac{1}{288} \cdot \frac{4\,bh^3 \cdot 1800000}{l^2} = 25000 \cdot \frac{bh^3}{l^2}.$$

For cast iron $\frac{a}{l} = \frac{1}{480}$, and $E = 17000000$, hence:

$$P = \frac{1}{480} \cdot \frac{4\,bh^3}{l^2} \cdot 17000000 = 142000 \cdot \frac{bh^3}{l^2}.$$

If further we take for cast iron $\frac{a}{l} = \frac{1}{480}$, and $E = 29000000$ lbs., we obtain for a rectangular beam of this material:

$$P = 242000 \frac{bh^3}{l^2}.$$

The co-efficients 25000, 142000, 242000 must be multiplied by

$3\pi = 9{,}42$, and h and b be replaced by r, for cylindrical beams as round axles, &c. The following table gives the dimensions of the transverse sections, l being expressed in feet, b, h, r in inches, and P in pounds.

Substances.	Rectangular section.	Circular section.
Wood	$bh^3 = \frac{P l^2}{170}$	$r^4 = \frac{P l^2}{1600}$
Cast iron	$bh^3 = \frac{P l^2}{980}$	$r^4 = \frac{P l^2}{9250}$
Wrought iron . .	$bh^3 = \frac{P l^2}{1680}$	$r^4 = \frac{P l^2}{15800}$

If the load Q be uniformly distributed over the beam, P must be replaced by $\frac{5}{8}\, Q$, § 190, and if the weight of the beam be taken into account, by $P + \frac{5}{8}\, G$. If it be the case of a beam which is fixed at one extremity and loaded at the other, P and l must then be doubled, therefore, $P l^2$ to be multiplied by eight; if, lastly, the beam fixed at one extremity sustains a load Q uniformly distributed, for $P l^2$, we must substitute $\frac{3}{8} \cdot 8\, Q l^2 = 3\, Q l^2$ for $P l^2$.

Examples.—1. What load will a wooden beam, 20 feet long, 7 inches thick and 9 deep, reposing on both its extremities, sustain for a length of time? This load is $P = 170 \cdot \frac{b h^3}{l^2} = 170 \cdot \frac{7 \cdot 9^3}{20^2} = 1190 \cdot \frac{729}{400} = 2170$ lbs.

In § 198 P was found $= 1890$ lbs.—2. What thickness must an iron axle, 12 feet long, be cast, if the same has to sustain a uniformly distributed load $Q = 40000$ lbs., without any detrimental flexure? $r^4 = \frac{\frac{5}{8}\, Q l^2}{15800}$, therefore here $r^4 = \frac{5}{8} \cdot \frac{40000 \cdot 12^2}{15800} = 228$, and $r = \sqrt[4]{228} = 3{,}89$ inches; consequently, the thickness of the axle $2\,r = 7{,}78$, or about $7\frac{3}{4}$ inches. By the formula for strength, if the modulus of tenacity of wrought iron be taken at $\frac{14}{9}$ times that of cast iron: $r^3 = \frac{Q l}{8 \cdot \frac{14}{9} \cdot 4700} = \frac{40000 \cdot 144 \cdot 9}{8 \cdot 14 \cdot 4700} = 98{,}5$; hence, $r = \sqrt[3]{98{,}5} = 4{,}62$ inches, and $2\,r = 9{,}24$ inches.

§ 206. *Rupture by Compression.*—If prismatic bodies are so strongly compressed in the direction of their axes, as to amount to rupture, their resistance to compression has to be overcome. This rupture may take place in two ways. If the body be short, if it approximates to a cube, it will fall to pieces without undergoing flexure, but if the body is longer than it is broad and thick, flexure similar to that which takes place will precede the rupture. The one kind of rupture consists in a crushing, bruising, transverse strain, or splitting asunder of the body or its parts; the other, in a fracture or destruction of a section of the body. Hence *a distinction is made between the crushing strength and strength of rupture under compression.*

The resistance to crushing is, for similar sections, proportional to their areas; for regular sections, however, somewhat greater than for irregular, and greatest of all for circular sections. It is besides independent for the most part of the length of the body. Short wooden prisms split asunder in the direction of their length, or form a bulge; stones break into several pieces or separate along an inclined plane. Ten times the absolute strength is given to wood and stones; to iron, only one of five times; and to walls of rough stones, twenty times. If K be the modulus of resistance to crushing, and F the transverse section of the bodies, the working load will be

$P = FK_1$ and $F = \frac{P}{K_1}$, where for K_1, $\frac{1}{5}K$ to $\frac{1}{20}K$ must be substituted.

TABLE

OF THE MODULUS OF RESISTANCE TO CRUSHING.e

Names of substances.	Modulus K.	Names of substances.	Modulus K.
Basalt . . .	27000	Brick . . .	580 to 2200
Gneiss . . .	5100	Oak . . .	2800 " 6800
Granite . . .	6000 to 11000	Pine . . .	6800 " 8000
Limestone . .	1500 " 6000	Fir . . .	2000
Marble . . .	3200e" 12000	Cast iron . .	146000
Mortar . . .	450e" 900	Wrought iron .	72000
Sandstone . .	1400 " 13000	Copper . . .	60000

The values of K contained in the preceding table are not unfrequently, especially for wooden columns, applicable even when the bodies are very long, only it has been found necessary to diminish these values by one, two, or three-sixths, when the columns are twelve,etwenty-four or forty-eight times as long as they are thick. Accordingly, for a column of oak, one foot thick and twenty-four long, K must be taken at from $2800\,(1 - \frac{2}{6}) = 1900$ lbs. to $6800 \cdot \frac{4}{6} = 4500$ lbs. The formulæ developed in § 185 for the transverse section of bodies of considerable weight, and of bodies of the strongest form, here find their application.*

