REGULARLY VARYING RANDOM FIELDS

LIFAN WU AND GENNADY SAMORODNITSKY

ABSTRACT. We study the extremes of multivariate regularly varying random fields. The crucial tools in our study are the tail field and the spectral field, notions that extend the tail and spectral processes of Basrak and Segers (2009). The spatial context requires multiple notions of extremal index, and the tail and spectral fields are applied to clarify these notions and other aspects of extremal clusters. An important application of the techniques we develop is to the Brown-Resnick random fields.

1. Introduction

An $\mathbb{R}^d$-valued random vector $\mathbf{X}$ is said to have a multivariate regularly varying distribution with exponent $\alpha > 0$ if there exists a regularly varying with exponent α function $V : \mathbb{R}_+ \to \mathbb{R}_+$, and a nonzero Radon measure μ on $(\overline{\mathbb{R}})^d \backslash \{\mathbf{0}\} = [-\infty, \infty]^d \backslash \{\mathbf{0}\}$ that does not charge infinite points, such that

$$\frac{\mathbb{P}(x^{-1}\mathbf{X} \in \cdot)}{V(x)} \xrightarrow{v} \mu(\cdot) \tag{1.1}$$

(vaguely) as $x \to \infty$. The limiting measure μ is called the tail measure of $\mathbf{X}$ and it possesses the scaling property $\mu(uA) = u^{-\alpha}\mu(A)$ for any $\alpha > 0$ and a measurable set $A \subset (\overline{\mathbb{R}})^d \backslash \{\mathbf{0}\}$; see e.g. Resnick (1987, 2007). It is usual to say simply that $\mathbf{X}$ is regularly varying.

Infinite-dimensional notions of regular variation are more complicated, but they have been developed as well. The notion of regularly varying stochastic process with sample paths in $\mathbb{D}([0,1])$ was introduced in Hult and Lindskog (2005), and it was extended to random fields with sample paths in $\mathbb{D}([0,1]^d)$ in Davis and Mikosch (2008).

When stationarity is present, each observation of the stochastic process is equally likely to be an extreme, and it is of interest to determine how these extremes cluster or, in other words, how these extremes differ from the extremes of i.i.d. observations with the same marginal distributions. The extremal index of a stationary process, introduced by Leadbetter (1983), measures the sizes of extremal clusters. Under the additional assumption of multivariate regular variation, Davis and Mikosch (2009) introduced the extremogram to capture the dependence of the extremes in a stationary regularly varying stochastic process. In order to describe the extremal dependence of an entire stochastic process, an unpublished work of Owada and Samorodnitsky (2014) introduced the notion of a tail measure for a regularly varying stochastic process, and better known notions are those of the tail and spectral processes developed by Basrak and Segers (2009).

Extending some of these notions to random fields is challenging due to the lack of natural order in the time domain. Choi (2002) proved the existence of a spatial extremal index under the

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coordinate-wise mixing condition introduced by Leadbetter and Rootzén (1998), while Ferreira and Pereira (2008) proposed a way to compute it. Recently, Cho et al. (2016) formulated the notion of an extremogram for random fields. In this paper, we extend the theory of the tail and spectral processes of Basrak and Segers (2009) to $\mathbb{R}^d$-valued regularly varying random fields with parameter space $\mathbb{Z}^k$. At the same time and independently, a part of this extension was also done in Basrak and Planinić (2018), but the goals of that paper are different. We will mention the similarities in the sequel.

The structure of this paper is as follows. In Section 2 we introduce the notion of the tail field corresponding to a stationary regularly varying random field. Its properties are studied Section 3, where the notion of the spectral field is also introduced. These two notions are analogous to the notions of the tail and spectral processes of Basrak and Segers (2009). A general discussion of the possible notions of the spatial extremal index is in Section 4. The point process description of the extremal clusters is extended from the case of one-dimensional time to random fields in Section 5. An application to Brown-Resnick random fields is in Section 6.

Notation. As usual, letters such as X, stand for random variables, while bold letters, such as $\mathbf{X}$, stand for random vectors. Similarly, $\mathbf{i}$ and i stand for the indices in $\mathbb{Z}^k$ and $\mathbb{Z}$, correspondingly. We use the notation $\mathbb{N}_0$ for $\mathbb{N} \cup \{0\}$. For a pair of indices $\mathbf{i}$ and $\mathbf{j}$, we say that $\mathbf{i} \le \mathbf{j}$ if $i_\ell \le j_\ell$ for all $\ell = 1, \dots, k$, in which case $\mathbf{X}([\mathbf{i} : \mathbf{j}])$ is the random vector $(\mathbf{X}(\mathbf{t}) : \mathbf{i} \le \mathbf{t} \le \mathbf{j})$. The hypercubes $[(-\mathbf{n}+\mathbf{1}) : \mathbf{n}-\mathbf{1}]$ and $[\mathbf{0} : \mathbf{n}-\mathbf{1}]$ are denoted by $\mathcal{R}_{\mathbf{n}}$ and $\mathcal{R}_{\mathbf{n}}^+$, respectively.

For a random field $(\mathbf{X}(\mathbf{t}) : \mathbf{t} \in \mathbb{Z}^k)$ and a finite set $A \subset \mathbb{Z}^k$ we write $M_X(A)$ for $\max_{\mathbf{t}\in A} \|\mathbf{X}(\mathbf{t})\|$. Also, we write $\mathbf{0}$ and $\mathbf{1}$ for the vectors of all 0's and 1's, respectively. Finally, all the vector operations in this paper are performed element-wise.

2. The Tail Field

Let $(\mathbf{X}(\mathbf{t}) : \mathbf{t} \in \mathbb{Z}^k)$ be an $\mathbb{R}^d$-valued random field. It is said to be jointly regularly varying if the random vector $(\mathbf{X}(\mathbf{t}_1), \dots, \mathbf{X}(\mathbf{t}_n))$ is regularly varying in $\mathbb{R}^{nd}$ for any $\mathbf{t}_1, \dots, \mathbf{t}_n \in \mathbb{Z}^k$. The following result is an extension of Theorem 2.1 in Basrak and Segers (2009) to random fields. We will see that only a partial extension is possible.

Theorem 2.1. *An $\mathbb{R}^d$-valued stationary random field $(\mathbf{X}(\mathbf{t}) : \mathbf{t} \in \mathbb{Z}^k)$ is jointly regularly varying with index $\alpha > 0$ if and only if there exists a random field $(\mathbf{Y}(\mathbf{t}) : \mathbf{t} \in \mathbb{Z}^k)$ such that*

$$\mathcal{L}\left(x^{-1}\mathbf{X}(\mathbf{t}) : \mathbf{t} \in \mathbb{Z}^k \,\middle|\, \|\mathbf{X}(\mathbf{0})\| > x\right) \to \mathcal{L}(\mathbf{Y}(\mathbf{t}) : \mathbf{t} \in \mathbb{Z}^k) \tag{2.1}$$

as $x \to \infty$ in the sense of convergence of the finite-dimensional distributions, and $\mathbb{P}(\|\mathbf{Y}(\mathbf{0})\| > y) = y^{-\alpha}$ for $y \ge 1$.

Extending the terminology of Basrak and Segers (2009), we call the limiting random field $(\mathbf{Y}(\mathbf{t}) : \mathbf{t} \in \mathbb{Z}^k)$ *the tail field* of the stationary field $(\mathbf{X}(\mathbf{t}) : \mathbf{t} \in \mathbb{Z}^k)$.

Proof of Theorem 2.1. The argument is similar to the case of the one-dimensional time. Suppose first that $(\mathbf{X}(\mathbf{t}) : \mathbf{t} \in \mathbb{Z}^k)$ is jointly regularly varying. Then for arbitrary index pairs $\mathbf{i} \le \mathbf{j}$, $\mathbf{X}([\mathbf{i} : \mathbf{j}])$ is a regularly varying vector with index α, By stationarity, the function V in (1.1) can be chosen to be $V(x) = \mathbb{P}(\|\mathbf{X}(\mathbf{0})\| > x)$ regardless of $\mathbf{i}, \mathbf{j}$, so there exists a Radon measure $\mu_{\mathbf{i},\mathbf{j}}$ on $\left(\overline{\mathbb{R}}^d\right)^{\prod_{\ell=1}^k (j_\ell - i_\ell + 1)} \setminus \{\mathbf{0}\}$ such that

$$\frac{1}{\mathbb{P}(\|\mathbf{X}(\mathbf{0})\| > x)} \mathbb{P}\left(x^{-1}\mathbf{X}([\mathbf{i} : \mathbf{j}]) \in \cdot\right) \xrightarrow{v} \mu_{\mathbf{i},\mathbf{j}}(\cdot) \tag{2.2}$$

as $x \to \infty$. The restriction $\nu_{\mathbf{i},\mathbf{j}}$ to the set $\{\mathbf{y}([\mathbf{i}:\mathbf{j}]) \mid \|\mathbf{y}(\mathbf{0})\| > 1\}$ is, by definition, a probability measure, and the collection of the probability measures $\left(\nu_{\mathbf{i},\mathbf{j}}\right)$ is, clearly, consistent, in the sense that, if $\mathbf{i}_2 \le \mathbf{i}_1 \le \mathbf{j}_1 \le \mathbf{j}_2$ then the measure $\nu_{\mathbf{i}_1,\mathbf{j}_1}$ is obtained from the measure $\nu_{\mathbf{i}_2,\mathbf{j}_2}$ by integrating out the redundant dimensions. By the Kolmogorov extension theorem there is a random field $(\mathbf{Y}(\mathbf{t}) : \mathbf{t} \in \mathbb{Z}^k)$ whose finite-dimensional distributions are determined by the family $\left(\nu_{\mathbf{i},\mathbf{j}}\right)$. Then (2.1) follows from (2.2), and the Pareto distribution of $\|\mathbf{Y}(\mathbf{0})\|$ follows as in the case of the one-dimensional time.

In the opposite direction, suppose that (2.1) holds for all $\mathbf{i} \le \mathbf{j} \in \mathbb{Z}^k$, and $\mathbb{P}(\|\mathbf{Y}(\mathbf{0})\| > y) = y^{-\alpha}$. As in the case of the one-dimensional time, for $y \ge 1$ we have

$$\frac{\mathbb{P}(\|\mathbf{X}(\mathbf{0})\| > xy)}{\mathbb{P}(\|\mathbf{X}(\mathbf{0})\| > x)} = \mathbb{P}(\|x^{-1}\mathbf{X}(\mathbf{0})\| > y \mid \|\mathbf{X}(\mathbf{0})\| > x) \to \mathbb{P}(\|\mathbf{Y}(\mathbf{0})\| > y) = y^{-\alpha}$$

as $x \to \infty$, so that $\|\mathbf{X}(\mathbf{0})\|$ is a regularly varying variable with index α. We need to show that for any $\mathbf{i} \le \mathbf{j} \in \mathbb{Z}^k$,

$$\frac{1}{\mathbb{P}(\|\mathbf{X}(\mathbf{0})\| > x)} \mathbb{P}\left(x^{-1}\mathbf{X}([\mathbf{i}:\mathbf{j}]) \in \cdot\right)$$

converges vaguely as $x \to \infty$, and by the already established regular variation of $\|\mathbf{X}(\mathbf{0})\|$, it is enough to establish weak convergence on the set of vectors for which the norm of $\mathbf{X}(\mathbf{t})$ is at least 1 for some fixed $\mathbf{i} \le \mathbf{t} \le \mathbf{j}$. We will, in fact, show weak convergence to the law of the random vector $\mathbf{Y}([\mathbf{i}-\mathbf{t}:\mathbf{j}-\mathbf{t}])$. Indeed, on the relevant set, by stationarity,

$$\begin{aligned}
&\frac{1}{\mathbb{P}(\|\mathbf{X}(\mathbf{0})\| > x)} \mathbb{P}\left(x^{-1}\mathbf{X}([\mathbf{i}:\mathbf{j}]) \in \cdot\right) \\
=&\frac{1}{\mathbb{P}(\|\mathbf{X}(\mathbf{0})\| > x)} \mathbb{P}\left(x^{-1}\mathbf{X}([\mathbf{i}:\mathbf{j}]) \in \cdot\,, \|x^{-1}\mathbf{X}(\mathbf{t})\| > 1\right) \\
=&\mathbb{P}\left(x^{-1}\mathbf{X}([\mathbf{i}:\mathbf{j}]) \in \cdot \mid \|x^{-1}\mathbf{X}(\mathbf{t})\| > 1\right) \\
=&\mathbb{P}\left(x^{-1}\mathbf{X}([\mathbf{i}-\mathbf{t}:\mathbf{j}-\mathbf{t}]) \in \cdot \mid \|\mathbf{X}(\mathbf{0})\| > x\right) \\
\to&\mathbb{P}\left(\mathbf{Y}([\mathbf{i}-\mathbf{t}:\mathbf{j}-\mathbf{t}]) \in \cdot\right)
\end{aligned}$$

as $x \to \infty$, as required. □

Remark 2.2. A similar statement is in Theorem 3.1 of Basrak and Planinić (2018). When the time is one-dimensional, Basrak and Segers (2009) proved that the weak convergence on the set of nonnegative times,

$$\mathcal{L}\left(x^{-1}\mathbf{X}(t) :\ t \in \mathbb{N}_0 \middle| \|\mathbf{X}(0)\| > x\right) \to \mathcal{L}\left(\mathbf{Y}(t) :\ t \in \mathbb{N}_0\right), \tag{2.3}$$

sufficed to guarantee the joint regular variation of the original process. Interestingly, the obvious analogue of this statement for random fields is false, as the following example of a scalar-valued random field with 2-dimensional time illustrates.

Example 2.3. Let (Z_1, Z_2) be a random vector such that $(Z_1, Z_2) \stackrel{d}{=} (Z_2, Z_1)$, Z_1 is regularly varying with index α, but the random vector (Z_1, Z_2) itself is not regularly varying. For completeness, we will construct an example of such a vector below.

Let $\left(Z_1^{(j)}, Z_2^{(j)}\right)$, $j \in \mathbb{Z}$, be iid copies of (Z_1, Z_2). We define a scalar-valued random field $(X(\mathbf{t}) : \mathbf{t} \in \mathbb{Z}^2)$ by letting

$$X(\mathbf{t}) = \begin{cases} Z_1^{(t_1+t_2)}, & \text{if } t_1 \text{ is odd} \\ Z_2^{(t_1+t_2)}, & \text{if } t_1 \text{ is even} \end{cases}.$$

It is clearly stationary. We claim that

$$\mathcal{L}\left(x^{-1}X(\mathbf{t}):\mathbf{t}\in\mathbb{N}_0^2\Big||X(\mathbf{0})|>x\right)\to\mathcal{L}\left(Y(\mathbf{t}):\mathbf{t}\in\mathbb{N}_0^2\right) \tag{2.4}$$

as $x\to\infty$, where $Y(\mathbf{0})$ has the Pareto(α) distribution, and $Y(\mathbf{t})=0$ for each $\mathbf{t}\neq\mathbf{0}$. Indeed, since $\mathbf{0}$ is the only point in $\mathbb{N}_0^2$ on the line $t_1+t_2=0$, $X(\mathbf{t})$ is independent of $X(\mathbf{0})$ for each $\mathbf{t}\in\mathbb{N}_0^2\setminus\{\mathbf{0}\}$. Therefore, for any such $\mathbf{t}$ we have $\mathcal{L}(x^{-1}X(\mathbf{t})|X(\mathbf{0})>x)\to\delta_{\mathbf{0}}$ as $x\to\infty$. Therefore, (2.4) follows since $X(\mathbf{0})$ is regularly varying with index α because of the assumed regular variation of Z_1. Therefore, (2.4) holds. Note that the latter is the obvious analogue of (2.3) for a random field.

