SECTION II.

MECHANICS, OR THE PHYSICAL SCIENCE OF MOTION IN GENERAL.

CHAPTER I.

FUNDAMENTAL PRINCIPLES OF MECHANICS.

§ 44. *Mechanics.*—Mechanics is the science which treats of the laws of the motion of material bodies. It is an application of phoronomics to the bodies of the external world, in so far as the latter is concerned with the motion only of geometrical bodies.

Mechanics is a part of natural philosophy, or of the doctrine of laws according to which changes take place in the material world, viz., that part which considers the changes in bodies resulting from measureable motions.

§ 45. *Force.*—Force is the cause of motion or change of motion in material bodies. Every change of motion, viz., every change in the velocity of a body must be regarded as the effect of a force. For this reason we measure the force called gravity by a body falling freely, because the same incessantly changes its velocity. On the other hand, rest, or the invariability of the state of motion of a body, must not be attributed to the absence of forces, for opposite forces destroy each other and produce no effect. The gravity with which a body falls to the ground still acts, though the body rest upon a table; but this action is counteracted by the solidity of the table or of the support.

§ 46. A body is in equilibrium, or the forces acting upon a body are in equilibrium, when there is no residuary effect, no motion produced or changed, or when each neutralizes the other. In a body suspended by a thread, the strength of the thread is in equilibrium with gravity. In forces, equilibrium is destroyed, and motion arises if one of the forces be removed, or in any way counteracted; for instance, a steel spring, bent by a weight, enters into motion when the weight is taken away, because the force of the spring, called elasticity, then comes into action.

Statics is that part of mechanics which treats of the equilibrium of forces. Dynamics, on the other hand, treats of forces in so far as they produce motion.

§ 47. *Division of Forces.*—According to their effects, forces are either *moving forces* or *resistances;* that is, as motion is brought about or impeded. Gravity, the elasticity of a steel spring, &c., belong to motive forces. Friction, the solidity of bodies, &c., are resisting forces or resistances, because by them motion is either diminished or destroyed, and can by no means be brought about. Moving forces are divided into accelerating and retarding; the first produces a positive, the second a negative acceleration; by the one an accelerating, by the other a retarding motion is produced. Resistances are retarding forces, but a retarding force is not always a resistance. Gravity, for example, acts upon a body projected vertically upwards to retard it; but gravity, on this account, is no resisting force; for, by the consequent falling down of the body, it then again becomes a motive one.

There is a distinction between constant and variable forces. While constant forces always act in the same way, and, therefore, produce like effects in like particles of time, *i. e.* equal increments or decrements of velocity, the effects of variable forces are different at different times; while the former bring about a uniformly variable motion, to the latter corresponds a variably accelerated or a variably retarded one.

§ 48. *Pressure.*—*Pressure* and *traction* are the first effects of forces upon material bodies. By means of them, bodies are compressed and extended, and especially changed in their form. The pressure in traction brought about by gravity, acting vertically downwards, which the support of a heavy body, or the string to which a body is attached has to sustain, is called the weight of the body.

Pressure and traction, and weight also, are magnitudes of a particular kind, which can only virtually be compared with each other, as the action of forces serves for their measurement. The simplest, and on that account the most general, means of measuring forces is by weights.

§ 49. *Equality of Forces.*—Two weights, or two pressures, or tractions, and also the forces which correspond to these last, are equal, when one may be replaced by the other, without producing different effects. If, for example, a steel spring be bent by a weight G, as by another G_1, then are these weights, and therefore the gravities in both bodies, equal. If a loaded balance be made to vibrate as much by a weight G as by another G_1, substituted for G, these two weights G, G_1 are equal; in this case, the arms of the balance may be equal or unequal, and the remaining load great or small.

A pressure or weight (force) is 2, 3, 4, &c., times as great as another pressure, &c., if it produces the same effect as 2, 3, 4 . . . n pressures together of the second kind. If a balance, otherwise loaded at will, is brought into the same vibration by a weight (G) as by the addition of 2, 3, 4, equal weights (G_1), the weight (G) is 2, 3, 4, &c., times as great as the weight (G_1).

§ 50. *Matter.*—Matter is that by means of which bodies belonging to the external world, which in contradistinction to geometical bodies we term material or physical, act upon our senses. Mass is the quantity of matter composing a body.

Bodies of equal volume, or equal geometrical contents, have generally different weights when they consist of different kinds of matter. We cannot, therefore, infer the weight of a body from its volume until we first know the weight of a unit of volume, for instance, a cubic foot or cubic centimetre of the matter of the body.