Examples.—1. What load can a round column of pine, 12 feet long and 11 inches diameter, sustain? $F = \frac{\pi \cdot 11^2}{4} = 95$ square inches; if we now take for K a mean value $= \frac{6800 + 8000}{2} = 7400$, and diminish the value one-sixth, because the length is 13 times that of the thickness, and therefore put $K = 7400 \cdot \frac{5}{6} = 6200$ lbs.; and give a ten-times security, we shall then have $P = \frac{6200 \cdot F}{10} = 620 \cdot 95 = 58900$ lbs.—2. How thick must be the foundation walls of a massive building of 20000000 lbs. weight, 60 feet outer length, and 40 feet breadth, if for this purpose we use well finished blocks of gneiss? Let x be the requisite thickness, $60 - x$ is the mean length, and $40 - x$ the breadth; therefore the mean perimeter $2\,(60 - x + 40 - x) = 200 - 4x$; if we multiply this by x, we obtain the base of the walls $(200 - 4x)\,x$ square feet $= 144\,(200 - 4x)\,x = 576\,(50 - x)\,x$ square inches. For a twenty-fold security, a square inch of gneiss sustains a pressure $= \frac{5100}{20} = 255$ lbs.; hence we have to put $255 \cdot 576\,(50 - x)\,x = 20000000$, or $50\,x - x^2 = \frac{20000000}{146880} = 136$. Now $x = \frac{136 + x^2}{50}$, or about $x = \frac{136}{50}$

* See Appendix.

$= 2{,}7$ feet. Now x^2 being put $= 2{,}7^2 = 7$, more accurately $x = \frac{136 + 7}{50} = \frac{143}{50}$ $= 2{,}86$ feet, for which we may take $2{,}9$ feet, $= 35$ inches.

§ 207. *Rupture under Compression.*—If a prismatic body $ABCD$, Fig. 236, be fixed at one extremity, and at the other be acted on by a pressure P, which acts in the direction of the axis of the body, the relations of deflexion will come out otherwise than when the pressure acts perpendicular to the axis. The neutral axis KL assumes another form, because the arms of the pressure P are not formed by the abscisses, but by the ordinates, as HK. From § 188, we have for the angles of curvature LML_1, $L_1M_1L_2$, &c., of the neutral axis KL, Fig.

Fig. 236.

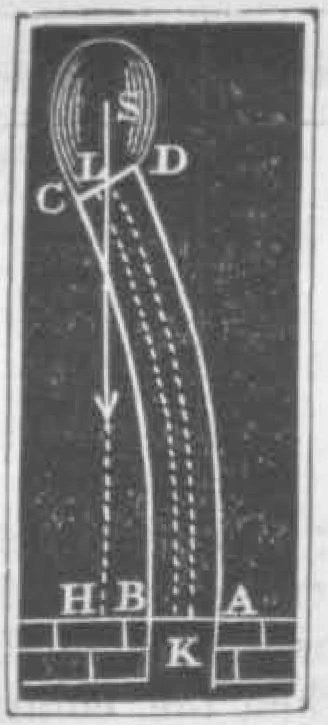
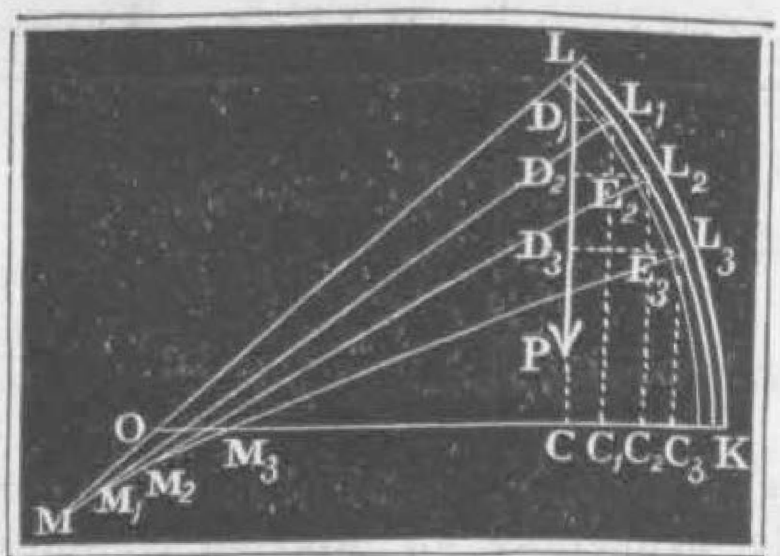


Fig. 237.