However, the random field $(X(\mathbf{t}):\mathbf{t}\in\mathbb{Z}^2)$ is not regularly varying. To see this, note that with

$$\mathbf{t}_1=\begin{pmatrix}0\\0\end{pmatrix},\ \mathbf{t}_2=\begin{pmatrix}-1\\1\end{pmatrix}$$

we have $\big(X(\mathbf{t}_1),X(\mathbf{t}_2)\big)\overset{d}{=}(Z_1,Z_2)$, which, by the assumption, is not regularly varying.

It remains to construct a random vector (Z_1,Z_2) such that $(Z_1,Z_2)\overset{d}{=}(Z_2,Z_1)$, Z_1 is regularly varying with index α, but the random vector (Z_1,Z_2) itself is not regularly varying. Let $a_n=n!$, $n=1,2,\ldots$. Let $Z\geq 1$ have the standard Pareto(α) distribution. If $Z\in[a_{2n-1},a_{2n})$ for some $n=1,2,\ldots$, set $Z_1=Z_2=Z$. If $Z\in[a_{2n},a_{2n+1})$ for some $n=1,2,\ldots$, take Z_1 and Z_2 be standard Pareto(α) random variables conditioned on being in the interval $[a_{2n},a_{2n+1})$ but otherwise independent. Formally, for any two-dimensional Borel set A,

$$\begin{aligned}\mathbb{P}\big((Z_1,Z_2)\in A\big)=&\sum_{n=1}^{\infty}\int_{a_{2n-1}}^{a_{2n}}\mathbb{1}\big((z,z)\in A\big)az^{-(\alpha+1)}\,dz\\&+\sum_{n=1}^{\infty}\frac{1}{a_{2n}^{-\alpha}-a_{2n+1}^{-\alpha}}\int_{a_{2n}}^{a_{2n+1}}\int_{a_{2n}}^{a_{2n+1}}\mathbb{1}\big((z_1,z_2)\in A\big)az_1^{-(\alpha+1)}az_2^{-(\alpha+1)}\,dz_1\,dz_2\,.\end{aligned}$$

By construction, $(Z_1,Z_2)\overset{d}{=}(Z_2,Z_1)$, and each coordinate of the random vector has the standard Pareto(α) distribution. It remains to show that the random vector (Z_1,Z_2) is not regularly varying. Note that

$$\mathbb{P}\left(a_{2n-1}^{-1}(Z_1,Z_2)\in(1,2]\times(1,2]\right)\sim\mathbb{P}\left(Z\in(a_{2n-1},2a_{2n-1}]\right)\sim(1-2^{-\alpha})a_{2n-1}^{-\alpha}$$

as $n\to\infty$. On the other hand,

$$\begin{aligned}\mathbb{P}\left(a_{2n}^{-1}(Z_1,Z_2)\in(1,2]\times(1,2]\right)\leq&\frac{1}{a_{2n}^{-\alpha}-a_{2n+1}^{-\alpha}}\left(\int_{a_{2n}}^{2a_{2n}}az^{-(\alpha+1)}\,dz\right)^2\\&+\mathbb{P}\left(Z\geq a_{2n+1}\right)\sim(1-2^{-\alpha})^2a_{2n}^{-\alpha}\end{aligned}$$

as $n\to\infty$. Therefore, (1.1) cannot hold.

3. Properties of the tail field

This section describes the properties of the tail field introduced in the previous section. These are similar, but not identical, to the properties of the tail process. In particular, we introduce an object parallel to that of the spectral process of Basrak and Segers (2009), which we call the spectral field. The latter is defined as $\mathbf{\Theta}(\mathbf{t})=\mathbf{Y}(\mathbf{t})/\|\mathbf{Y}(\mathbf{0})\|$, $\mathbf{t}\in\mathbb{Z}^k$, where $(\mathbf{Y}(\mathbf{t}):\mathbf{t}\in\mathbb{Z}^k)$ is the tail field of an $\mathbb{R}^d$-valued stationary random field $(\mathbf{X}(\mathbf{t}) : \mathbf{t} \in \mathbb{Z}^k)$ that is jointly regularly varying with index $\alpha > 0$. As in the one-dimensional case, it is easy to check that

$$\text{the spectral field is independent of } \|\mathbf{Y}(\mathbf{0})\| \,. \tag{3.1}$$

The following proposition can be proved in the same way as for the one-dimensional time, so we do not include the proof. See also Theorem 3.1 in Basrak and Planinić (2018). Note, however, that a part of Corollary 3.2 in Basrak and Segers (2009) fails in the case of random fields; see Example 2.3.

Proposition 3.1. *Let $(\mathbf{X}(\mathbf{t}) : \mathbf{t} \in \mathbb{Z}^k)$ be an $\mathbb{R}^d$-valued stationary random field, and $\|\mathbf{X}(\mathbf{0})\|$ be a regularly varying variable with index α for some $\alpha \in (0, \infty)$. Then $(\mathbf{X}(\mathbf{t}) : \mathbf{t} \in \mathbb{Z}^k)$ is jointly regularly varying with index α if and only if there exists a random field $(\mathbf{\Theta}(\mathbf{t}) : \mathbf{t} \in \mathbb{Z}^k)$ such that*

$$\mathcal{L}\left(\frac{\mathbf{X}(\mathbf{t})}{\|\mathbf{X}(\mathbf{0})\|} : \mathbf{t} \in \mathbb{Z}^k \,\middle|\, \|\mathbf{X}(\mathbf{0})\| > x \right) \to \mathcal{L}(\mathbf{\Theta}(\mathbf{t}) : \mathbf{t} \in \mathbb{Z}^k) \tag{3.2}$$

as $x \to \infty$.

Even though neither the tail field nor the spectral field is generally stationary, the stationarity of the original random field $(\mathbf{X}(\mathbf{t}) : \mathbf{t} \in \mathbb{Z}^k)$ makes itself felt in the former fields. In particular, it leads to a "change-of-time" property for these fields. A similar result in the case of one-dimensional time is a part of Theorem 3.1 in Basrak and Segers (2009). We present this property in a somewhat more general form.

Theorem 3.2. *Let $(\mathbf{Y}(\mathbf{t}) : \mathbf{t} \in \mathbb{Z}^k)$ be the tail field corresponding to an $\mathbb{R}^d$-valued stationary random field $(\mathbf{X}(\mathbf{t}) : \mathbf{t} \in \mathbb{Z}^k)$ that is jointly regularly varying with index $\alpha > 0$, and let $(\mathbf{\Theta}(\mathbf{t}) : \mathbf{t} \in \mathbb{Z}^k)$ be the corresponding spectral field. Let $g : (\mathbb{R}^d)^{\mathbb{Z}^k} \to \mathbb{R}$ be a bounded measurable function. Take any $\mathbf{s} \in \mathbb{Z}^k$. Then the following identities hold:*

$$\mathbb{E}\big[g(\mathbf{Y}(\cdot - \mathbf{s}))\mathbb{1}\big(\mathbf{Y}(-\mathbf{s}) \neq \mathbf{0}\big)\big] = \int_0^\infty \mathbb{E}[g(r\mathbf{\Theta}(\cdot))\mathbb{1}(r\|\mathbf{\Theta}(\mathbf{s})\| > 1)]\, d(-r^{-\alpha})\,, \tag{3.3}$$

$$\mathbb{E}\big[g(\mathbf{\Theta}(\cdot - \mathbf{s}))\mathbb{1}\big(\mathbf{\Theta}(-\mathbf{s}) \neq \mathbf{0}\big)\big] = \mathbb{E}\left[g\left(\frac{\mathbf{\Theta}(\cdot)}{\|\mathbf{\Theta}(\mathbf{s})\|} \right) \|\mathbf{\Theta}(\mathbf{s})\|^\alpha \right]. \tag{3.4}$$

Proof. Since a probability measure on $(\mathbb{R}^d)^{\mathbb{Z}^k}$ is uniquely determined by its finite-dimensional distributions, for (3.3) it is enough to prove that for any $\mathbf{i} \le \mathbf{j} \in \mathbb{Z}^k$ and any bounded measurable function $g : (\mathbb{R}^d)^{\prod_{\ell=1}^k (j_\ell - i_\ell + 1)} \to \mathbb{R}$, we have

$$\mathbb{E}\big[g(\mathbf{Y}([\mathbf{i} - \mathbf{s} : \mathbf{j} - \mathbf{s}]))\mathbb{1}\big(\mathbf{Y}(-\mathbf{s}) \neq \mathbf{0}\big)\big] = \int_0^\infty \mathbb{E}[g(r\mathbf{\Theta}([\mathbf{i} : \mathbf{j}]))\mathbb{1}(r\|\mathbf{\Theta}(\mathbf{s})\| > 1)]\, d(-r^{-\alpha})\,. \tag{3.5}$$

Suppose first that g is bounded and continuous. Let $\varepsilon > 0$. By (2.1) and stationarity, the argument of Basrak and Segers (2009) gives us

$$\mathbb{E}\big[g(\mathbf{Y}([\mathbf{i} - \mathbf{s} : \mathbf{j} - \mathbf{s}]))\mathbb{1}\big(\|\mathbf{Y}(-\mathbf{s})\| > \varepsilon\big)\big] = \int_\varepsilon^\infty \mathbb{E}[g(r\mathbf{\Theta}([\mathbf{i} : \mathbf{j}]))\mathbb{1}(r\|\mathbf{\Theta}(\mathbf{s})\| > 1)]\, d(-r^{-\alpha})\,.$$

If g is, in addition, nonnegative, then we can let $\varepsilon \downarrow 0$ in this relation, so that monotone convergence theorem gives us (3.5) for nonnegative bounded and continuous g. The assumption of nonnegativity can now be removed by writing g as the difference of its positive and negative parts. Since integrals of bounded continuous functions uniquely determine a finite measure, we see that (3.5) holds without the assumption of continuity. As in Basrak and Segers (2009), (3.4) follows from (3.3) by defining a new bounded measurable function on $(\mathbb{R}^d)^{\mathbb{Z}^k}$ as $\tilde{g}(\mathbf{y}) = g(\mathbf{y}/\|\mathbf{y}(\mathbf{s})\|)\mathbb{1}(\mathbf{y}(\mathbf{s}) \neq 0)$ and applying (3.3) to this function. $\square$

If h is a bounded measurable function on the unit sphere $\mathbb{S}^{d-1}$ in $\mathbb{R}^d$, then choosing $g(\mathbf{y}) = h(\mathbf{y}(\mathbf{s}))$ if $\|\mathbf{y}(\mathbf{s})\| = 1$ and $g(\mathbf{y}) = 0$ otherwise produces a bounded measurable function on $\big(\overline{\mathbb{R}}^d\big)^{\mathbb{Z}^k}$. Applying (3.4) to this function gives us the identity

$$\mathbb{E}[h(\mathbf{\Theta}(\mathbf{s})/\|\mathbf{\Theta}(\mathbf{s})\|)\|\mathbf{\Theta}(\mathbf{s})\|^\alpha] = \mathbb{E}[h(\mathbf{\Theta}(\mathbf{0}))\mathbb{1}(\mathbf{\Theta}(-\mathbf{s}) \neq \mathbf{0})]\,. \tag{3.6}$$

The value of $\mathbb{E}[\|\mathbf{\Theta}(\mathbf{s})\|^\alpha]$ is a measure of the effect of changing the "origin" of the spectral field from $\mathbf{0}$ to $\mathbf{s}$ (recall that $\|\mathbf{\Theta}(\mathbf{0})\| = 1$ by the definition). With $h(\cdot) \equiv 1$, (3.6) reduces to $\mathbb{E}[\|\mathbf{\Theta}(\mathbf{s})\|^\alpha] = \mathbb{P}(\mathbf{\Theta}(-\mathbf{s}) \neq \mathbf{0})$ (so $\mathbb{E}[\|\mathbf{\Theta}(\mathbf{s})\|^\alpha] \leq 1$). In particular, for $\delta > 0$,

$$\begin{aligned}\lim_{x\to\infty} \mathbb{P}\big(\|\mathbf{X}(\mathbf{0})\| > \delta x \,|\, \|\mathbf{X}(\mathbf{s})\| > x\big) &= \lim_{x\to\infty} \mathbb{P}\big(\|\mathbf{X}(-\mathbf{s})\| > \delta x \,|\, \|\mathbf{X}(\mathbf{0})\| > x\big)\\ &= \mathbb{P}(\|\mathbf{Y}(-\mathbf{s})\| > \delta)\,,\end{aligned}$$

so

$$\begin{aligned}\mathbb{E}[\|\mathbf{\Theta}(\mathbf{s})\|^\alpha] &= \mathbb{P}(\mathbf{\Theta}(-\mathbf{s}) \neq \mathbf{0})\\ &= \mathbb{P}(\mathbf{Y}(-\mathbf{s}) \neq \mathbf{0}) = \lim_{\delta\downarrow 0} \mathbb{P}(\|\mathbf{Y}(-\mathbf{s})\| > \delta)\\ &= \lim_{\delta\downarrow 0}\lim_{x\to\infty} \mathbb{P}\big(\|\mathbf{X}(\mathbf{0})\| > \delta x \,|\, \|\mathbf{X}(\mathbf{s})\| > x\big)\,,\end{aligned}$$

thus providing an intuitive interpretation of the quantity $\mathbb{E}[\|\mathbf{\Theta}(\mathbf{s})\|^\alpha]$. Furthermore, assuming that $\mathbb{E}[\|\mathbf{\Theta}(\mathbf{s})\|^\alpha] > 0$, (3.6) says that the two probability measures on $\mathbb{S}^{d-1}$,

$$\begin{aligned}\mathbb{P}_1(\cdot) &= \frac{1}{\mathbb{E}[\|\mathbf{\Theta}(\mathbf{s})\|^\alpha]}\mathbb{E}[\|\mathbf{\Theta}(\mathbf{s})\|^\alpha \mathbb{1}(\mathbf{\Theta}(\mathbf{s})/\|\mathbf{\Theta}(\mathbf{s})\| \in \cdot)]\,,\\ \mathbb{P}_2(\cdot) &= \frac{\mathbb{P}(\mathbf{\Theta}(\mathbf{0}) \in \cdot,\, \mathbf{\Theta}(-\mathbf{s}) \neq \mathbf{0})}{\mathbb{P}(\mathbf{\Theta}(-\mathbf{s}) \neq \mathbf{0})}\,,\end{aligned}$$

are equal. Therefore, a necessary and sufficient condition for $\mathbb{E}[\|\mathbf{\Theta}(\mathbf{s})\|^\alpha] = 1$ is

$$\mathbb{P}(\mathbf{\Theta}(\mathbf{0}) \in \cdot) = \mathbb{E}[\|\mathbf{\Theta}(\mathbf{s})\|^\alpha \mathbb{1}(\mathbf{\Theta}(\mathbf{s})/\|\mathbf{\Theta}(\mathbf{s})\| \in \cdot)]\,.$$

The above discussion is an extension of the ideas in Basrak and Segers (2009) in the case of on-dimensional time to random fields.