§ 51. *Unit of Weight.*—The measurement of weights and forces consists in a comparison of them with some given invariable weight, taken as unity. The choice of this unit of weight or force is perfectly arbitrary; it is nevertheless advantageous in practice, that the weight of a volume of some universally diffused body, equivalent to that of the unit, should be chosen.

The units of weight or pressure are different in different countries. In England, the unit of pressure from which all the rest are derived is the weight of 22,185 cubic inches of distilled water (at a temp. 62° Fahr. taken in air, and the height of barometer at 30 inches). This weight is equal to 5760 grains; which again is equal to one pound troy, and 7000 such grains constitute the pound avoirdupois. The gramme is the weight of a cubic centimetre of pure water in a state of maximum density (at a temperature of 4° C.). The Prussian pound is also a unit referred to a weight of water. A Prussian cubic foot of distilled water in vacuo, and at a temperature 15° R. weighs 66 Prussian pounds. Now a Prussian foot = 139,13 Paris lines = 0,3137946 metres = 1,029722 English feet: hence it follows that a Prussian pound = 467,711 grammes = 1,031114 pounds English.*

§ 52. *Inertia.*—Inertia is that property of matter, in consequence of which it can of itself alone neither acquire nor change motion. Every material body remains at rest so long as no force acts upon it, and every material body once set into motion maintains a *uniform rectilinear motion*, so long as it is not subjected to the action of a force. Hence, when a change takes place in the condition of motion of a body, when it changes its direction of motion, or when it acquires a greater or less velocity, this is not to be attributed to the body as a certain quantum of matter, but to the agency of some foreign cause or force. In as much as a development of force takes place at every change in the motion of a material body, in so far inertia may be ranked amongst forces.

If we could entirely remove the forces acting upon a mass in motion, it would move on uniformly without ceasing, but we find nowhere such a uniform motion, because it is not possible for us to withdraw a mass from the action of every force. When a body moves upon an horizontal table, gravity, which is then counteracted by the table, exerts upon the body no immediate action, except that from the pressure of the body against the table there arises a resistance, which we shall consider more closely in the sequel under the name of fric-

* In the United States, the standard weight is the pound troy, the original of which is the mint pound, constructed by Capt. Kater at the request of Mr. Gallatin.—Am. Ed.

tion, which incessantly abstracts velocity from the moving body, imparts to it a retarded motion, and brings it finally to rest.

The air likewise opposes resistance to a moving body, and from this resistance, if the friction of the body were entirely put aside, a gradual diminution of velocity would ensue. But we find that the loss of velocity becomes the less, and that the motion also approximates more and more to a uniform one, the more we diminish the number and strength of these resistances; and hence we may conclude, that, by the removal of all moving forces and resistances, an entirely uniform motion must take place.

§ 53. *Measure of Forces.*—The force (P) which accelerates an inert mass (M) is proportional to the acceleration (p), and to the mass itself (M): it increases in equal masses as the increment of velocity in infinitely small times, and increases by equal increments of velocity in the same ratio as the masses become greater. The $m$tuple acceleration of one and the same mass, or of equal masses requires an $m$tuple force, and an $n$tuple mass for the same acceleration, an $n$tuple force.

As we have not yet chosen a measure of the mass, we may, therefore, at once, put $P=Mp$, *i. e.* the force equal to the product of the mass and the acceleration, and, at the same time, in place of the power, its effect, *i. e.* the pressure produced by it.

The correctness of this general law of motion may be readily proved by direct experiment: for example, by letting equal and differently movable masses be impelled upon an horizontal table by means of bent springs; and, it is obvious, from this, too, that all the consequences deduced, and all the laws developed from them for compound motions, fully correspond with observation and the phenomena of nature.

§ 54. *Mass.*—All bodies fall at one and the same place of the earth, and in vacuo equally fast, viz., with an invariable acceleration $g = 9{,}81$ metres $= 32{,}2$ feet (§ 15); if, therefore, the mass of a body $= M$, and the weight measuring its gravity $= G$, we have from the last formula

$$G=Mg, \text{ i. e.}$$

the weight of a body is a product of its mass and the acceleration of gravity, and inversely:

$$M=\frac{G}{g}, \text{ i. e.}$$

the mass of a body is its weight divided by the acceleration of gravity, or the mass is that weight which a body would otherwise have if the acceleration of gravity were = to unity, as a metre, a foot, &c. At a point upon, or in the vicinity of the earth, or of any other heavenly body, where bodies do not fall with 9,81 metres = 32,2 feet, but with a velocity (after the first second) of one metre = $3\frac{1}{3}$ ft., the mass, or rather its measure, is from hence immediately given by the weight of the body.