237, $\phi_1 = \frac{M_1 \,.\, \overline{LL_1}}{WE}$, $\phi_2 = \frac{M_2 \,.\, \overline{L_1L_2}}{WE}$, &c., but here the moments are $M_1 = P \,.\, \overline{D_1L_1}$, $M_2 = P \,.\, \overline{D_2L_2}$, &c., hence we have the measures of the angles: $\phi_1 = \frac{P \,.\, \overline{D_1L_1} \,.\, \overline{LL_1}}{WE}$, $\phi_2 = \frac{P \,.\, \overline{D_2L_2} \,.\, \overline{L_1L_2}}{WE}$, &c. If we introduce the tangential angles $L_1LD_1 = KOL = \alpha$, $L_2L_1E_2 = \alpha_1 = \alpha - \phi_1$, $L_3L_2E_3 = \alpha_2 = \alpha_1 - \phi_2 = \alpha - \phi_1 - \phi_2$, &c., and if we suppose only a small curvature, we may then write: $LL_1 = \frac{D_1L_1}{\alpha}$, $L_1L_2 = \frac{E_2L_2}{\alpha_1} = \frac{E_2L_2}{\alpha - \phi_1}$, $L_2L_3 = \frac{E_3L_3}{\alpha_2}$, &c.; if further we divide the entire height of the arc $CK = a$ into n equal parts, we may then put: $D_1L_1 = CC_1 = E_2L_2 = C_1C_2$, &c. $= \frac{a}{n}$, but $D_2L_2 = \frac{2a}{n}$, $D_3L_3 = \frac{3a}{n}$, &c., and by the substitution of these values it follows that:

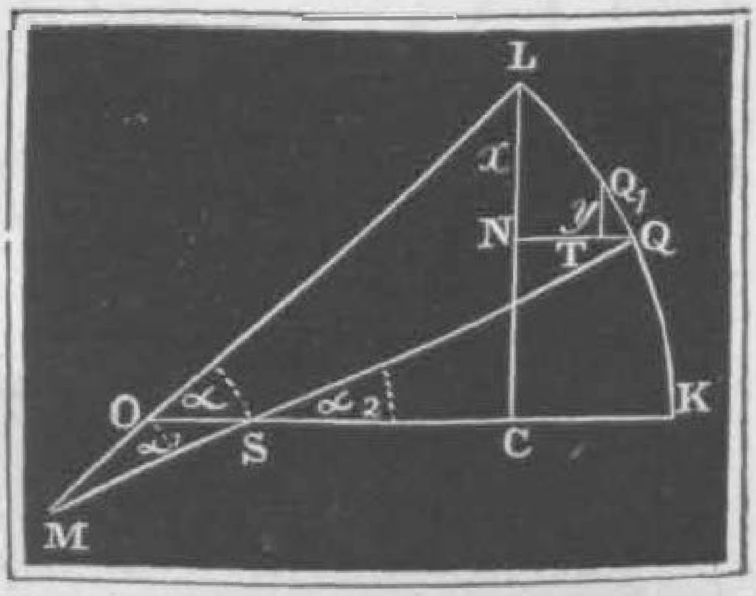
$$\phi_1 = \frac{P \,.\, \frac{a}{n} \,.\, \frac{a}{n\alpha}}{WE} = \frac{Pa^2}{WE\, n^2\alpha}, \phi^2 = \frac{P \,.\, \frac{2a}{n} \,.\, \frac{a}{n\,\alpha_1}}{WE} = \frac{2\,P\,a^2}{WE\, n^2\alpha_1},$$

$\phi_3 = \frac{3\,Pa^2}{WE\,n^2 a_2}$, &c., or $\phi_1\, a = \frac{Pa^2}{WE\,n^2}$, $\phi_2\, a_1 = \frac{2\,Pa^2}{WE\,n^2}$, $\phi_3\, a_2 = \frac{3\,Pa^2}{WE\,n^2}$, &c.

The sum $\phi_1 a + \phi_2\, a_1 + \phi_3\, a_2 + \ldots = \frac{Pa^2}{WE\,n^2}(1 + 2 + 3 + \ldots + n) = \frac{Pa^2}{WE\,.\,n^2}\cdot\frac{n^2}{2} = \frac{Pa^2}{2\,WE}$, and may be also found, if a be divided into m equal parts, and any such part $\frac{a}{m}$, be put $= \phi_1 = \phi_2 = \phi_3$, &c. We shall then obtain $\phi_1\, a + \phi_2\, a_1 + \phi_3\, a_2 + \ldots = \frac{a}{m}\,.\,a + \frac{a}{m}\left(a - \frac{a}{m}\right) + \frac{a}{m}\left(a - \frac{2\,a}{m}\right) + \ldots + \frac{a}{m}\,.\,\frac{a}{m}$ by taking out the common factor $\left(\frac{a}{m}\right)^2$, and writing it in an inverse order $= \left(\frac{a}{m}\right)^2(1 + 2 + \ldots + m) = \frac{a^2}{m^2}\cdot\frac{m^2}{2} = \frac{a^2}{2}$, and by making these two sums equal to each other $\alpha^2 = \frac{Pa^2}{WE}$, an equation between the angle of curvature $LOKP = \alpha^\circ$, and the height of the arc $CK = a$.

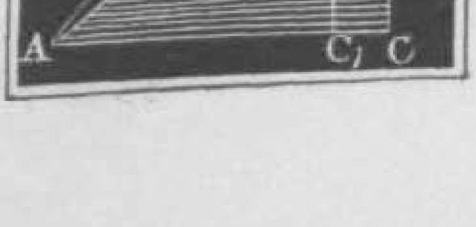
For the equation of the elastic line LK, Fig. 238, let us take $LN = x$ and $NQ = y$ as co-ordinates, and put the corresponding angle of curvature $LMQ = \alpha_1$. In the last equation, if we put a for y, and y for a, we then have to replace the sum $\phi_1\, a + \phi_2\, a_1 + \phi_3\, a_2 + \ldots$ by $\frac{\alpha^2 - (\alpha - \alpha_1)^2}{2}$; hence, if we represent the supplementary angle $QSK = \alpha - \alpha_1$ by α_2, we afterwards obtain $\alpha^2 - \alpha_2^2 = \frac{Py^2}{WE}$, or $\alpha_2^2 = \alpha^2 - \frac{Py^2}{WE}$, and so $\alpha_2 = \sqrt{\frac{P}{WE}}\,.\,\sqrt{a^2 - y^2}$. But since $\alpha_2^\circ = \angle\, QQ_1T$, and *tang.* $QQ_1T = \frac{TQ}{TQ_1} = \frac{\text{element } \delta \text{ of the ordinate } y}{\text{element } \epsilon \text{ of the absciss } x}$, we have $\frac{\delta}{\epsilon} = \sqrt{\frac{P}{WE}}\,.\,\sqrt{a^2 - y^2}$.