The important "change-of-time" property (3.4) has recently been shown in Janßen (2018) to be equivalent, in the case of the one-dimensional time, to a certain distributional invariance property of the spectral process. As we explain below, this equivalence extends to random fields. We start with a simple extension of Lemma 2.2 *ibid.* It describes a rather unexpected property of the spectral field. The argument requires the notion of invariant order. A complete order $\prec$ on $\mathbb{Z}^k$ is called invariant if $\mathbf{s} \prec \mathbf{t}$ for $\mathbf{s}, \mathbf{t} \in \mathbb{Z}^k$ implies that $\mathbf{s} + \mathbf{i} \prec \mathbf{t} + \mathbf{i}$ for any $\mathbf{i} \in \mathbb{Z}^k$. An example of an invariant order is the lexicographic (or dictionary) order: for $\mathbf{s}, \mathbf{t} \in \mathbb{Z}^k$, we say that $\mathbf{s} \prec \mathbf{t}$ if either (1) $s_1 < t_1$, or (2) there exists $2 \leq \ell \leq k$ such that $s_i = t_i$ for all $i = 1, \ldots, \ell - 1$, and $s_\ell < t_\ell$.

Lemma 3.3. *Let* $(\mathbf{\Theta}(\mathbf{t}) : \mathbf{t} \in \mathbb{Z}^k)$ *be an* $\mathbb{R}^d$*-valued random field such that* $\mathbb{P}(\|\mathbf{\Theta}(\mathbf{0})\| = 1) = 1$ *and satisfies* (3.4). *Then* $\|\mathbf{\Theta}(\mathbf{t})\| \to 0$ *a.s. as* $\|\mathbf{t}\| \to \infty$ *if and only if* $\sum_{\mathbf{t}\in\mathbb{Z}^k} \|\mathbf{\Theta}(\mathbf{t})\|^\alpha < \infty$ *a.s.*

Proof. Trivially, the summability condition implies that the values of the field vanish at infinity. In the other direction, fix an invariant order on $\mathbb{Z}^k$, and suppose that the event $\{\|\mathbf{\Theta}(\mathbf{t})\| \to 0$ as $\|\mathbf{t}\| \to \infty\}$ has probability 1. On this event there is, clearly, a finite number of points in $\mathbb{Z}^k$ over which $\|\mathbf{\Theta}(\mathbf{i})\|$ achieves the supremum $\sup_{\mathbf{t}\in\mathbb{Z}^k} \|\mathbf{\Theta}(\mathbf{t})\|$. Therefore, on this event we can define a $\mathbb{Z}^k$-valued random variable $\mathbf{T}$ such that $\|\mathbf{\Theta}(\mathbf{T})\| = \sup_{\mathbf{t}\in\mathbb{Z}^k} \|\mathbf{\Theta}(\mathbf{t})\|$ and any other point of $\mathbb{Z}^k$ with this property succeeds $\mathbf{T}$ in the invariant order. If, to the contrary, we have $\mathbb{P}(\sum_{\mathbf{t}\in\mathbb{Z}^k}\|\mathbf{\Theta}(\mathbf{t})\|^\alpha=\infty)>0$, then there exists $\mathbf{i}\in\mathbb{Z}^k$ such that $P(\sum_{\mathbf{t}\in\mathbb{Z}^k}\|\mathbf{\Theta}(\mathbf{t})\|^\alpha=\infty,\mathbf{T}=\mathbf{i})>0$, which gives us

$$\infty=\mathbb{E}\left[\sum_{\mathbf{t}\in\mathbb{Z}^k}\|\mathbf{\Theta}(\mathbf{t})\|^\alpha\mathbb{1}(\mathbf{T}=\mathbf{i})\right]=\sum_{\mathbf{t}\in\mathbb{Z}^k}\mathbb{E}\left[\|\mathbf{\Theta}(\mathbf{t})\|^\alpha\mathbb{1}(\mathbf{T}=\mathbf{i})\right].$$

For each $\mathbf{i}\in\mathbb{Z}^k$ we define a function $g_{\mathbf{i}}:(\overline{\mathbb{R}}^d)^{\mathbb{Z}^k}\to\mathbb{R}$ as follows. If $(\boldsymbol{\theta}(\mathbf{s}),\,\mathbf{s}\in\mathbb{Z}^k)$ is such that

$$\|\boldsymbol{\theta}(\mathbf{j})\|<\|\boldsymbol{\theta}(\mathbf{i})\|\text{ for }\mathbf{j}\prec\mathbf{i},\ \|\boldsymbol{\theta}(\mathbf{j})\|\le\|\boldsymbol{\theta}(\mathbf{i})\|\text{ for }\mathbf{j}\succeq\mathbf{i}\,,$$

then we set $g_{\mathbf{i}}(\boldsymbol{\theta}(\mathbf{s}),\,\mathbf{s}\in\mathbb{Z}^k)=1$. Otherwise we set $g_{\mathbf{i}}(\boldsymbol{\theta}(\mathbf{s}),\,\mathbf{s}\in\mathbb{Z}^k)=0$. Clearly, each $g_{\mathbf{i}}$ is a bounded measurable function. Then by the "change of time property" (3.4),

$$\begin{aligned}\infty&=\sum_{\mathbf{t}\in\mathbb{Z}^k}\mathbb{E}\left[\|\mathbf{\Theta}(\mathbf{t})\|^\alpha g_{\mathbf{i}}(\mathbf{\Theta}(\mathbf{s}):\mathbf{s}\in\mathbb{Z}^k)\right]=\sum_{\mathbf{t}\in\mathbb{Z}^k}\mathbb{E}\left[\|\mathbf{\Theta}(\mathbf{t})\|^\alpha g_{\mathbf{i}}\left(\frac{\mathbf{\Theta}(\mathbf{s})}{\|\mathbf{\Theta}(\mathbf{t})\|}:\mathbf{s}\in\mathbb{Z}^k\right)\right]\\&=\sum_{\mathbf{t}\in\mathbb{Z}^k}\mathbb{E}\left[g_{\mathbf{i}}\left(\mathbf{\Theta}(\mathbf{s}-\mathbf{t}):\mathbf{s}\in\mathbb{Z}^k\right)\mathbb{1}(\mathbf{\Theta}(-\mathbf{t})\neq\mathbf{0})\right]\\&=\sum_{\mathbf{t}\in\mathbb{Z}^k}\mathbb{E}\left[\mathbb{1}(\|\mathbf{\Theta}(\mathbf{j})\|<\|\mathbf{\Theta}(\mathbf{i}-\mathbf{t})\|,\,\mathbf{j}\prec\mathbf{i}-\mathbf{t})\mathbb{1}(\|\mathbf{\Theta}(\mathbf{j})\|\le\|\mathbf{\Theta}(\mathbf{i}-\mathbf{t})\|,\,\mathbf{j}\succeq\mathbf{i}-\mathbf{t})\mathbb{1}(\mathbf{\Theta}(-\mathbf{t})\neq\mathbf{0})\right]\\&\le\sum_{\mathbf{t}\in\mathbb{Z}^k}\mathbb{E}\left[\mathbb{1}(\|\mathbf{\Theta}(\mathbf{j})\|<\|\mathbf{\Theta}(\mathbf{i}-\mathbf{t})\|,\,\mathbf{j}\prec\mathbf{i}-\mathbf{t})\mathbb{1}(\|\mathbf{\Theta}(\mathbf{j})\|\le\|\mathbf{\Theta}(\mathbf{i}-\mathbf{t})\|,\,\mathbf{j}\succeq\mathbf{i}-\mathbf{t})\right]\\&=\sum_{\mathbf{t}\in\mathbb{Z}^k}P(\mathbf{T}=\mathbf{i}-\mathbf{t})=1,\end{aligned}$$

which leads to a contradiction. Hence, $\sum_{\mathbf{t}\in\mathbb{Z}^k}\|\mathbf{\Theta}(\mathbf{t})\|^\alpha<\infty$ a.s.. □

As in the one-dimensional case, the spectral field vanishes a.s. at infinity under Condition 5.1 below; see Theorem 5.2. Lemma 3.3 shows that under Condition 5.1 the spectral process also satisfies the stronger summability statement.

The next theorem is a version of Theorem 2.4 in Janßen (2018) for random fields. It establishes a certain invaraince property of the law of a spectral fields satsifying the equaivalent conditions of Lemma 3.3.

Theorem 3.4. *Let $(\mathbf{\Theta}(\mathbf{t}):\mathbf{t}\in\mathbb{Z}^k)$ be an $\mathbb{R}^d$-valued random field such that $0<\sum_{\mathbf{t}\in\mathbb{Z}^k}\|\mathbf{\Theta}(\mathbf{t})\|^\alpha<\infty$ a.s.. Let $\boldsymbol{\mathcal{I}}$ be an $\mathbb{Z}^k$-valued random element such that*

$$\mathbb{P}(\boldsymbol{\mathcal{I}}=\mathbf{i}\mid(\mathbf{\Theta}(\mathbf{t}):\mathbf{t}\in\mathbb{Z}^k))=\frac{\|\mathbf{\Theta}(\mathbf{i})\|^\alpha}{\sum_{\mathbf{t}\in\mathbb{Z}^k}\|\mathbf{\Theta}(\mathbf{t})\|^\alpha}.\tag{3.7}$$

for $\mathbf{i}\in\mathbb{Z}^k$. Define

$$\mathbf{\Theta}^{RS}(\mathbf{t})=\frac{\mathbf{\Theta}(\mathbf{t}+\boldsymbol{\mathcal{I}})}{\|\mathbf{\Theta}(\boldsymbol{\mathcal{I}})\|},\ \mathbf{t}\in\mathbb{Z}^k\,.$$

Then a necessary and sufficient condition for the equality of the laws

$$\mathcal{L}((\mathbf{\Theta}^{RS}(\mathbf{t}):\mathbf{t}\in\mathbb{Z}^k))=\mathcal{L}((\mathbf{\Theta}(\mathbf{t}):\mathbf{t}\in\mathbb{Z}^k))\tag{3.8}$$

is that $(\mathbf{\Theta}(\mathbf{t}):\mathbf{t}\in\mathbb{Z}^k)$ satisfies (3.4) and $\mathbb{P}(\|\mathbf{\Theta}(\mathbf{0})\|=1)=1$.

Proof. Suppose first (3.8) holds. Since $\|\boldsymbol{\Theta}^{\mathrm{RS}}(\mathbf{0})\| = 1$, we must have $\mathbb{P}(\|\boldsymbol{\Theta}(\mathbf{0})\| = 1) = 1$. Now, let $g : (\mathbb{R}^d)^{\mathbb{Z}^k} \to \mathbb{R}$ be a bounded measurable function. Denoting $\|\boldsymbol{\Theta}\|_\alpha = \left(\sum_{\mathbf{t}\in\mathbb{Z}^k} \|\boldsymbol{\Theta}(\mathbf{t})\|^\alpha\right)^{1/\alpha}$, we have

$$
\begin{aligned}
&\mathbb{E}[g(\boldsymbol{\Theta}(\cdot - \mathbf{s}))\mathbb{1}(\boldsymbol{\Theta}(-\mathbf{s}) \neq \mathbf{0})] = \mathbb{E}\left[g(\boldsymbol{\Theta}^{\mathrm{RS}}(\cdot - \mathbf{s}))\mathbb{1}(\boldsymbol{\Theta}^{\mathrm{RS}}(-\mathbf{s}) \neq \mathbf{0})\right] \\
=&\mathbb{E}\left[\mathbb{E}\left[g\left(\frac{\boldsymbol{\Theta}(\cdot - \mathbf{s} + \boldsymbol{\mathcal{I}})}{\|\boldsymbol{\Theta}(\boldsymbol{\mathcal{I}})\|}\right)\mathbb{1}\left(\frac{\boldsymbol{\Theta}(-\mathbf{s} + \boldsymbol{\mathcal{I}})}{\|\boldsymbol{\Theta}(\boldsymbol{\mathcal{I}})\|} \neq \mathbf{0}\right)\middle|(\boldsymbol{\Theta}(\mathbf{t}) : \mathbf{t} \in \mathbb{Z}^k)\right]\right] \\
=&\mathbb{E}\left[\sum_{\mathbf{i}\in\mathbb{Z}^k} \frac{\|\boldsymbol{\Theta}(\mathbf{i})\|^\alpha}{\|\boldsymbol{\Theta}\|_\alpha^\alpha} g\left(\frac{\boldsymbol{\Theta}(\cdot - \mathbf{s} + \mathbf{i})}{\|\boldsymbol{\Theta}(\mathbf{i})\|}\right)\mathbb{1}\left(\boldsymbol{\Theta}(-\mathbf{s} + \mathbf{i}) \neq \mathbf{0}\right)\right].
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\mathbb{E}\left[g\left(\frac{\boldsymbol{\Theta}(\cdot)}{\|\boldsymbol{\Theta}(\mathbf{s})\|}\right)\|\boldsymbol{\Theta}(\mathbf{s})\|^\alpha\right] \\
=&\mathbb{E}\left[g\left(\frac{\boldsymbol{\Theta}^{\mathrm{RS}}(\cdot)}{\|\boldsymbol{\Theta}^{\mathrm{RS}}(\mathbf{s})\|}\right)\|\boldsymbol{\Theta}^{\mathrm{RS}}(\mathbf{s})\|^\alpha\right] \\
=&\mathbb{E}\left[\sum_{\mathbf{j}\in\mathbb{Z}^k} \frac{\|\boldsymbol{\Theta}(\mathbf{j})\|^\alpha}{\|\boldsymbol{\Theta}\|_\alpha^\alpha} g\left(\frac{\boldsymbol{\Theta}(\cdot + \mathbf{j})}{\|\boldsymbol{\Theta}(\mathbf{s} + \mathbf{j})\|}\right)\left\|\frac{\boldsymbol{\Theta}(\mathbf{s} + \mathbf{j})}{\|\boldsymbol{\Theta}(\mathbf{j})\|}\right\|^\alpha \mathbb{1}(\boldsymbol{\Theta}(\mathbf{j}) \neq \mathbf{0})\right] \\
=&\mathbb{E}\left[\sum_{\mathbf{i}\in\mathbb{Z}^k} \frac{\|\boldsymbol{\Theta}(\mathbf{i})\|^\alpha}{\|\boldsymbol{\Theta}\|_\alpha^\alpha} g\left(\frac{\boldsymbol{\Theta}(\cdot - \mathbf{s} + \mathbf{i})}{\|\boldsymbol{\Theta}(\mathbf{i})\|}\right)\mathbb{1}(\boldsymbol{\Theta}(-s + \mathbf{i}) \neq \mathbf{0})\right]
\end{aligned}
$$

by substituting $\mathbf{j} = \mathbf{i} - \mathbf{s}$. The equal results of these two calculations show that the random field has the property (3.4).

In the other direction, suppose that the random field $(\boldsymbol{\Theta}(\mathbf{t}) : \mathbf{t} \in \mathbb{Z}^k)$ satisfies (3.4) and $\mathbb{P}(\|\boldsymbol{\Theta}(\mathbf{0})\| = 1) = 1$. For any bounded measurable function g,

$$
\mathbb{E}[g(\boldsymbol{\Theta}^{\mathrm{RS}}(\mathbf{t}) : \mathbf{t} \in \mathbb{Z}^k)] = \sum_{\mathbf{i}\in\mathbb{Z}^k} \mathbb{E}\left[g\left(\frac{\boldsymbol{\Theta}(\mathbf{t} - \mathbf{i})}{\|\boldsymbol{\Theta}(-\mathbf{i})\|}\right)\frac{\|\boldsymbol{\Theta}(-\mathbf{i})\|^\alpha}{\|\boldsymbol{\Theta}\|_\alpha^\alpha}\mathbb{1}(\boldsymbol{\Theta}(-\mathbf{i}) \neq \mathbf{0})\right].
$$

Define a new function $\bar{g}$ by

$$
\bar{g}\left(\boldsymbol{\theta}(\mathbf{t}) :\, \mathbf{t} \in \mathbb{Z}^k\right) = g\left(\frac{\boldsymbol{\theta}(\mathbf{t})}{\|\boldsymbol{\theta}(\mathbf{0})\|} : \mathbf{t} \in \mathbb{Z}^k\right)\frac{\|\boldsymbol{\theta}(\mathbf{0})\|^\alpha}{\|\boldsymbol{\theta}\|_\alpha^\alpha}
$$

if $\boldsymbol{\theta}(\mathbf{0}) \neq \mathbf{0}$ and $\|\boldsymbol{\theta}\|_\alpha^\alpha = \sum_{\mathbf{i}\in\mathbb{Z}^k} \|\boldsymbol{\theta}(\mathbf{i})\|^\alpha < \infty$. If these conditions do not hold, set $\bar{g} = 0$. Since $\bar{g}$ is a bounded and measurable function, we have by (3.4) and the fact that $\mathbb{P}(\|\boldsymbol{\Theta}(\mathbf{0})\| = 1) = 1$,

$$
\begin{aligned}
\mathbb{E}[g(\boldsymbol{\Theta}^{\mathrm{RS}}(\mathbf{t}) : \mathbf{t} \in \mathbb{Z}^k)] &= \sum_{\mathbf{i}\in\mathbb{Z}^k} \mathbb{E}\left[\bar{g}(\boldsymbol{\Theta}(\mathbf{t} - \mathbf{i}) : \mathbf{t} \in \mathbb{Z}^k)\mathbb{1}(\boldsymbol{\Theta}(-\mathbf{i}) \neq \mathbf{0})\right] \\
&= \sum_{\mathbf{i}\in\mathbb{Z}^k} \mathbb{E}\left[\bar{g}\left(\frac{\boldsymbol{\Theta}(\mathbf{t})}{\|\boldsymbol{\Theta}(\mathbf{i})\|} : \mathbf{t} \in \mathbb{Z}^k\right)\|\boldsymbol{\Theta}(\mathbf{i})\|^\alpha\right] \\
&= \sum_{\mathbf{i}\in\mathbb{Z}^k} \mathbb{E}\left[g(\boldsymbol{\Theta}(\mathbf{t}) : \mathbf{t} \in \mathbb{Z}^k)\frac{\|\boldsymbol{\Theta}(\mathbf{i})\|^\alpha}{\|\boldsymbol{\Theta}\|_\alpha^\alpha}\right] \\
&=\mathbb{E}\left[g(\boldsymbol{\Theta}(\mathbf{t}) : \mathbf{t} \in \mathbb{Z}^k)\right],
\end{aligned}
$$

proving (3.8). $\square$

4. Extremal index of a random field

The extremal index is one of the major ways to characterize how the extremes of a stationary sequence cluster; it was introduced in Leadbetter (1983) and extensively studied and used ever since. The corresponding notion for random fields appeared in Ferreira and Pereira (2008). One of the attractive features of the notion of the extremal index is that it admits multiple interpretations. These different points of view on extremal index, however, turn out to be equivalent only under appropriate technical conditions (and the equivalences turn out to be even more strained for random fields). In fact, the original definition of the extremal index itself includes an assumption of its existence. For jointly regularly varying random fields, the tail field sheds new light on the notion of the extremal index. Importantly, no assumptions of existence are required for the tail field-based notions of the extremal index (apart, of course, from the regular variation). In order to clarify the situation, we keep the definitions distinct.

Definition 4.1. An $\mathbb{R}^d$-valued stationary random field $(\mathbf{X}(\mathbf{t}) : \mathbf{t} \in \mathbb{Z}^k)$ has a **classical extremal index** $\theta_{\rm cl}$ if for each $\tau > 0$ and any array $(u_{\mathbf{n}}(\tau))$ satisfying

$$\left(\prod_{\ell=1}^{k} n_\ell\right) \mathbb{P}(\|\mathbf{X}(\mathbf{0})\| > u_{\mathbf{n}}(\tau)) \to \tau \tag{4.1}$$

as $\mathbf{n} \to \boldsymbol{\infty}$, it also holds that

$$\mathbb{P}\left(M_X(\mathcal{R}^+_{\mathbf{n}}) \le u_{\mathbf{n}}(\tau)\right) \to e^{-\theta_{\rm cl}\tau}. \tag{4.2}$$

Remark 4.2. It is common to formulate the definition of the classical extremal index by requiring that (4.1) and (4.2) hold for some array $(u_{\mathbf{n}}(\tau))$. This appears to tie the notion to a particular choice of the array, and does not seem to broaden the applicability of the definition.

Definition 4.3. An $\mathbb{R}^d$-valued stationary random field $(\mathbf{X}(\mathbf{t}) : \mathbf{t} \in \mathbb{Z}^k)$ has a **block extremal index** $\theta_{\rm b}$ if for some array $(\mathbf{r}_{\mathbf{n}})$ increasing to $\boldsymbol{\infty}$ such that $\mathbf{r}_{\mathbf{n}}/\mathbf{n} \to \mathbf{0}$, for each $\tau > 0$ and any array $(u_{\mathbf{n}}(\tau))$ satisfying (4.1), it holds that

$$\theta_{\rm b} = \lim_{\mathbf{n}\to\boldsymbol{\infty}} \frac{\mathbb{P}(M_X(\mathcal{R}^+_{\mathbf{r}_{\mathbf{n}}}) > u_{\mathbf{n}}(\tau))}{\left(\prod_{\ell=1}^{k} r_{n_\ell}\right) \mathbb{P}(\|\mathbf{X}(\mathbf{0})\| > u_{\mathbf{n}}(\tau))} \tag{4.3}$$

Under certain conditions the block extremal index coincides with the classical extremal index, assuming the latter exists. One such set of conditions is the so called coordinatewise tail mixing condition; see Proposition 3.2 in Pereira et al. (2017).

The next definition of the extremal index is well known in the case of the one-dimensional time, but does not seem to have been formulated for random fields. It concentrates on the conditional probability of the random field being free of exceedances over the rest of a hypercube given an exceedance at one of the corners of the hypercube.

Definition 4.4. An $\mathbb{R}^d$-valued stationary random field $(\mathbf{X}(\mathbf{t}) : \mathbf{t} \in \mathbb{Z}^k)$ has a **run extremal index** $\theta_{{\rm run},\mathbf{i}}$ with respect to $\mathbf{i} \in \{0,1\}^k$ if for some array $(\mathbf{r}_{\mathbf{n}})$ increasing to $\boldsymbol{\infty}$ such that $\mathbf{r}_{\mathbf{n}}/\mathbf{n} \to \mathbf{0}$, and any array $(u_{\mathbf{n}}(\tau))$ satisfying (4.1) for some $\tau > 0$, it holds that

$$\theta_{{\rm run},\mathbf{i}} = \lim_{\mathbf{n}\to\boldsymbol{\infty}} \mathbb{P}(M_X(\mathcal{R}^+_{\mathbf{r}_{\mathbf{n}}} \backslash \{\mathbf{t}_{\mathbf{n},\mathbf{i}}\}) \le u_{\mathbf{n}}(\tau) \mid X(\mathbf{t}_{\mathbf{n},\mathbf{i}}) > u_{\mathbf{n}}(\tau))\,, \tag{4.4}$$

where $(t_{\mathbf{n},\mathbf{i}})_l = r_{n_l} - 1$ if $i_{n_l} = 1$ and 0 if $i_{n_l} = 0$.

When the time is one-dimensional, the hypercube has two corners, and the stationarity implies that the run extremal index, if it exists, is the same for the two corners. Indeed,

$$\begin{aligned}
&\mathbb{P}\left(\max_{t=1,\dots,r_n-1} X(t) \le u_n(\tau) \mid X(0) > u_n(\tau)\right)\\
=&\frac{\mathbb{P}\left(\max_{t=1,\dots,r_n-1} X(t) \le u_n(\tau), X(0) > u_n(\tau)\right)}{\mathbb{P}(X(0) > u_n(\tau))}\\
=&\frac{\mathbb{P}\left(\max_{t=0,\dots,r_n-2} X(t) \le u_n(\tau)\right) - \mathbb{P}\left(\max_{t=0,\dots,r_n-1} X(t) \le u_n(\tau)\right)}{\mathbb{P}(X(r_n-1) > u_n(\tau))}\\
=&\frac{\mathbb{P}\left(\max_{t=0,\dots,r_n-2} X(t) \le u_n(\tau), X(r_n-1) > u_n(\tau)\right)}{\mathbb{P}(X(r_n-1) > u_n(\tau))}\\
=&\mathbb{P}\left(\max_{t=0,\dots,r_n-2} X(t) \le u_n(\tau) \,\middle|\, X(r_n-1) > u_n(\tau)\right).
\end{aligned}$$

This, however, is no longer necessarily the case that for random fields the run extremal index is independent of the corner of the hypercube used to define it, as will be seen in Example 4.8 below. When the time is one-dimensional, under certain conditions the run extremal index coincides with the classical extremal index; one such set of conditions being the AIM conditions of O'Brien (1987). As the previous discussion and Example 4.8 indicate, this is no longer the case for random fields.

Definition 4.5. Let $(\mathbf{X}(\mathbf{t}) : \mathbf{t} \in \mathbb{Z}^k)$ be a stationary jointly regularly varying $\mathbb{R}^d$-valued random field $(\mathbf{X}(\mathbf{t}) : \mathbf{t} \in \mathbb{Z}^k)$ with the tail field $(\mathbf{Y}(\mathbf{t}) : \mathbf{t} \in \mathbb{Z}^k)$. Its **tail field extremal index** $\theta_{\mathrm{tf},\mathbf{i}}$ with respect to $\mathbf{i} \in \{0,1\}^k$ is

$$\theta_{\mathrm{tf},\mathbf{i}} = \mathbb{P}\left(\sup_{\mathbf{t}:\,\mathbf{t}(\mathbf{1}-2\mathbf{i})\ge\mathbf{0},\,\mathbf{t}\ne\mathbf{0}} \|\mathbf{Y}(\mathbf{t})\| \le 1\right).$$

Under appropriate conditions, similar to those of Basrak and Segers (2009) in the one-dimensional time case, the tail field extremal index coincides with the run extremal index and, in particular, the latter exists.

Proposition 4.6. *Let $(\mathbf{X}(\mathbf{t}) : \mathbf{t} \in \mathbb{Z}^k)$ be a stationary jointly regularly varying random field with the tail field $(\mathbf{Y}(\mathbf{t}) : \mathbf{t} \in \mathbb{Z}^k)$. Let $\mathbf{i} \in \{0,1\}^k$. Suppose that for any array $(u_{\mathbf{n}}(\tau))$ satisfying (4.1) for some $\tau > 0$ and some array $(\mathbf{r_n})$ increasing to $\boldsymbol{\infty}$ such that $\mathbf{r_n}/\mathbf{n} \to \mathbf{0}$,*

$$\lim_{M\to\infty}\limsup_{\mathbf{n}\to\boldsymbol{\infty}} \mathbb{P}\left(M_X(\mathcal{R}^+_{A_{M,\mathbf{i}}}\setminus\{t_{\mathbf{n},\mathbf{i}}\}) \le u_{\mathbf{n}}(\tau),\ M_X(\mathcal{R}^+_{\mathbf{r_n}}\setminus\mathcal{R}^+_{A_{M,\mathbf{i}}}) > u_{\mathbf{n}}(\tau) \,\middle|\, \|\mathbf{X}(\mathbf{0})\| > u_{\mathbf{n}}(\tau)\right) = 0\,,$$

where $A_{M,\mathbf{i}} = \{\mathbf{x} \in \mathcal{R}^+_{\mathbf{r_n}} : |x_l - (t_{\mathbf{n},\mathbf{i}})_l| \le M,\ l = 1,\dots,d\}$. Then the run extremal index $\theta_{\mathrm{run},\mathbf{i}}$ exists and is equal to the tail field extremal index $\theta_{\mathrm{tf},\mathbf{i}}$.

Proof. It is enough to consider the case $\mathbf{i} = \mathbf{0}$, in which case the condition in the proposition reduces to

$$\lim_{M\to\infty}\limsup_{\mathbf{n}\to\boldsymbol{\infty}} \mathbb{P}\left(M_X(\mathcal{R}^+_{M\mathbf{1}}\setminus\{\mathbf{0}\}) \le u_{\mathbf{n}}(\tau),\ M_X(\mathcal{R}^+_{\mathbf{r_n}}\setminus\mathcal{R}^+_{M\mathbf{1}}) > u_{\mathbf{n}}(\tau) \,\middle|\, \|\mathbf{X}(\mathbf{0})\| > u_{\mathbf{n}}(\tau)\right) = 0\,. \tag{4.5}$$

We have for any $M = 1, 2, \dots$, any $(\mathbf{r_n})$ increasing to $\boldsymbol{\infty}$,

$$\mathbb{P}\left(\sup_{\mathbf{0}\le\mathbf{t}\le M\mathbf{1},\,\mathbf{t}\ne\mathbf{0}} \|\mathbf{Y}(\mathbf{t})\| \le 1\right)$$

$$= \lim_{\mathbf{n}\to\infty} \mathbb{P}\left(\sup_{\mathbf{0}\le \mathbf{t}\le M\mathbf{1},\, \mathbf{t}\neq\mathbf{0}} \frac{1}{u_{\mathbf{n}}(\tau)} \|\mathbf{X}(\mathbf{t})\| \le 1 \,\middle|\, \|\mathbf{X}(\mathbf{0})\| > u_{\mathbf{n}}(\tau) \right)$$
$$\ge \limsup_{\mathbf{n}\to\infty} \mathbb{P}(M_X(\mathcal{R}^+_{\mathbf{r_n}} \backslash \{\mathbf{0}\}) \le u_{\mathbf{n}}(\tau) \mid \|\mathbf{X}(\mathbf{0})\| > u_{\mathbf{n}}(\tau)) .$$

Letting $M \to \infty$, we obtain

$$\mathbb{P}\left(\sup_{\mathbf{t}\ge\mathbf{0},\, \mathbf{t}\neq\mathbf{0}} \|\mathbf{Y}(\mathbf{t})\| \le 1 \right) \ge \limsup_{\mathbf{n}\to\infty} \mathbb{P}(M_X(\mathcal{R}^+_{\mathbf{r_n}} \backslash \{\mathbf{0}\}) \le u_{\mathbf{n}}(\tau) \mid \|\mathbf{X}(\mathbf{0})\| > u_{\mathbf{n}}(\tau)) . \tag{4.6}$$

Furthermore, we can write for $\mathbf{n}$ large enough,

$$\begin{aligned} &\mathbb{P}(M_X(\mathcal{R}^+_{\mathbf{r_n}} \backslash \{\mathbf{0}\}) \le u_{\mathbf{n}}(\tau) \mid \|\mathbf{X}(\mathbf{0})\| > u_{\mathbf{n}}(\tau)) \\ =&\mathbb{P}\left(\frac{1}{u_{\mathbf{n}}(\tau)} M_X(\mathcal{R}^+_{M\mathbf{1}} \backslash \{\mathbf{0}\}) \le 1 \,\middle|\, \|\mathbf{X}(\mathbf{0})\| > u_{\mathbf{n}}(\tau) \right) \\ -&\mathbb{P}\left(M_X(\mathcal{R}^+_{M\mathbf{1}} \backslash \{\mathbf{0}\}) \le u_{\mathbf{n}}(\tau),\, M_X(\mathcal{R}^+_{\mathbf{r_n}} \backslash \{\mathbf{0}\}) > u_{\mathbf{n}}(\tau) \,\middle|\, \|\mathbf{X}(\mathbf{0})\| > u_{\mathbf{n}}(\tau) \right) . \end{aligned}$$

By (4.5), letting first $\mathbf{n} \to \infty$ and then $M \to \infty$ gives us

$$\mathbb{P}\left(\sup_{\mathbf{t}\ge\mathbf{0},\, \mathbf{t}\neq\mathbf{0}} \|\mathbf{Y}(\mathbf{t})\| \le 1 \right) \le \liminf_{\mathbf{n}\to\infty} \mathbb{P}(M_X(\mathcal{R}^+_{\mathbf{r_n}} \backslash \{\mathbf{0}\}) \le u_{\mathbf{n}}(\tau) \mid \|\mathbf{X}(\mathbf{0})\| > u_{\mathbf{n}}(\tau)) ,$$

which, in conjunction with (4.6), proves both existence of $\theta_{\text{run},\mathbf{0}}$ and the fact that it is equal to $\theta_{\text{tf},\mathbf{0}}$. □

Another version of a tail field based extremal index arises naturally in limit theorems discussed in the next section. Let $\prec$ be an invariant order on $\mathbb{Z}^k$.