Fig. 238.



If in a rectangular triangle ABC, Fig. 239, with the hypothenuse $AB = a$, the angle CAB increases by a small amount $BAB_1 = \psi^\circ$, the perpendicular $BC = y$ increases by the amount

Fig. 239.

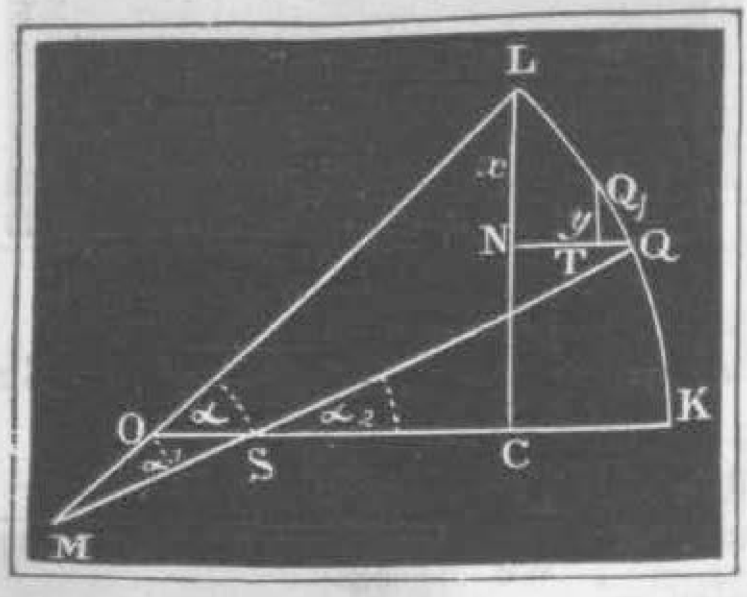


$DB_1 = \delta$, and the base AC decreases by the amount $DB = C_1C$, then $\frac{DB_1}{BB_1} = \frac{AC}{AB}$, *i. e.* $\frac{\delta}{a\psi} = \frac{\sqrt{a^2 - y^2}}{a}$; and hence $\frac{\delta}{\psi} = \sqrt{a^2 - y^2}$. By comparing the two expressions $\frac{\delta}{\varepsilon} = \sqrt{\frac{P}{WE}} \cdot \sqrt{a^2 - y^2}$ and $\frac{\delta}{\psi} = \sqrt{a^2 - y^2}$, we obtain the equation $\psi = \varepsilon \sqrt{\frac{P}{WE}}$ or $\frac{\varepsilon}{\psi} = \sqrt{\frac{WE}{P}}$. Therefore the ratio of the element ε of the absciss to the element of the arc ψ is invariable, and $= \sqrt{\frac{WE}{P}}$, and hence, also, the ratio of the absciss x to the whole arc $A = \sqrt{\frac{WE}{P}}$, *i. e.* $\frac{x}{A} = \sqrt{\frac{WE}{P}}$, and $A = x\sqrt{\frac{P}{WE}}$. If, finally, we substitute this value of A in the equation $BC = AB \sin. A$; *i. e.* $y = a \sin. A$, we obtain the equation sought:

$$y = a \sin. \left(x \sqrt{\frac{P}{WE}} \right).$$

With the assistance of this last formula we may find the ordinate $NQ = y$ corresponding to any absciss $LN = x$, Fig. 240. If in this we put $x = l$ and $y = a$, we then obtain $a = a \sin. \left(l \sqrt{\frac{P}{WE}} \right)$, *i. e.* $1 = \sin. \left(l \sqrt{\frac{P}{WE}} \right)$, whence it follows that $l\sqrt{\frac{P}{WE}} = \frac{\pi}{2}$ and $P = \left(\frac{\pi}{2l}\right)^2 WE$.

Fig. 240.



As this formula does not contain the height of the arc a, it follows that the force P is capable of maintaining equilibrium for every deflexion of the body. This remarkable circumstance is explained by the fact that an increase of the arm or of the statical moment is combined with an increase of the deflexion. Hence, therefore, the force for rupture is:

$$P = \left(\frac{\pi}{2l}\right)^2 WE.$$

§ 208. *Columns.*—If we put in the formula $P = \left(\frac{\pi}{2l}\right)^2 \cdot WE$ for $W = \frac{bh^3}{12}$, we then obtain in P the resisting strength of a rectangular column $P = \frac{\pi^2}{48} \cdot \frac{bh^3}{l^2} E$.

The strength, therefore, of a parallelopiped increases as the breadth or greater dimension, and the cube of the thickness or less dimension of the transverse section, and inversely as the square of the length.

If, on the other hand, we put $W = \frac{\pi}{4} r^4$, then for a cylindrical column we have $P = \frac{\pi^3}{16} \cdot \frac{r^4 E}{l^2}$.