Definition 4.7. Let $(\mathbf{X}(\mathbf{t}) : \mathbf{t} \in \mathbb{Z}^k)$ be a stationary jointly regularly varying $\mathbb{R}^d$-valued random field $(\mathbf{X}(\mathbf{t}) : \mathbf{t} \in \mathbb{Z}^k)$ with the tail field $(\mathbf{Y}(\mathbf{t}) : \mathbf{t} \in \mathbb{Z}^k)$. Its **half space extremal index** θ_{half} is

$$\theta_{\text{half}} = \mathbb{P}\left(\sup_{\mathbf{t}\prec\mathbf{0}} \|\mathbf{Y}(\mathbf{t})\| \le 1 \right) .$$

We will see in the next section that, under condition (4.5), the block extremal index exists and equals the half space extremal index. A corollary of this is that the half space extremal index is independent of the invariant order $\prec$ as long as (4.5) holds for some array $(\mathbf{r_n})$.

Example 4.8. A simple class of models is that of max-moving averages with local interaction. We consider one such model with two-dimensional time. Let $a_{-1,-1}, a_{-1,1}, a_{1,1}, a_{1,-1}$ be numbers in $[0, 1]$. Starting with i.i.d. standard Fréchet(1) random variables $(Z(\mathbf{t}) : \mathbf{t} \in \mathbb{Z}^2)$, we define a stationary random field $(X(\mathbf{t}) : \mathbf{t} \in \mathbb{Z}^2)$ by

$$X(\mathbf{t}) = \max\Big\{ Z(\mathbf{t}), a_{-1,-1}Z(\mathbf{t}-\mathbf{1}), a_{-1,1}Z(t_1-1, t_2+1), a_{1,1}Z(\mathbf{t}+\mathbf{1}), a_{1,-1}Z(t_1+1, t_2-1) \Big\} .$$

If F_Z denotes the c.d.f. of a standard Fréchet(1) random variable, then for any $u > 0$,

$$\mathbb{P}(M_X(\mathcal{R}^+_{\mathbf{r_n}}) \le u) = (F_Z(u))^{E(\mathbf{r},\mathbf{a})} ,$$

where

$$\begin{aligned} E(\mathbf{r},\mathbf{a}) = {} & r_{n_1} r_{n_2} + 3(a_{-1,-1} + a_{-1,1} + a_{1,1} + a_{1,-1}) + (r_{n_1} - 2)\big(\max(a_{-1,-1}, a_{1,-1}) + \max(a_{1,1}, a_{-1,1})\big) \\ & + (r_{n_2} - 2)\big(\max(a_{-1,-1}, a_{-1,1}) + \max(a_{1,1}, a_{1,-1})\big) , \end{aligned}$$

while

$$\mathbb{P}(X(\mathbf{0}) \le u) = (F_Z(u))^{1+(a_{-1,-1}+a_{-1,1}+a_{1,1}+a_{1,-1})}.$$

By (4.1) and (4.2) we conclude that the classical extremal index exists, and

$$\theta_{\rm cl} = (1+s)^{-1}\,,$$

where

$$s = a_{-1,-1} + a_{-1,1} + a_{1,1} + a_{1,-1}\,,$$

and by (4.3), the block extremal index $\theta_{\rm b}$ also exists and is equal to the classical extremal index.

It is also easy to compute the run extremal index. We perform the computation for the corner determined by $\mathbf{i} = \mathbf{0}$, and it can be done analogously for the other corners. Notice that

$$\begin{aligned}
&\lim_{\mathbf{n}\to\infty} \mathbb{P}(M_X(\mathcal{R}^+_{\mathbf{r_n}}\backslash\{\mathbf{0}\}) > u_{\mathbf{n}}(\tau) \mid X(\mathbf{0}) > u_{\mathbf{n}}(\tau)) \\
&= \lim_{\mathbf{n}\to\infty} \mathbb{P}(M_X(\{\mathbf{1},(2,0),(0,2),(2,2)) > u_{\mathbf{n}}(\tau) \mid X(\mathbf{0}) > u_{\mathbf{n}}(\tau)) \\
&= \lim_{\mathbf{n}\to\infty} \Big[\mathbb{P}(Z(\mathbf{0}) > u_{\mathbf{n}}(\tau) \mid X(\mathbf{0}) > u_{\mathbf{n}}(\tau))\mathbb{P}(X(\mathbf{1}) > u_{\mathbf{n}}(\tau) \mid Z(\mathbf{0}) > u_{\mathbf{n}}(\tau)\big) \\
&+ \mathbb{P}(a_{-1,1}Z((-1,1)) > u_{\mathbf{n}}(\tau) \mid X(\mathbf{0}) > u_{\mathbf{n}}(\tau))\mathbb{P}(X((0,2)) > u_{\mathbf{n}}(\tau) \mid a_{-1,1}Z((-1,1)) > u_{\mathbf{n}}(\tau)\big) \\
&+ \mathbb{P}(a_{1,1}Z(\mathbf{1}) > u_{\mathbf{n}}(\tau) \mid X(\mathbf{0}) > u_{\mathbf{n}}(\tau))\mathbb{P}(X(\mathbf{1}) > u_{\mathbf{n}}(\tau) \mid a_{1,1}Z(\mathbf{1}) > u_{\mathbf{n}}(\tau)\big) \\
&+ \mathbb{P}(a_{1,-1}Z((1,-1)) > u_{\mathbf{n}}(\tau) \mid X(\mathbf{0}) > u_{\mathbf{n}}(\tau))\mathbb{P}(X((2,0)) > u_{\mathbf{n}}(\tau) \mid a_{1,-1}Z((1,-1)) > u_{\mathbf{n}}(\tau)\big)\Big] \\
&= \frac{1}{1+s}\lim_{\mathbf{n}\to\infty} \mathbb{P}(a_{-1,-1}Z(\mathbf{0}) > u_{\mathbf{n}}(\tau) \mid Z(\mathbf{0}) > u_{\mathbf{n}}(\tau)\big) \\
&+ \frac{a_{-1,1}}{1+s}\lim_{\mathbf{n}\to\infty} \mathbb{P}(a_{-1,-1}Z((-1,1)) > u_{\mathbf{n}}(\tau) \mid a_{-1,1}Z((-1,1)) > u_{\mathbf{n}}(\tau)\big) + \frac{a_{1,1}}{1+s} \\
&+ \frac{a_{1,-1}}{1+s}\lim_{\mathbf{n}\to\infty} \mathbb{P}(a_{-1,-1}Z((1,-1)) > u_{\mathbf{n}}(\tau) \mid a_{1,-1}Z((1,-1)) > u_{\mathbf{n}}(\tau)\big) \\
&= (1+s)^{-1}\Big[a_{-1,-1} + \min(a_{-1,1}, a_{-1,-1}) + a_{1,1} + \min(a_{1,-1}, a_{-1,-1})\Big]\,,
\end{aligned}$$

which equals, by definition, to $1 - \theta_{{\rm run},\mathbf{0}}$.

Choosing $a_{-1,-1} = .1, a_{-1,1} = .7, a_{1,1} = .6, a_{1,-1} = .1$ results in $\theta_{{\rm run},\mathbf{0}} = .64, \theta_{{\rm run},\mathbf{1}} = .44, \theta_{{\rm run},(0,1)} = .4, \theta_{{\rm run},(1,0)} = .6$, so the run extremal index is different at all 4 corners. In this case also $\theta_{\rm cl} = .4$. However, taking the equal weight mixture of the above model with the model corresponding to $a_{-1,-1} = .6, a_{-1,1} = .2, a_{1,1} = .6, a_{1,-1} = .1$ results in all 5 different indices: $\theta_{\mathbf{0}} = .52, \theta_{\mathbf{1}} = .42, \theta_{(0,1)} = .56, \theta_{(1,0)} = .7$ and $\theta_{\rm cl} = .4$.

Finally, because of the local interaction, condition (4.5) holds in this case for any $(\mathbf{r_n})$ increasing to $\boldsymbol{\infty}$ such that $\mathbf{r_n}/\mathbf{n} \to \mathbf{0}$, and so by Proposition 4.6, the tail field extremal indices coincide with the run extremal indices.

5. Extremal index and limit theorems for point processes

Armed with the understanding of the spatial extremal indices developed in the previous section, we now proceed to study the extremal clusters.

Let $(\mathbf{X}(\mathbf{t}) : \mathbf{t} \in \mathbb{Z}^k)$ be an $\mathbb{R}^d$-valued stationary random field, jointly regularly varying with index $\alpha > 0$, and let $(\mathbf{Y}(\mathbf{t}) : \mathbf{t} \in \mathbb{Z}^k)$, $(\boldsymbol{\Theta}(\mathbf{t}) : \mathbf{t} \in \mathbb{Z}^k)$ be its associated tail field and spectral field, respectively. Let, once again, $(\mathbf{r_n})$ and $(u_\mathbf{n}(\tau))$ be arrays such that $\mathbf{r_n}/\mathbf{n} \to \mathbf{0}$, and (4.1) holds for $\tau > 0$. Consider the spatial point process (on $(\overline{\mathbb{R}})^d$, from which we remove the origin) defined by

$$C_\mathbf{n} = \sum_{\mathbf{t} \in \mathcal{R}^+_{\mathbf{r_n}}} \delta_{u_\mathbf{n}(\tau)^{-1}\mathbf{X}(\mathbf{t})}. \tag{5.1}$$

We call it the cluster process, and we are interested in the weak limit of the conditional law of $C_\mathbf{n}$, given that it does not vanish, i.e. given the event that $M_X(\mathcal{R}^+_{\mathbf{r_n}}) > u_\mathbf{n}(\tau)$. We view the weak limit of the cluster process as describing, asymptotically, a single extreme cluster of the random field. Theorem 5.2 describes the latter under the following assumption, which implies, at once, the condition of Proposition 4.6 for every corner of the hypercube.

Condition 5.1. For any array $(u_\mathbf{n}(\tau))$ satisfying (4.1) for some $\tau > 0$ and some array $(\mathbf{r_n})$ increasing to $\boldsymbol{\infty}$ such that $\mathbf{r_n}/\mathbf{n} \to \mathbf{0}$,

$$\lim_{M\to\boldsymbol{\infty}} \limsup_{\mathbf{n}\to\boldsymbol{\infty}} \mathbb{P}\left(M_X(\mathcal{R}_{\mathbf{r_n}} \backslash \mathcal{R}_{M\mathbf{1}}) > u_\mathbf{n}(\tau) \mid \|\mathbf{X}(\mathbf{0})\| > u_\mathbf{n}(\tau)\right) = 0. \tag{5.2}$$

Let $\prec$ be an arbitrary invariant order on $\mathbb{Z}^k$. The argument in the following theorem follows a logic similar to that in Theorem 4.3 of Basrak and Segers (2009).

Theorem 5.2. *Let $(\mathbf{X}(\mathbf{t}) : \mathbf{t} \in \mathbb{Z}^k)$ be a jointly regularly varying with index $\alpha > 0$, $\mathbb{R}^d$-valued stationary random field, satisfying Condition 5.1. Then $\mathbb{P}(\lim_{\|\mathbf{t}\|_\infty\to\infty} \|\mathbf{Y}(\mathbf{t})\| = 0) = 1$. Moreover, the block extremal index θ_b exists, is positive, and*

$$\theta_\mathrm{b} = \theta_\mathrm{half} = \mathbb{E}\left[\max_{\mathbf{t}\succeq\mathbf{0}} \|\boldsymbol{\Theta}(\mathbf{t})\|^\alpha - \max_{\mathbf{t}\succ\mathbf{0}} \|\boldsymbol{\Theta}(\mathbf{t})\|^\alpha\right]. \tag{5.3}$$

Furthermore, the conditional law of $C_\mathbf{n}$ converges weakly in the space of Radon measures on $(\overline{\mathbb{R}})^d \setminus \{\mathbf{0}\}$ to the conditional law of the point process

$$C = \sum_{\mathbf{t}\in\mathbb{Z}^k} \delta_{\mathbf{Y}(\mathbf{t})} \tag{5.4}$$

given that $\max_{\mathbf{t}\prec\mathbf{0}} \|\mathbf{Y}(\mathbf{t})\| \le 1$. The Laplace functional of C under this conditional law can be expressed as

$$\begin{aligned}\Psi_C(f) =& \mathbb{E}\left[\exp\left\{-\sum_{\mathbf{t}\in\mathbb{Z}^k} f(\mathbf{Y}(\mathbf{t}))\right\} \middle| \max_{\mathbf{t}\prec\mathbf{0}} \|\mathbf{Y}(\mathbf{t})\| \le 1\right] \\ =& \theta_\mathrm{half}^{-1} \int_0^\infty \mathbb{E}\left[\exp\left\{-\sum_{\mathbf{t}\succeq\mathbf{0}} f(y\boldsymbol{\Theta}(\mathbf{t}))\right\} \mathbb{1}\left(y \max_{\mathbf{t}\succeq\mathbf{0}} \|\boldsymbol{\Theta}(\mathbf{t})\| > 1\right)\right. \\ & \left. - \exp\left\{-\sum_{\mathbf{t}\succ\mathbf{0}} f(y\boldsymbol{\Theta}(\mathbf{t}))\right\} \mathbb{1}\left(y \max_{\mathbf{t}\succ\mathbf{0}} \|\boldsymbol{\Theta}(\mathbf{t})\| > 1\right)\right] d(-y^{-\alpha})\end{aligned} \tag{5.5}$$

for any nonnegative continuous f on $(\overline{\mathbb{R}})^d \setminus \{\mathbf{0}\}$ with a compact support.

Proof. For any $v > 0$, by Condition 5.1 and the regular variation of $\|\mathbf{X}(\mathbf{0})\|$, it holds that

$$\lim_{M\to\infty} \limsup_{\mathbf{n}\to\boldsymbol{\infty}} \mathbb{P}\left(M_X(\mathcal{R}_{\mathbf{r_n}} \backslash \mathcal{R}_{M\mathbf{1}}) > u_\mathbf{n}(\tau)v \mid \|\mathbf{X}(\mathbf{0})\| > u_\mathbf{n}(\tau)\right) = 0.$$

Therefore, for any $\epsilon > 0$ and $v > 0$, there exists $M > 0$ such that for all $K > M$,

$$\mathbb{P}\left(M_Y(\mathcal{R}_{K\mathbf{1}} \backslash \mathcal{R}_{M\mathbf{1}}) > v\right) \le \epsilon.$$

This implies that $\mathbb{P}(\lim_{\|\mathbf{t}\|_\infty\to\infty}\|\mathbf{Y}(\mathbf{t})\|=0)=1$. Next, choose an integer M so large that

$$\limsup_{\mathbf{n}\to\boldsymbol{\infty}} \mathbb{P}\left(M_X(\mathcal{R}_{\mathbf{r_n}}\backslash\mathcal{R}_{M\mathbf{1}}) > u_{\mathbf{n}}(\tau) \mid \|\mathbf{X}(\mathbf{0})\| > u_{\mathbf{n}}(\tau)\right) \le 1/2\,.$$

Let $\gamma_\ell = \lfloor (r_{\mathbf{n}})_\ell/M\rfloor$, $\ell=1,\dots,k$, and fit into the hypercube $\mathcal{R}^+_{\mathbf{r_n}}$ the $\prod_{\ell=1}^k\gamma_\ell$ smaller hypercubes with M points on each side. We decompose the event that a value exceeding $u_{\mathbf{n}}(\tau)$ is attained at one of the points of the resulting grid according to the last point of the grid (in the lexicographic order) at which a value exceeding $u_{\mathbf{n}}(\tau)$ is attained. For a point $M\mathbf{p}$ on this grid, let $A_{M\mathbf{p}}$ denote the set of the points of the grid larger than $M\mathbf{p}$. By stationarity,

$$\begin{aligned}
&\mathbb{P}(M_X(\mathcal{R}^+_{\mathbf{r_n}}) > u_{\mathbf{n}}(\tau))\\
&\ge \sum_{p_1=0}^{\gamma_1-1}\cdots\sum_{p_k=0}^{\gamma_k-1}\mathbb{P}\left(\|\mathbf{X}(M\mathbf{p})\|>u_{\mathbf{n}}(\tau), M_X(A_{M\mathbf{p}})\le u_{\mathbf{n}}(\tau)\right)\\
&= \sum_{p_1=0}^{\gamma_1-1}\cdots\sum_{p_k=0}^{\gamma_k-1}\left[\mathbb{P}\left(\|\mathbf{X}(M\mathbf{p})\|>u_{\mathbf{n}}(\tau)\right) - \mathbb{P}\left(\|\mathbf{X}(M\mathbf{p})\|>u_{\mathbf{n}}(\tau), M_X(A_{M\mathbf{p}})> u_{\mathbf{n}}(\tau)\right)\right]\\
&\ge \left(\prod_{\ell=1}^k\gamma_\ell\right)\left[\mathbb{P}(\|\mathbf{X}(\mathbf{0})\|>u_{\mathbf{n}}(\tau)) - \mathbb{P}\left(\|\mathbf{X}(\mathbf{0})\|>u_{\mathbf{n}}(\tau), M_X(\mathcal{R}_{\mathbf{r_n}}\backslash\mathcal{R}_{M\mathbf{1}})> u_{\mathbf{n}}(\tau)\right)\right]\,,
\end{aligned}$$

so

$$\liminf_{n\to\infty}\frac{\mathbb{P}(M_X(\mathcal{R}^+_{\mathbf{r_n}})>u_{\mathbf{n}}(\tau))}{\left(\prod_{\ell=1}^k r_{n_\ell}\right)\mathbb{P}(\|\mathbf{X}(\mathbf{0})\|>u_{\mathbf{n}}(\tau))} \ge 2^{-1}M^{-k} > 0,. \tag{5.6}$$

Next, we decompose the event $M_X(\mathcal{R}^+_{\mathbf{r_n}})>u_{\mathbf{n}}(\tau)$ using the order $\prec$, we have

$$\begin{aligned}
&\mathbb{E}\left[\exp\left\{-\sum_{\mathbf{i}\in\mathcal{R}^+_{\mathbf{r_n}}} f(u_{\mathbf{n}}(\tau)^{-1}\mathbf{X}(\mathbf{i}))\right\}\mathbb{1}(M_X(\mathcal{R}^+_{\mathbf{r_n}})>u_{\mathbf{n}}(\tau))\right]\\
&=\sum_{\mathbf{t}\in\mathcal{R}^+_{\mathbf{r_n}}}\mathbb{E}\left[\exp\left\{-\sum_{\mathbf{i}\in\mathcal{R}^+_{\mathbf{r_n}}} f(u_{\mathbf{n}}(\tau)^{-1}\mathbf{X}(\mathbf{i}))\right\}\mathbb{1}\left(\sup_{\mathbf{s}\prec\mathbf{t},\mathbf{s}\in\mathcal{R}^+_{\mathbf{r_n}}}\|\mathbf{X}(\mathbf{s})\|\le u_{\mathbf{n}}(\tau)<\|\mathbf{X}(\mathbf{t})\|\right)\right]\,,
\end{aligned}$$

with the convention that the supremum over the empty set is defined to be equal to zero. Denote

$$\theta_{\mathbf{n}} = \frac{\mathbb{P}(M_X(\mathcal{R}^+_{\mathbf{r_n}})>u_{\mathbf{n}}(\tau))}{\left(\prod_{\ell=1}^k r_{n_\ell}\right)\mathbb{P}(\|\mathbf{X}(\mathbf{0})\|>u_{\mathbf{n}}(\tau))}\,,$$

so that, if the array $(\theta_{\mathbf{n}})$ has a limit as $\mathbf{n}\to\boldsymbol{\infty}$, the limit is the block extremal index. It follows from (5.6) that every subsequential limit of this array is strictly positive. Fix $\mathbf{m}\in\mathbb{N}^k$, and choose $\mathbf{n}$ large enough so that $\mathbf{r_n}\ge 2\mathbf{m}-\mathbf{1}$. Then

$$\begin{aligned}
&\Bigg|\mathbb{E}\left[\exp\left\{-\sum_{\mathbf{i}\in\mathcal{R}^+_{\mathbf{r_n}}} f(u_{\mathbf{n}}(\tau)^{-1}\mathbf{X}(\mathbf{i}))\right\}\,\middle|\, M_X(\mathcal{R}^+_{\mathbf{r_n}})>u_{\mathbf{n}}(\tau)\right]\\
&-\theta_{\mathbf{n}}^{-1}\,\mathbb{E}\left[\exp\left\{-\sum_{\mathbf{i}\in\mathcal{R}_{\mathbf{m}}} f(u_{\mathbf{n}}(\tau)^{-1}\mathbf{X}(\mathbf{i}))\right\}\mathbb{1}\left(\max_{\mathbf{t}\prec\mathbf{0},\mathbf{t}\in\mathcal{R}_{\mathbf{m}}}\|\mathbf{X}(\mathbf{t})\|\le u_{\mathbf{n}}(\tau)\right)\,\middle|\,\|\mathbf{X}(\mathbf{0})\|>u_{\mathbf{n}}(\tau)\right]\Bigg|
\end{aligned}$$

(5.7)

$$
\begin{aligned}
\le & \frac{1}{\mathbb{P}(M_X(\mathcal{R}^+_{\mathbf{r_n}}) > u_{\mathbf{n}}(\tau))} \\
& \sum_{\mathbf{t}\in\mathcal{R}^+_{\mathbf{r_n}}} \left| \mathbb{E}\left[\exp\left\{ -\sum_{\mathbf{i}\in\mathcal{R}^+_{\mathbf{r_n}}} f(u_{\mathbf{n}}(\tau)^{-1}\mathbf{X}(\mathbf{i})) \right\} \mathbb{1}\left(\max_{\mathbf{s}\preceq\mathbf{t},\mathbf{s}\in\mathcal{R}^+_{\mathbf{r_n}}} \|\mathbf{X}(\mathbf{s})\| \le u_{\mathbf{n}}(\tau) < \|\mathbf{X}(\mathbf{t})\| \right) \right] \right. \\
& \left. - \mathbb{E}\left[\exp\left\{ -\sum_{\mathbf{i}\in\mathcal{R}_{\mathbf{m}}} f(u_{\mathbf{n}}(\tau)^{-1}\mathbf{X}(\mathbf{i})) \right\} \mathbb{1}\left(\max_{\mathbf{s}\prec\mathbf{0},\mathbf{s}\in\mathcal{R}_{\mathbf{m}}} \|\mathbf{X}(\mathbf{s})\| \le u_{\mathbf{n}}(\tau) < \|\mathbf{X}(\mathbf{0})\| \right) \right] \right| .
\end{aligned}
$$

Let $\mathcal{I} = [\mathbf{m}-\mathbf{1} : \mathbf{r_n}-\mathbf{m}]$. By stationarity and invariance of the order,

$$
\begin{aligned}
& \sum_{\mathbf{t}\in\mathcal{I}} \left| \mathbb{E}\left[\exp\left\{ -\sum_{\mathbf{i}\in\mathcal{R}^+_{\mathbf{r_n}}} f(u_{\mathbf{n}}(\tau)^{-1}\mathbf{X}(\mathbf{i})) \right\} \mathbb{1}\left(\max_{\mathbf{s}\preceq\mathbf{t},\mathbf{s}\in\mathcal{R}^+_{\mathbf{r_n}}} \|\mathbf{X}(\mathbf{s})\| \le u_{\mathbf{n}}(\tau) < \|\mathbf{X}(\mathbf{t})\| \right) \right] \right. \\
& \left. - \mathbb{E}\left[\exp\left\{ -\sum_{\mathbf{i}\in\mathcal{R}_{\mathbf{m}}} f(u_{\mathbf{n}}(\tau)^{-1}\mathbf{X}(\mathbf{i})) \right\} \mathbb{1}\left(\max_{\mathbf{s}\prec\mathbf{0},\mathbf{s}\in\mathcal{R}_{\mathbf{m}}} \|\mathbf{X}(\mathbf{s})\| \le u_{\mathbf{n}}(\tau) < \|\mathbf{X}(\mathbf{0})\| \right) \right] \right| \\
= & \sum_{\mathbf{t}\in\mathcal{I}} \left| \mathbb{E}\left(\left[\exp\left\{ -\sum_{\mathbf{i}\in\mathcal{R}^+_{\mathbf{r_n}}} f(u_{\mathbf{n}}(\tau)^{-1}\mathbf{X}(\mathbf{i})) \right\} \mathbb{1}\left(\max_{\mathbf{s}\preceq\mathbf{t},\mathbf{s}\in\mathcal{R}^+_{\mathbf{r_n}}} \|\mathbf{X}(\mathbf{s})\| \le u_{\mathbf{n}}(\tau) < \|\mathbf{X}(\mathbf{t})\| \right) \right] \right. \right. \\
& \left. \left. - \left[\exp\left\{ -\sum_{\mathbf{i}\in\mathcal{R}_{\mathbf{m}}(\mathbf{t})} f(u_{\mathbf{n}}(\tau)^{-1}\mathbf{X}(\mathbf{i})) \right\} \mathbb{1}\left(\max_{\mathbf{s}\prec\mathbf{t},\mathbf{s}\in\mathcal{R}_{\mathbf{m}}(\mathbf{t})} \|\mathbf{X}(\mathbf{s})\| \le u_{\mathbf{n}}(\tau) < \|\mathbf{X}(\mathbf{t})\| \right) \right] \right| \right) .
\end{aligned}
$$

Since f vanishes in a neighbourhood of the origin, there is $0 < v \le 1$ such that $f(\mathbf{x}) = 0$ when $\|\mathbf{x}\| \le v$. Therefore, for each fixed $\mathbf{t} \in \mathcal{I}$, the difference in the sum above will be nonzero only if $u_{\mathbf{n}}(\tau)^{-1}\|\mathbf{X}(\mathbf{s})\| > v$ for some $\mathbf{s} \in (\mathcal{R}^+_{\mathbf{r_n}} \backslash \mathcal{R}_{\mathbf{m}}(\mathbf{t}))$. By stationarity, this sum is upper bounded by

$$
\begin{aligned}
& \sum_{\mathbf{t}\in\mathcal{I}} \mathbb{P}\left(M_X(\mathcal{R}^+_{\mathbf{r_n}} \backslash \mathcal{R}_{\mathbf{m}}(\mathbf{t})) > u_{\mathbf{n}}(\tau)v, \|\mathbf{X}(\mathbf{t})\| > u_{\mathbf{n}}(\tau) \right) \\
\le & \left(\prod_{\ell=1}^k r_{n_\ell} \right) \mathbb{P}\left(M_X(\mathcal{R}_{\mathbf{r_n}} \backslash \mathcal{R}_{\mathbf{m}}) > u_{\mathbf{n}}(\tau)v, \|\mathbf{X}(\mathbf{0})\| > u_{\mathbf{n}}(\tau) \right).
\end{aligned}
$$

On the other hand, for each $\mathbf{t} \in \mathcal{R}^+_{\mathbf{r_n}} \backslash \mathcal{I}$, the summand is upper bounded by

$$
\mathbb{E}\left[\exp\left\{ -\sum_{\mathbf{s}\in\mathcal{R}_{\mathbf{m}}} f(u_{\mathbf{n}}(\tau)^{-1}\mathbf{X}(\mathbf{s})) \right\} \mathbb{1}\left(\max_{\mathbf{s}\prec\mathbf{0},\mathbf{s}\in\mathcal{R}_{\mathbf{m}}} \|\mathbf{X}(\mathbf{s})\| \le u_{\mathbf{n}}(\tau) < \|\mathbf{X}(\mathbf{0})\| \right) \right] \le \mathbb{P}(\|\mathbf{X}(\mathbf{0})\| > u_{\mathbf{n}}(\tau)).
$$