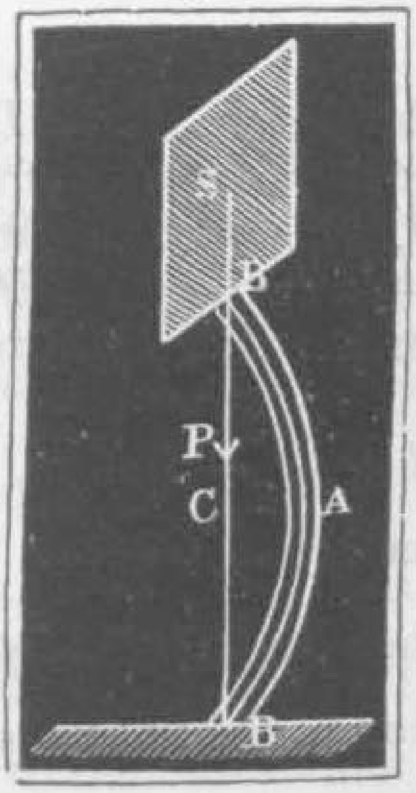
The strength of a cylinder increases, therefore, as the fourth power of the diameter, and inversely as the square of the length.

For a hollow column with the radii r_1, and r_2

$$P = \frac{\pi^3}{16} \cdot \frac{(r_1^4 - r_2^4) E}{l^2}.$$

If the column be not fixed at the lower extremity, it will assume a curvature BAB_1, Fig. 241, by which the lower half will be as strongly deflected as the upper, and the greatest curvature take place in the middle. Therefore this beam must be regarded as the double of one imbedded in a wall, and for l, $\frac{l}{2}$ must be substituted, so that for the rectangular and for the cylindrical columns,

Fig. 241.



$$P = \frac{\pi^2}{12} \cdot \frac{b h^3}{l^2} E, \quad P = \frac{\pi^3}{4} \cdot \frac{r^4}{l^2} E;$$

in both cases, however, there is a fourfold tenacity. These formulæ, when the columns are not very long, give generally a greater tenacity than the formula for the crushing strength, wherefore the ratios of the sections are often determined from the last. It is at least advisable only to make use of the formula for rupture under compression when the length is at least twenty times that of the thickness, and then, further, to allow a twenty-fold security.*

Examples.—1. For a column of fir, 12 feet long and 11 inches thick, the tenacity is $P = \frac{\pi^3}{16} \cdot \frac{r^4}{l^2} \cdot \frac{E}{20} = 31 . \left(\frac{11}{24}\right)^4 . \frac{1800000}{20} = 31 . 0{,}044e. 90000 = 123000$ lbs.; in Example No. 1, of § 206, 58900 lbs. only, therefore about $\frac{1}{2}$ of the above, was found.—2. How thick must a column of oak, 30 feet high, be in order to be able to bear a load of 60000 lbs.?

Here $r = \sqrt[4]{\frac{4 P l^2}{\pi^3 E}} = \sqrt[4]{\frac{4 . 60000 . (30 . 12)^2}{31 . \frac{1800000}{20}}} = \sqrt[4]{\frac{8 . 6 . 360^2}{31 . 18}} = 10{,}3$ inches; consequently, the thickness is about 21 inches. The strength of crushing requires, if K be put $= \frac{1}{10} \cdot \frac{4}{6} \cdot \frac{2800 + 6800}{2} = 320$ lbs., the transverse section $F = \frac{60000}{320} = 188$ square inches; whence, $r = \sqrt{\frac{188}{\pi}} = 13{,}7 . 0{,}564 = 7{,}7$ inches, and the thickness should be $15\frac{1}{2}$ inches. For this case the first value must be taken.

§ 209. *Torsion.*—When a body ABC, Fig. 242, fixed at one ex-

* See Appendix.

tremity, is acted upon by a force whose direction lies in the plane normal to the axis, and, therefore, endeavors to turn the body about the axis, or when two forces of revolution P and Q act in different normal planes upon a body AB, fixed by its axis, Fig. 243, the fibres running parallel to the axis undergo a wrenching or torsion, the amount of which we wish to determine. Let AB, Fig. 242, be a

Fig. 242.

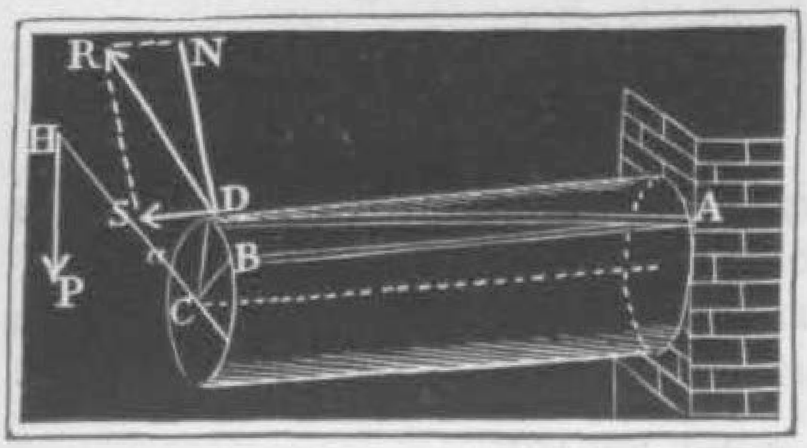
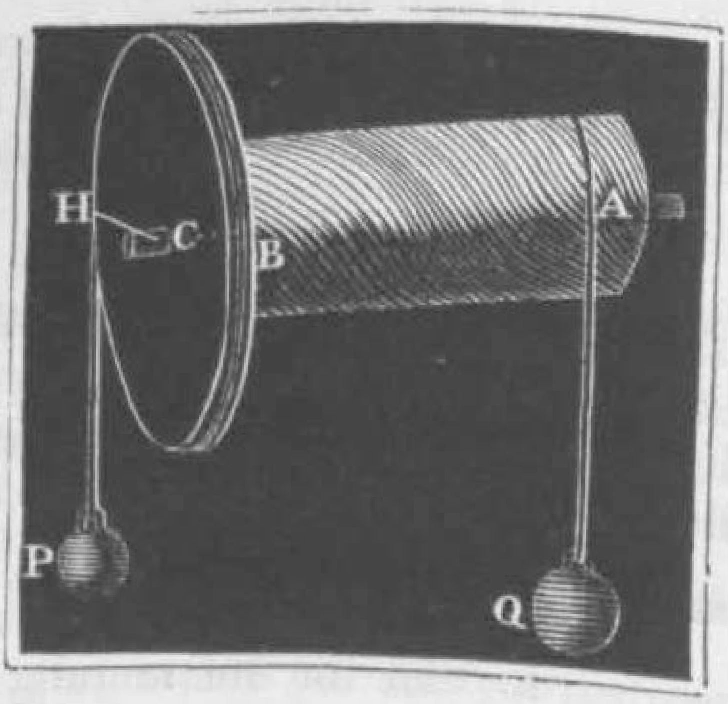


Fig. 243.



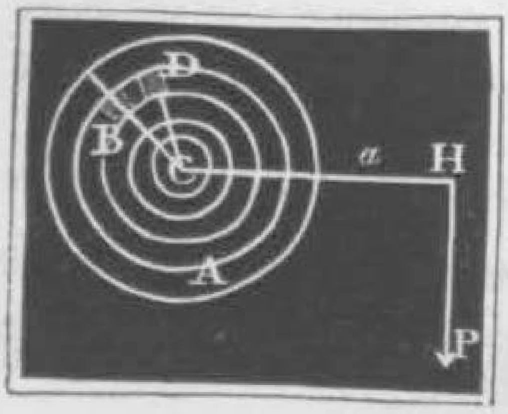
fibre before, and AD the same fibre during the torsion, and, therefore, let the extremity of the fibre B be advanced by the force of torsion to D. If now l be the initial length AB, and λ its extension, therefore $l + \lambda$ the length AD during the torsion, and if s be the corresponding torsion BD, we have after the Pythagorean law to put

$$\overline{AD}^2 = \overline{AB}^2 + \overline{BD}^2$$

$(l + \lambda)^2 = l^2 + s^2$, or $l^2 + 2\,l\lambda + \lambda^2 = l^2 + s^2$, may be put approximately $= \lambda = \frac{s^2}{2l}$. If further F be the section of such a fibre, we then have for the force required to produce this extension in the direction of the fibre $S = \frac{s^2}{2l^2} \,.\, F \,.\, E$. But this force or tension (S) of a fibre is only a component of the force of torsion R, which produces besides a further pressure N, normal to the fibres. From the similarity of the triangles RDS and BDA, it follows that $S : R = s : l$, hence $S = \frac{Rs}{l}$, and by equating both values of S:

$$R = \frac{s}{2l} \,.\, F \,.\, E.$$

Fig. 244.



Therefore the force of torsion of a fibre increases as the torsion (s), and the transverse section F, and inversely as the length (l) of the fibre.

To find the force of torsion of a cylindrical axle CBA, Fig. 244, let us divide its radius r into n equal parts, and suppose concentric circles passing through the points of division, so that the transverse section becomes decomposed into annular elements of the thickness $\frac{r}{n}$, and radii $\frac{r}{n}, \frac{2r}{n}, \frac{3r}{n} \ldots \frac{nr}{n}$. The solid contents of these elements are $F_1 = 2\pi \cdot \frac{r}{n} \cdot \frac{r}{n} = 2\pi\left(\frac{r}{n}\right)^2$, $F_2 = 2\pi \cdot \frac{2r}{n} \cdot \frac{r}{n} = 4\pi\left(\frac{r}{n}\right)^2$, $F_3 = 2\pi \cdot \frac{3r}{n} \cdot \frac{r}{n} = 6\pi\left(\frac{r}{n}\right)^2$, &c. If all the fibres are twisted by the angle $BCD = \alpha^{\circ}$, they have the corresponding torsions $s_1 = \frac{r}{n}\alpha$, $s_2 = \frac{2r}{n}\alpha$, $s_3 = \frac{3r}{n}\alpha$, and hence the forces of torsion are $R_1 = \frac{r\alpha}{2nl} \cdot 2\pi\left(\frac{r}{n}\right)^2 E = \frac{\alpha\pi}{l}\left(\frac{r}{n}\right)^3 E$, $R_2 = \frac{4\alpha\pi}{l}\left(\frac{r}{n}\right)^3 E$, $P_3 = \frac{9\alpha\pi}{l}\left(\frac{r}{n}\right)^3 E$, &c. If further we multiply these forces by the arms $\frac{r}{n}, \frac{2r}{n}, \frac{3r}{n}$, and add together the values so obtained, we have for the moments of torsion $Pa = \frac{\alpha\pi}{l}\left(\frac{r}{n}\right)^4 (1^3 + 2^3 + 3^3 + \ldots + n^3)\, E$, *i. e.* $Pa = \frac{\alpha\pi}{l}\left(\frac{r}{n}\right)^4 \cdot \frac{n^4}{4} = \frac{\alpha\pi r^4}{4\,l} \cdot E$, and inversely, the measure of the angle of torsion:

$$\alpha = \frac{4\,l \,.\, Pa}{\pi\, r^4\, E}.$$

If the axle be hollow and have radii r_1 and r_2, we have then

$$Pa = \frac{\alpha\pi}{4l} E\,(r_1^4 - r_2^4), \text{ therefore } \alpha = \frac{4\,Pal}{\pi E\,(r_1^4 - r_2^4)}.$$