Combining the two parts, we see that the difference in (5.7) does not exceed

$$
\begin{aligned}
& \frac{1}{\mathbb{P}(M_X(\mathcal{R}^+_{\mathbf{r_n}}) > u_{\mathbf{n}}(\tau))} \left[\left(\prod_{\ell=1}^k r_{n_\ell} \right) \mathbb{P}\left(M_X(\mathcal{R}_{\mathbf{r_n}} \backslash \mathcal{R}_{\mathbf{m}}) > u_{\mathbf{n}}(\tau)v, \|\mathbf{X}(\mathbf{0})\| > u_{\mathbf{n}}(\tau) \right) \right. \\
& \left. + \mathrm{Card}(\mathcal{R}^+_{\mathbf{r_n}} \backslash \mathcal{I}) \mathbb{P}(\|\mathbf{X}(\mathbf{0})\| > u_{\mathbf{n}}(\tau)) \right] \\
& = \frac{1}{\theta_{\mathbf{n}}} \left[\mathbb{P}\left(M_X(\mathcal{R}_{\mathbf{r_n}} \backslash \mathcal{R}_{\mathbf{m}}) > u_{\mathbf{n}}(\tau)v \mid \|\mathbf{X}(\mathbf{0})\| > u_{\mathbf{n}}(\tau) \right) + \frac{\mathrm{Card}(\mathcal{R}^+_{\mathbf{r_n}} \backslash \mathcal{I})}{\prod_{\ell=1}^k r_{n_l}} \right] \to 0
\end{aligned}
$$

as $\mathbf{n} \to \boldsymbol{\infty}, \mathbf{m} \to \boldsymbol{\infty}$, where we have used (5.6). Therefore, for any sequence $(\mathbf{n}_k)$ converging to $\boldsymbol{\infty}$, along which $\theta_{\mathbf{n}}$ has a (positive) limit, say, L,

$$\begin{aligned}
&\lim_{k\to\infty} \mathbb{E}\left[\exp\left\{-\sum_{\mathbf{t}\in\mathcal{R}^+_{\mathbf{r}_{\mathbf{n}_k}}} f(u_{\mathbf{n}_k}(\tau)^{-1}\mathbf{X}(\mathbf{t}))\right\}\middle| M_X(\mathcal{R}^+_{\mathbf{r}_{\mathbf{n}_k}}) > u_{\mathbf{n}_k}(\tau)\right] \\
=&\lim_{\mathbf{m}\to\boldsymbol{\infty}} \lim_{k\to\infty} \theta^{-1}_{\mathbf{n}_k}\mathbb{E}\left[\exp\left\{-\sum_{\mathbf{t}\in\mathcal{R}_{\mathbf{m}}} f(u_{\mathbf{n}_k}(\tau)^{-1}\mathbf{X}(\mathbf{t}))\right\}\right. \\
&\qquad \left.\mathbb{1}\left(\max_{\mathbf{t}\prec\mathbf{0},\mathbf{t}\in\mathcal{R}_{\mathbf{m}}} \|\mathbf{X}(\mathbf{t})\| \le u_{\mathbf{n}_k}(\tau)\right) \middle| \|\mathbf{X}(\mathbf{0})\| > u_{\mathbf{n}_k}(\tau)\right] \\
=&L^{-1}\mathbb{E}\left[\exp\left\{-\sum_{\mathbf{t}\in\mathbb{Z}^k} f(\mathbf{Y}(\mathbf{t}))\right\}\mathbb{1}\left(\max_{\mathbf{t}\prec\mathbf{0}}\|\mathbf{Y}(\mathbf{t})\|\le 1\right)\right].
\end{aligned}$$

Choosing $f = 0$ gives us

$$L = \mathbb{P}\left(\max_{\mathbf{t}\prec\mathbf{0}}\|\mathbf{Y}(\mathbf{t})\|\le 1\right),$$

which implies several things. First of all, it implies that all subsequential limits L are equal, so the array $(\theta_{\mathbf{n}})$ has a limit as $\mathbf{n} \to \boldsymbol{\infty}$. Therefore the block extremal index exists and is positive, and $\theta_{\mathrm{b}} = \theta_{\mathrm{half}}$. This also proves the convergence of the Laplace transform of the cluster process computed under its conditional law:

$$\begin{aligned}
&\lim_{\mathbf{n}\to\boldsymbol{\infty}} \mathbb{E}\left[\exp\left\{-\sum_{\mathbf{i}\in\mathcal{R}^+_{\mathbf{r}_{\mathbf{n}}}} f(u_{\mathbf{n}}(\tau)^{-1}\mathbf{X}(\mathbf{i}))\right\}\middle| M_X(\mathcal{R}^+_{\mathbf{r}_{\mathbf{n}}}) > u_{\mathbf{n}}(\tau)\right] \\
=&\mathbb{E}\left[\exp\left\{-\sum_{\mathbf{t}\in\mathbb{Z}^k} f(\mathbf{Y}(\mathbf{t}))\right\}\middle| \max_{\mathbf{t}\prec\mathbf{0}}\|\mathbf{Y}(\mathbf{t})\|\le 1\right]
\end{aligned}$$

for any nonnegative continuous f on $(\overline{\mathbb{R}})^d \setminus \{\mathbf{0}\}$ with a compact support. This, of course, proves the stated weak convergence of the conditional laws of the cluster process.

One shows that the Laplace transform $\Psi_C(f)$ of the limiting point process computed under its conditional law has the expression in the right hand side of (5.5) using the same argument as in Basrak and Segers (2009), using the invariant order $\prec$. Finally, (5.3) follows from (5.5) applied to the zero function. □

Remark 5.3. It is elementary to check that, if $f(\mathbf{x}) = 0$ whenever $\|\mathbf{x}\| \le 1$, then the obvious analogue of an alternative expression (4.6) in Basrak and Segers (2009) for the Laplace transform $\Psi_C(f)$ holds as well. Furthermore, under both Condition 5.1 and the asymptotic independence of extremal clusters condition

$$\mathbb{E}\left[\exp\left\{-\sum_{\mathbf{t}\in\mathcal{R}^+_{\mathbf{n}}} f(u_{\mathbf{n}}(1)^{-1}\mathbf{X}(\mathbf{t}))\right\}\right] - \left(\mathbb{E}\left[\exp\left\{-\sum_{\mathbf{t}\in\mathcal{R}^+_{\mathbf{r}_{\mathbf{n}}}} f(u_{\mathbf{n}}(1)^{-1}\mathbf{X}(\mathbf{t}))\right\}\right]\right)^{\prod_{\ell=1}^k \lfloor n_\ell/r_{n_\ell}\rfloor} \to 0$$

for every continuous function f with a compact support, one also obtains a picture of exceedance clusters on a larger scale, as in Theorem 4.5 *ibid.* For the point process

$$N_n = \sum_{\mathbf{t}\in\mathcal{R}_{\mathbf{n}}^+} \delta_{u_{\mathbf{n}}(1)^{-1}\mathbf{X}(\mathbf{t})}$$

one obtains weak convergence in the space of Radon measures on $(\overline{\mathbb{R}})^d \setminus \{\mathbf{0}\}$ to a cluster Poisson point process whose restriction to the set $\{\mathbf{x} : \|\mathbf{x}\| > a\}$, $a > 0$, has the representation

$$\sum_{i=1}^{P_a} \sum_{\mathbf{t}\in\mathbb{Z}^k} \delta_{a\mathbf{Z}_i(\mathbf{t})} \mathbf{1}\big(\|\mathbf{Z}_i(\mathbf{t})\| > 1\big)\,,$$

where $\big(\mathbf{Z}_i(\mathbf{t}),\, \mathbf{t}\in\mathbb{Z}^k\big)$, $i = 1, 2, \ldots$ are i.i.d. copies of the single cluster limiting process in Theorem 5.2, independent of a mean $\theta_{\mathrm{b}} u^{-\alpha}$ Poisson random variable P_a. A different, and very detailed, representation of the entire limiting point process is in Basrak and Planinić (2018).

6. Brown-Resnick Random Fields

The tail field is a convenient formalism to describe the extremes of a jointly regularly varying stationary random field. It is useful, in particular, in describing the extremal clusters, and it can be used to define versions of the extremal index. In order to make it concrete, in this section, we focus on the class of the so-called Brown-Resnick random fields. For simplicity we will keep the values of the field one-dimensional, with the standard Fréchet marginal distributions.

Let $(W(\mathbf{t}) : \mathbf{t}\in\mathbb{R}^k)$ be a stationary increment (real-valued) zero-mean Gaussian random field, with variance $\sigma^2(\mathbf{t})$ and variogram $\gamma(\mathbf{t}) = E(W(\mathbf{t}) - W(\mathbf{0}))^2$, $\mathbf{t}\in\mathbb{R}^k$. The stationarity of the increments means that $E(W(\mathbf{t}) - W(\mathbf{s}))^2 = \gamma(\mathbf{t}-\mathbf{s})$ for all $\mathbf{t},\mathbf{s}\in\mathbb{R}^k$. Let $(W_i(\mathbf{t}) : \mathbf{t}\in\mathbb{R}^k), i\in\mathbb{N}$ be i.i.d. copies of this random field, independent of a Poisson point process $\sum_{i=1}^\infty \delta_{U_i}$ on $\mathbb{R}_+$ with intensity du/u^2. The Brown-Resnick random field associated with the Gaussian random field $(W(\mathbf{t}) : \mathbf{t}\in\mathbb{R}^k)$ is defined by

$$X(\mathbf{t}) = \max_{i=1,2,\ldots} U_i \exp\{W_i(\mathbf{t}) - \sigma^2(\mathbf{t})/2\}\,. \tag{6.1}$$

Since $E\big(\exp\{W(\mathbf{t}) - \sigma^2(\mathbf{t})/2\}\big) = 1$ for each t, this is a well defined max-stable random field with the standard Fréchet marginal distributions; see de Haan (1984). Furthermore, it is a stationary random field (even when the Gaussian random field $(W(\mathbf{t}) : \mathbf{t}\in\mathbb{R}^k)$ itself is not stationary); see Theorem 2 and Remark 3 in Kabluchko et al. (2009). As any max-stable random field with the standard Fréchet marginal distributions, the Brown-Resnick random field is multivariate regular varying (with $\alpha = 1$). This fact is also seen from the following proposition, that computes the law of the tail field of this random field.

Proposition 6.1. *The Brown-Resnick random field $(X(\mathbf{t}) : \mathbf{t}\in\mathbb{Z}^k)$ is multivariate regularly varying, and the finite-dimensional distributions of its tail field $(Y(\mathbf{t}) : \mathbf{t}\in\mathbb{Z}^k)$ can be computed by*

$$\begin{aligned}
&\mathbb{P}(Y(\mathbf{t}_1)\le y_1,\ldots,Y(\mathbf{t}_n)\le y_n)\\
=&\mathbb{E}\left[\max_{i=1,\ldots,n}\left(\frac{1}{y_i}\exp\left\{W(\mathbf{t}_i)-\frac{\sigma^2(\mathbf{t}_i)}{2}\right\},\exp\left\{W(\mathbf{0})-\frac{\sigma^2(\mathbf{0})}{2}\right\}\right)\right]\\
&-\mathbb{E}\left[\max_{i=1,\ldots,n}\frac{1}{y_i}\exp\left\{W(\mathbf{t}_i)-\frac{\sigma^2(\mathbf{t}_i)}{2}\right\}\right]
\end{aligned} \tag{6.2}$$

for $\mathbf{t}_1, \ldots, \mathbf{t}_n \in \mathbb{Z}^k$ and positive $y_1, \ldots, y_n$. In particular, the marginal distributions of the tail field are given by

$$\mathbb{P}(Y(\mathbf{t}) \leq y) = \Phi\left(\frac{2\ln y + \gamma(\mathbf{t})}{2\sqrt{\gamma(\mathbf{t})}}\right) - \frac{1}{y}\Phi\left(\frac{2\ln y - \gamma(\mathbf{t})}{2\sqrt{\gamma(\mathbf{t})}}\right), \tag{6.3}$$

for $\mathbf{t} \in \mathbb{Z}^k$ and $y > 0$. Here $\Phi(\cdot)$ is the standard normal cdf.

Proof. Let $V_i(\mathbf{t}) = \exp\{W_i(\mathbf{t}) - \sigma^2(\mathbf{t})/2\}$. Then for any finite set of points in $\mathbb{Z}^k$ and positive numbers,

$$\mathbb{P}(X(\mathbf{t}_1) \leq x_1, \ldots, X(\mathbf{t}_n) \leq x_n) = \exp\left\{-\mathbb{E}\left[\max\left(\frac{V(\mathbf{t}_1)}{x_1}, \ldots, \frac{V(\mathbf{t}_n)}{x_n}\right)\right]\right\}, \tag{6.4}$$

so

$$\begin{aligned}
&\mathbb{P}(x^{-1}X(\mathbf{t}_1) \leq y_1, \ldots, x^{-1}X(\mathbf{t}_n) \leq y_n \mid X(\mathbf{0}) > x) \\
=&\frac{\mathbb{P}(X(\mathbf{t}_1) \leq xy_1, \ldots, X(\mathbf{t}_n) \leq xy_n) - \mathbb{P}(X(\mathbf{t}_1) \leq xy_1, \ldots, X(\mathbf{t}_n) \leq xy_n, X(\mathbf{0}) \leq x)}{P(X(\mathbf{0}) > x)} \\
=&\frac{\exp\left\{-\mathbb{E}\left[\max\left(\frac{V(\mathbf{t}_1)}{xy_1}, \ldots, \frac{V(\mathbf{t}_n)}{xy_n}\right)\right]\right\} - \exp\left\{-\mathbb{E}\left[\max\left(\frac{V(\mathbf{t}_1)}{xy_1}, \ldots, \frac{V(\mathbf{t}_n)}{xy_n}, \frac{V(\mathbf{0})}{x}\right)\right]\right\}}{1 - e^{-1/x}} \\
\sim& x\left[\exp\left\{-\frac{1}{x}\mathbb{E}\left[\max_{i=1,\ldots,n} \frac{1}{y_i}\exp\left\{W(\mathbf{t}_i) - \frac{\sigma^2(\mathbf{t}_i)}{2}\right\}\right]\right\}\right. \\
&\left. - \exp\left\{-\frac{1}{x}\mathbb{E}\left[\max_{i=1,\ldots,n}\left(\frac{1}{y_i}\exp\left\{W(\mathbf{t}_i) - \frac{\sigma^2(\mathbf{t}_i)}{2}\right\}, \exp\left\{W(\mathbf{0}) - \frac{\sigma^2(\mathbf{0})}{2}\right\}\right)\right]\right\}\right],
\end{aligned}$$

which converges, as $x \to \infty$, to the expression in the right hand side of (6.2). In particular, the marginal distributions satisfy

$$\begin{aligned}
&\mathbb{P}(Y(\mathbf{t}) \leq y) \\
=&\mathbb{E}\left[\max\left(\frac{1}{y}\exp\left\{W(\mathbf{t}) - \frac{\sigma^2(\mathbf{t})}{2}\right\}, \exp\left\{W(\mathbf{0}) - \frac{\sigma^2(\mathbf{0})}{2}\right\}\right)\right] - \mathbb{E}\left[\frac{1}{y}\exp\left\{W(\mathbf{t}) - \frac{\sigma^2(\mathbf{t})}{2}\right\}\right],
\end{aligned}$$

and (6.3) follows by straightforward calculations with lognormal random variables; see e.g. Lien (1986). □

We will investigate the extremal behaviour of the restriction of the Brown-Resnick random field to the integer grid $\mathbb{Z}^k$. The first question is whether this field satisfies Condition 5.1 (and, hence, also the assumption (4.5)). The answer is given in the following proposition.