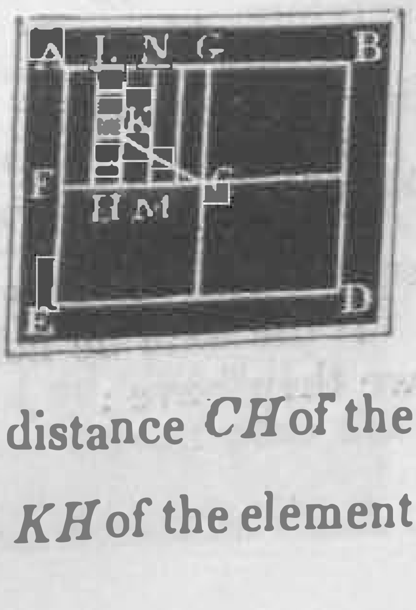
The application of hollow axles gives also with respect to torsion a saving in material, for if we put $r_2 = r$, and $r_1 = r\sqrt{2}$, we then obtain for the hollow axle, which has the same section as a solid one, the moment of torsion:

$$= \frac{\alpha\pi E}{4l}(4r^4 - r^4) = 3 \cdot \frac{\alpha\pi r^4 E}{4l}, \text{ i. e.}$$

thrice as great as for the solid axle.

§ 210. For a shaft or axle of a rectangular section $ABDE$, Fig. 245, the moment of torsion is found in the following manner. If we divide half the breadth $AG = b$ into n equal parts, and carry through the points of division the parallel planes HL, MN, &c., we obtain elements of equal sections, each $= \frac{b}{n} \cdot h$, where h represents half the height $AF = GC$ of the section. If now we divide one of these elementary strips into m equal parts, we have for its area $\frac{1}{m} \cdot \frac{bh}{n} = \frac{bh}{mn}$. Let the normal distance CH of the strip HL from the centre C, $= c$, and the distance KH of the element

Fig. 245.



K from the normal $CH, = e$, then the distance of the element from the axis is $CK = \sqrt{c^2+e^2}$, accordingly the arc of torsion $= \alpha \sqrt{c^2+e^2}$, and the moment of torsion

$$= \frac{\alpha \sqrt{c^2 + e^2}}{2\,l} \cdot \frac{bh}{mn} \sqrt{c^2 + e^2} \,.\, E = \frac{\alpha bh}{2mnl} (c^2 + e^2)\, E.$$

If now we successively put $e = \frac{1}{m} h, \frac{2}{m} h, \frac{3}{m} h$, &c., and sum the results, we have the moment of the strip:

$$HL = \frac{\alpha bh}{2mnl} (c^2 + \frac{h^2}{m^2} + c^2 + \frac{4h^2}{m^2} + c^2 + \frac{9h^2}{m^2} + \ldots) E$$

$$= \frac{\alpha bh}{2mnl} \left[mc^2 + \left(\frac{h^2}{m^2}\right) (1 + 4 + 9 + \ldots + m^2)\right] E.$$

But $1+4+9+\ldots+m^2 = \frac{m^3}{3}$, hence the moment of the strip $= \frac{\alpha bh}{2nl}\left(c^2 + \frac{h^2}{3}\right) E$. To obtain the moments of all the strips, let us again put $c = \frac{b}{n}, \frac{2b}{n}, \frac{3b}{n}$, &c., and again sum the results, we shall then have:

$$\frac{\alpha bh}{2\,nl} \left[\left(\frac{b}{n}\right)^2 + 4\left(\frac{b}{n}\right)^2 + 9\left(\frac{b}{n}\right)^2 + \ldots + \frac{nh^2}{3}\right] E =$$

$$\frac{\alpha bh}{2nl} \left[\left(\frac{b}{n}\right)^2 (1 + 4 + \ldots n^2) + \frac{nh^2}{3}\right] E = \frac{\alpha bh}{2\,nl}\left(\frac{b^2}{n^2} \cdot \frac{n^3}{3} + \frac{nh^2}{3}\right) E$$

$$= \frac{\alpha bh}{2\,l} \left(\frac{b^2 + h^2}{3}\right) E.$$

Generally the sections are square, and therefore $b = h$. As we have only considered a fourth part of the shaft, it follows for the whole shaft that:

$$Pa = \frac{4\,\alpha\, b^4}{3\,l} E.$$

For a cylindrical axle $P_1 a_1 = \frac{\alpha \pi r^4}{4\,l} E$; if we put $b = r$ we then obtain $Pa = \frac{4}{3} \cdot \frac{4}{\pi} \cdot \frac{\alpha \pi r^4}{4\,l} E = \frac{16}{3\pi} P_1 a_1$, the moment of the square is, therefore, $= \frac{16}{3\pi} = 1{,}756$ times as great as that of the round axle. But if we make $4b^2 = \pi r^2$, and, therefore, both sections equal, we then obtain $Pa = \frac{\alpha \,.\, \pi^2 r^4}{4\,.\,3l} E = \frac{\pi^2}{4.3} \cdot \frac{4}{\pi} \,.\, P_1 a_1 = \frac{\pi}{3} P_1 a_1$, therefore the square shaft is only a little stronger than the cylindrical axle.