Proposition 6.2. *Let $(X(\mathbf{t}) : \mathbf{t} \in \mathbb{Z}^k)$ be the Brown-Resnick random field (6.1) corresponding to a stationary increment zero-mean Gaussian random field with variance $\sigma^2(\mathbf{t})$. Then $(X(\mathbf{t}) : \mathbf{t} \in \mathbb{Z}^k)$ satisfied Condition 5.1 if and only if the Gaussian field satisfies*

$$\lim_{\mathbf{t}\to\boldsymbol{\infty},\, \mathbf{t}\in\mathbb{Z}^k} \big(W_i(\mathbf{t}) - \sigma^2(\mathbf{t})/2\big) = -\infty \quad a.s.. \tag{6.5}$$

Proof. Choose and fix the arrays $(u_{\mathbf{n}}(\tau))$ and $(\mathbf{r_n})$. By the inclusion-exclusion formula,

$$\begin{aligned}
&\mathbb{P}\left(M_X(\mathcal{R}_{\mathbf{r_n}} \backslash \mathcal{R}_{M\mathbf{1}}) > u_{\mathbf{n}}(\tau) \mid \|\mathbf{X}(\mathbf{0})\| > u_{\mathbf{n}}(\tau)\right) \\
&= 1 - \frac{\mathbb{P}\left(M_X(\mathcal{R}_{\mathbf{r_n}} \backslash \mathcal{R}_{M\mathbf{1}} \cup \{\mathbf{0}\}) > u_{\mathbf{n}}(\tau)\right) - \mathbb{P}\left(M_X(\mathcal{R}_{\mathbf{r_n}} \backslash \mathcal{R}_{M\mathbf{1}}) > u_{\mathbf{n}}(\tau)\right)}{\mathbb{P}\left(\|\mathbf{X}(\mathbf{0})\| > u_{\mathbf{n}}(\tau)\right)},
\end{aligned}$$

so Condition 5.1 is satisfied if and only if

$$\lim_{M\to\infty}\liminf_{\mathbf{n}\to\infty} u_{\mathbf{n}}(\tau)\left[\mathbb{P}\left(M_X(\mathcal{R}_{\mathbf{r_n}}\backslash\mathcal{R}_{M\mathbf{1}}\cup\{\mathbf{0}\})>u_{\mathbf{n}}(\tau)\right)-\mathbb{P}\left(M_X(\mathcal{R}_{\mathbf{r_n}}\backslash\mathcal{R}_{M\mathbf{1}})>u_{\mathbf{n}}(\tau)\right)\right]=1\,. \tag{6.6}$$

By (6.4), as $\mathbf{n}\to\boldsymbol{\infty}$,

$$\begin{aligned}
&\mathbb{P}\left(M_X(\mathcal{R}_{\mathbf{r_n}}\backslash\mathcal{R}_{M\mathbf{1}}\cup\{\mathbf{0}\})>u_{\mathbf{n}}(\tau)\right)-\mathbb{P}\left(M_X(\mathcal{R}_{\mathbf{r_n}}\backslash\mathcal{R}_{M\mathbf{1}})>u_{\mathbf{n}}(\tau)\right)\\
=&\exp\left\{-u_{\mathbf{n}}(\tau)^{-1}\mathbb{E}\max_{\mathbf{t}\in\mathcal{R}_{\mathbf{r_n}}\backslash\mathcal{R}_{M\mathbf{1}}}V(\mathbf{t})\right\}-\exp\left\{-u_{\mathbf{n}}(\tau)^{-1}\mathbb{E}\max_{\mathbf{t}\in\mathcal{R}_{\mathbf{r_n}}\backslash\mathcal{R}_{M\mathbf{1}}\cup\{\mathbf{0}\}}V(\mathbf{t})\right\}\\
\sim& u_{\mathbf{n}}(\tau)^{-1}\left[\mathbb{E}\max_{\mathbf{t}\in\mathcal{R}_{\mathbf{r_n}}\backslash\mathcal{R}_{M\mathbf{1}}\cup\{\mathbf{0}\}}V(\mathbf{t})-\mathbb{E}\max_{\mathbf{t}\in\mathcal{R}_{\mathbf{r_n}}\backslash\mathcal{R}_{M\mathbf{1}}}V(\mathbf{t})\right]\exp\left\{-u_{\mathbf{n}}(\tau)^{-1}\mathbb{E}\max_{\mathbf{t}\in\mathcal{R}_{\mathbf{r_n}}\backslash\mathcal{R}_{M\mathbf{1}}}V(\mathbf{t})\right\}\,.
\end{aligned}$$

Since

$$\begin{aligned}
0&\le u_{\mathbf{n}}(\tau)^{-1}\mathbb{E}\max_{\mathbf{t}\in\mathcal{R}_{\mathbf{r_n}}\backslash\mathcal{R}_{M\mathbf{1}}}V(\mathbf{t})\le u_{\mathbf{n}}(\tau)^{-1}\sum_{\mathbf{t}\in\mathcal{R}_{\mathbf{r_n}}\backslash\mathcal{R}_{M\mathbf{1}}}\mathbb{E}V(\mathbf{t})\\
&=u_{\mathbf{n}}(\tau)^{-1}\mathrm{Card}\big(\mathcal{R}_{\mathbf{r_n}}\backslash\mathcal{R}_{M\mathbf{1}}\big)\to 0\,,
\end{aligned}$$

(6.6) is equivalent to

$$\lim_{M\to\infty}\liminf_{\mathbf{n}\to\infty}\mathbb{E}\Big[\max_{\mathbf{t}\in\mathcal{R}_{\mathbf{r_n}}\backslash\mathcal{R}_{M\mathbf{1}}\cup\{\mathbf{0}\}}V(\mathbf{t})-\max_{\mathbf{t}\in\mathcal{R}_{\mathbf{r_n}}\backslash\mathcal{R}_{M\mathbf{1}}}V(\mathbf{t})\Big]=1\,. \tag{6.7}$$

Suppose first that (6.5) holds, i.e. that $V(\mathbf{t})\to 0$ a.s. as $\mathbf{t}\to\boldsymbol{\infty}$. Then

$$\begin{aligned}
&\lim_{M\to\infty}\liminf_{\mathbf{n}\to\infty}\mathbb{E}\Big[\max_{\mathbf{t}\in\mathcal{R}_{\mathbf{r_n}}\backslash\mathcal{R}_{M\mathbf{1}}\cup\{\mathbf{0}\}}V(\mathbf{t})-\max_{\mathbf{t}\in\mathcal{R}_{\mathbf{r_n}}\backslash\mathcal{R}_{M\mathbf{1}}}V(\mathbf{t})\Big]\\
=&\lim_{M\to\infty}\mathbb{E}\Big[\max_{\mathbf{t}\in(\mathcal{R}_{M\mathbf{1}})^c\cup\{\mathbf{0}\}}V(\mathbf{t})-\max_{\mathbf{t}\in(\mathcal{R}_{M\mathbf{1}})^c}V(\mathbf{t})\Big]=\mathbb{E}V(\mathbf{0})=1\,,
\end{aligned}$$

so (6.7) is satisfied.

Suppose, on the other hand, that (6.5) fails. Then there is $a>0$ and an event A of positive probability such that

$$\limsup_{\mathbf{t}\to\boldsymbol{\infty},\,\mathbf{t}\in\mathbb{Z}^k}V(\mathbf{t})>a\ \text{ on } A. \tag{6.8}$$

Therefore, on A, for all M,

$$\limsup_{\mathbf{n}\to\infty}\Big[\max_{\mathbf{t}\in\mathcal{R}_{\mathbf{r_n}}\backslash\mathcal{R}_{M\mathbf{1}}\cup\{\mathbf{0}\}}V(\mathbf{t})-\max_{\mathbf{t}\in\mathcal{R}_{\mathbf{r_n}}\backslash\mathcal{R}_{M\mathbf{1}}}V(\mathbf{t})\Big]\le\max(V(\mathbf{0}),a)-a\,. \tag{6.9}$$

Since for every $\mathbf{n}$

$$\max_{\mathbf{t}\in\mathcal{R}_{\mathbf{r_n}}\backslash\mathcal{R}_{M\mathbf{1}}\cup\{\mathbf{0}\}}V(\mathbf{t})-\max_{\mathbf{t}\in\mathcal{R}_{\mathbf{r_n}}\backslash\mathcal{R}_{M\mathbf{1}}}V(\mathbf{t})\le V(\mathbf{0})\,,$$

an integrable random variable, we can use Fatou's lemma in the form

$$\begin{aligned}
&\lim_{M\to\infty}\liminf_{\mathbf{n}\to\infty}\mathbb{E}\Big[\max_{\mathbf{t}\in\mathcal{R}_{\mathbf{r_n}}\backslash\mathcal{R}_{M\mathbf{1}}\cup\{\mathbf{0}\}}V(\mathbf{t})-\max_{\mathbf{t}\in\mathcal{R}_{\mathbf{r_n}}\backslash\mathcal{R}_{M\mathbf{1}}}V(\mathbf{t})\Big]\\
\le&\lim_{M\to\infty}\limsup_{\mathbf{n}\to\infty}\mathbb{E}\Big[\max_{\mathbf{t}\in\mathcal{R}_{\mathbf{r_n}}\backslash\mathcal{R}_{M\mathbf{1}}\cup\{\mathbf{0}\}}V(\mathbf{t})-\max_{\mathbf{t}\in\mathcal{R}_{\mathbf{r_n}}\backslash\mathcal{R}_{M\mathbf{1}}}V(\mathbf{t})\Big]\\
\le&\lim_{M\to\infty}\mathbb{E}\limsup_{\mathbf{n}\to\infty}\Big[\max_{\mathbf{t}\in\mathcal{R}_{\mathbf{r_n}}\backslash\mathcal{R}_{M\mathbf{1}}\cup\{\mathbf{0}\}}V(\mathbf{t})-\max_{\mathbf{t}\in\mathcal{R}_{\mathbf{r_n}}\backslash\mathcal{R}_{M\mathbf{1}}}V(\mathbf{t})\Big]\,.
\end{aligned}$$

The upper limit inside the expectation cannot exceed $V(\mathbf{0})$ and, by (6.9), it is strictly smaller than $V(\mathbf{0})$ on an event of a positive probability. Therefore,

$$\lim_{M\to\infty}\liminf_{\mathbf{n}\to\infty}\mathbb{E}\Big[\max_{\mathbf{t}\in\mathcal{R}_{\mathbf{r_n}}\backslash\mathcal{R}_{M\mathbf{1}}\cup\{\mathbf{0}\}}V(\mathbf{t})-\max_{\mathbf{t}\in\mathcal{R}_{\mathbf{r_n}}\backslash\mathcal{R}_{M\mathbf{1}}}V(\mathbf{t})\Big]<\mathbb{E}V(\mathbf{0})=1\,,$$

and (6.7) fails. $\square$

Since the condition (6.5) cannot be satisfied if the Gaussian random field $(W(\mathbf{t}) : \mathbf{t} \in \mathbb{R}^k)$ is stationary (and nontrivial), Condition 5.1 is not satisfied for the corresponding Brown-Resnick random field. Furthermore, denoting the constant variance of the Gaussian field by $\sigma^2 > 0$, we have by (6.3),

$$\mathbb{P}(Y(\mathbf{t}) > 1) = 2\Phi\left(-\frac{1}{2}\sqrt{\gamma(\mathbf{t})}\right) \geq 2\Phi(-\sigma).$$

Therefore, the tail field does not necessarily vanish as $\mathbf{t} \to \infty$, and the extremal clusters may last indefinitely.

The following corollary is an immediate consequence of propositions 6.1 and 6.2.

Corollary 6.3. *Let $(X(\mathbf{t}) : \mathbf{t} \in \mathbb{Z}^k)$ be the Brown-Resnick random field* (6.1) *corresponding to a stationary increment zero-mean Gaussian random field with variance $\sigma^2(\mathbf{t})$, satisfying* (6.5)*. Then the block extremal index θ_{b} exists, is positive and equal to θ_{half} for every invariant order on $\mathbb{Z}^k$, and can be computed by*

$$\theta_{\mathrm{b}} = \mathbb{E}\left[\max_{\mathbf{t}\preceq\mathbf{0}} \exp\left\{W(\mathbf{t}) - \sigma^2(\mathbf{t})/2\right\}\right] - \mathbb{E}\left[\max_{\mathbf{t}\prec\mathbf{0}} \exp\left\{W(\mathbf{t}) - \sigma^2(\mathbf{t})/2\right\}\right]. \tag{6.10}$$

Proof. By Proposition 6.2, Condition 5.1 is satisfied. By Theorem 5.2 the block extremal index θ_{b} exists, is positive and, using (6.2),

$$\begin{aligned}\theta_{\mathrm{b}} = \theta_{\mathrm{half}} &= \mathbb{P}\left(\max_{\mathbf{t}\prec\mathbf{0}} Y(\mathbf{t}) \leq 1\right)\\ &= \mathbb{E}\left[\max_{\mathbf{t}\preceq\mathbf{0}} \exp\left\{W(\mathbf{t}) - \sigma^2(\mathbf{t})/2\right\}\right] - \mathbb{E}\left[\max_{\mathbf{t}\prec\mathbf{0}} \exp\left\{W(\mathbf{t}) - \sigma^2(\mathbf{t})/2\right\}\right].\end{aligned}$$

$\square$

Since an exact simulation of Brown-Resnick random fields is not easy (see e.g. Dieker and Mikosch (2015); Oesting et al. (2012)), results of the type (6.10) can be used for numerical evaluation of the extremal index of the field. We demonstrate this on an example.

Example 6.4 (Brown-Resnick field corresponding to the additive Fractional Brownian motion)**.** Recall that the standard Fractional Brownian motion with Hurst paremeter $0 < H < 1$ is a stationary increment zero-mean Gaussian process on $\mathbb{R}$, vanishing at the origin, with the variogram $\gamma(t) = |t|^{2H}$, $t \in \mathbb{R}$. Let $\mathrm{fBm}_{H_i}(t) \in \mathbb{R}$, $i = 1, \ldots, k$ be independent standard Fractional Brownian motions, with respective Hurst parameters $H_1, \ldots, H_k$. Then

$$W(t_1, \ldots, t_k) = \mathrm{fBm}_{H_1}(t_1) + \cdots + \mathrm{fBm}_{H_k}(t_k),\ \mathbf{t} = (t_1, \ldots, t_k) \in \mathbb{R}^k$$

is a zero mean stationary increment Gaussian random field, the additive Fractional Brownian motion. It is elementary to check that each standard Fractional Brownian motion satisfies (6.5) (this follows, for example, by the Borel-Cantelli lemma). Therefore, so does the additive Fractional Brownian motion.

For $k = 2$ we have used (6.10) to calculate the block extremal index of the Brown-Resnick random field corresponding to the additive Fractional Brownian motion. In this calculation we truncated the domain of the additive Fractional Brownian motion to the square $[-200, 200] \times [-200, 200]$. The results are plotted on Figure 1 as a function of H_1 and H_2.

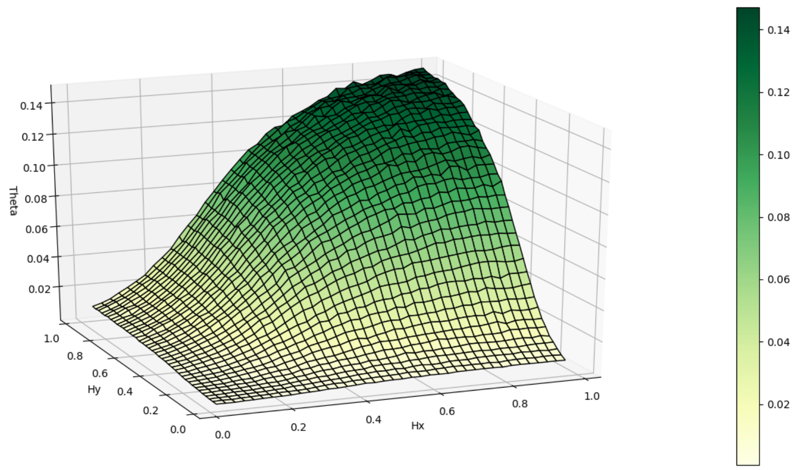

Figure 1. Extremal index vs. Hurst parameters

Figure 1 shows a positive relationship between θ_{b} and the Hurst parameters. This can be understood by noticing that, the smaller is the Hurst parameter, the slower is the variance increasing, the closer is the Fractional Brownian motion to the case of a constant variance, i.e. of stationarity. As we are discussing above, when the Gaussian random field is stationary, the extremal clusters of the corresponding Brown-Resnick random field can be very large.

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School of Operations Research and Information Engineering, Cornell University, Ithaca, NY 14853

Email address: lw529@cornell.edu

School of Operations Research and Information Engineering, and Department of Statistical Science, Cornell University, Ithaca, NY 14853

Email address: gs18@cornell.edu