If the axle be hollow, and the outer and inner radii be $2b_1$ and $2b_2$, we then have:

$$Pa = \frac{4\alpha E}{3\,l} (b_1{}^4 - b_2{}^4).$$

§ 211. *Breaking Twist.*—When the torsion exceeds a certain limit, the fibres are torn asunder, and the cylindrical axle is *twisted asunder.* For the moment of rupture of the fibre furthest from the axis $\frac{\lambda}{l} = \frac{K}{E}$, but $\frac{\lambda}{l}$ is also $= \frac{s^2}{2\,l^2} = \frac{a^2 r^2}{2\,l^2}$, hence it follows that $\frac{a\,r}{l} = \sqrt{\frac{2\,K}{E}}$. The statical moment of twisting for the round axle is:

$$Pa = \sqrt{\frac{2\,K}{E}} \cdot \frac{\pi r^3}{4} E = \frac{\pi r^3}{2} \sqrt{\frac{KEB}{2}},$$

but for the square shaft, where the greatest distance of a fibre is half the diagonal $b \sqrt{2}$, it follows that

$$\frac{K}{E} = \frac{2\,a^2\,b^2}{2\,l^2} \text{ since } \frac{a\,b}{l} = \sqrt{\frac{K}{E}} \text{ and } Pa = \frac{4\,b^3}{3} \sqrt{KE}.$$

Since the fibres are not only extended by torsion, but also compressed, and as we have only had regard to extension in our development, so it may be expected that the formulæ found do not in their quantitative relations quite correspond with experiment and, therefore, it is necessary to take the constants E and $\sqrt{KE}$ from experiments made especially to determine them.

If a be given in degrees, such observations admit of our putting for the torsion:

Substances.	Circular section.	Square section.
Wood	$Pa = 3500 \cdot \frac{a^0 r^4}{l}$.	$Pa = 5800 \frac{a^0 b^4}{l}$.
Cast iron	$Pa = 160000 \frac{a^0 r^4}{l}$.	$Pa = 280000 \frac{a^0 b^4}{l}$.
Steel and wrought iron .	$Pa = 280000 \frac{a^0 r^4}{l}$.	$Pa = 470000 \frac{a^0 b^4}{l}$.

In what relates to the strength of torsion, numerous experiments made upon cast iron have given $\sqrt{\frac{KE}{2}} = 30000$ to 66000 lbs., if therefore, a five-fold security be taken, then is $\frac{\pi}{2} \sqrt{\frac{KE}{2}} = 12600$ lbs. therefore, for the round cast iron axle $Pa = 12600\,r^3$, and for the square $= 15000\,b^3$.

The same formulæ hold good for axles of wrought iron, but for wooden ones we may put $Pa = 1260\,r^3$ and $= 1500\,b_3$, *i. e.* the modulus of strength $= \frac{1}{10}$ that of iron axles. The modulus of strength for steel $\sqrt{\frac{KE}{2}}$ must be taken at twice that of iron, and gun metal at one half.*

* See Appendix.

Examples.—1. The iron upright axle of a turbine exerts at the circumference of a toothed wheel of 15 inches radius reposing upon it, a force of 2500 lbs.; what thickness must be given to it? $Pa = 2500 . 15 = 37500$, and if we put $r^3 = \frac{Pa}{12600} = \frac{37500}{12600} = \frac{375}{126}$, we shall obtain $r = \sqrt[3]{\frac{375}{126}} = 1,44$ inches; hence, the thickness of the axle $2r = 2,88$ inches, for which 3 inches may be assumed. If the distance of the toothed wheel from the water wheel is 60 inches, the torsion of the axle $= \alpha^0 = \frac{Pal}{160000\, r^4} = \frac{37500e\ 60}{160000 . 1,44^4} = \frac{375 . 6}{160 . 4,28} = \frac{14,06}{4,28} = 3^0\ 3'$, therefore very considerable. —2. On a square axle of fir, a force $P = 500$ lbs., acts at an arm of 20 feet, whilst the load is applied at an arm of 2 feet, the distance measured in the direction of the axis $l = 10$ feet; how thick must this axle be made, and how great is the torsion? It is $Pa = Qb = 500 . 2 . 12 = 120000$ inch lbs.; but the load $Q = \frac{a}{b} P = 5000$ lbs.; half the side b of the axle is determined by $b^3 = \frac{Pa}{1500} = \frac{120000}{1500} = 80$; hence $b = \sqrt[3]{80} = 4,31$ inches, and the whole side $= 8,62$ inches. The torsion amounts to $\alpha^0 = \frac{Pal}{5800 . b^4} = \frac{120000 . 12 . 10}{5800e\ 4,31^4} = \frac{144000}{58 . 345} = 7^0$; therefore, here very considerable. In general, less torsion is allowed, and therefore the axles are made much stronger. Generally, this angle does not amount to $\frac{1}{2}$ a degree. If, however, we put $\alpha^0 = \frac{1}{2}^0$, for this case we shall obtain $b^4 = \frac{144000}{58 . \frac{1}{2}} = 4965$, hence $b = \sqrt[4]{4965} = 8,4$ inches, and $2b = 16,8$ inches. According to Gerstner, the angle of torsion of an axle ought not to amount to more than $0,1^0$.

Remark. If an axle has to sustain relative elasticity and that of torsion, we must make the calculation for both, and apply the greater ratio of the dimensions found.