According as we express the acceleration of gravity in metres or in feet, we have, therefore, the mass

$$M=\frac{G}{9,81}=0,1019\ G, \text{ or}$$

$$M=\frac{G}{32,2}=0,031\ G.$$

The mass of a 20 lb. heavy body, $M=0,031\times 20=0,62$ lb., and inversely the weight of a mass of 20 lbs. $G=32,2\times 20=644$ lbs.

§ 55. In so far as we assume the acceleration (g) of gravity as invariable, it follows that the mass of a body is exactly proportional to its weight, and that also for the masses M and M_1, with the weights G and G_1:

$$\frac{M}{M_1}=\frac{G}{G_1}.$$

We hence obtain the weight as a measure of the mass of a body; the greater the mass which a body measures, the greater is its weight.

The acceleration of gravity is, in fact, somewhat variable, it becomes greater the nearer we approach the poles of the earth, and diminishes the more we advance towards the earth's equator; it is greatest at the poles, and least at the equator. It also diminishes the more a body is above or below the level of the sea; and attains its greatest value at the level of the sea. But, since a mass, so long as nothing is added to, or taken from it, is invariable, so that at all points of the earth, as well as those beyond it, at the moon, for instance, it is still the same; it hence follows that the weights also of bodies are variable and dependent upon the place of the bodies, and must be altogether proportional to the acceleration of gravity, corresponding with the place, or

$$\frac{G}{G_1}=\frac{g}{g_1}.$$

One and the same steel spring is differently bent by one and the same weight at different places of the earth; it is least at the equator, on high mountains, and in deep mines; greatest in the vicinity of the poles, and at the level of the sea.

§ 56. *Density* is the intensity with which space is filled by matter. A body is so much the denser the more matter there is in its space. The natural measure of density is that quantity of matter (that mass) which fills a unit of volume, because matter can only be measured by weight, so that the weight of a unit of volume, a cubic metre, or cubic foot of some matter, serves as a measure of its density.

For example: the density of a cubic foot of water = 62,38 lb., and that of cast iron = 452,13 lb., because a cubic foot of water weighs 62,38 lb. = 998,08 oz. avd., and a cubic foot of cast iron weighs 452,13 lb.

From the volume V of a body and its density γ, its weight $G = V\gamma$. The volume multiplied by the density gives the weight of a body.

The density of bodies is either uniform or variable, according as equal volumes of the same body are of equal or of unequal weight. The density of metals, for instance, is uniform, or they are homogeneous, because equal and very small parts of them are of the same weight: on the other hand, granite is a body of variable density, because made up of parts of different densities.

Example.—1. If the density of lead be 708 lbs., 3,2 cubic feet of lead weigh $= 708 \times 3{,}2 = 2265$ lbs.—2. If the density of bar iron $= 485{,}8$ lbs.; a mass of it of 205 lbs. has a volume $V = \frac{G}{\gamma} = \frac{205}{502} = 0{,}4023$ cubic ft. $= 0.4083 \times 1728 = 705.54$ cubic inches.—3. 10,4 cubic feet of deal, perfectly saturated with water, weigh 577 lbs.; the density of this wood is therefore: $\gamma = \frac{G}{V} = \frac{577}{10{,}4} = 55{,}5$ lbs.

§ 57. *Specific Gravity.*—Specifie gravity or specific weight is the relation of the density of a body to that of the density of some other, generally water, taken for unity. Now the density is equal to the weight of a unit of volume: hence the specific gravity is also the relation of the weight of one body to that of another, viz. water, under the same volume.

In order not to confound the specific weight with thatewhich belongs to a body of a certain magnitude, the last is usually called the absolute weight.

If γ be the density of matter (of water) to which we refer the density of other matter, and γ_1 the density of any one kind of matter, whose specific gravity we will designate by ε, then the formula

$$\varepsilon = \frac{\gamma_1}{\gamma} \text{ and } \gamma_1 = \varepsilon \cdot \gamma.$$

holds good, and the density of a substance is equal to its specific gravity into the density of water.

The absolute weight G of a mass of volume V and specific gravity ε is: $G = V\gamma_1 = V\varepsilon\gamma$.

Example.—1. The density of pure silver is 653,368 lbs. and that of water $= 62{,}38$ lbs., consequently the specific gravity of the former $= \frac{653{,}368}{62{,}38} = 10{,}474$; *i.e.* each mass of silver is $10\frac{1}{2}$ times as heavy as a mass of water filling the same space.—2. The specific gravity of quicksilvert $= 13{,}598$; its density, therefore, is $= 13{,}598 \times 62{,}38 = 848{,}24$ lbs.; a mass of 35 cubic inches, therefore, weighs:

$$G = 848{,}24 \,.\, V = \frac{848 \times 35}{1728} = 17{,}18 \text{ lbs.}$$

Remark. In these calculations the use of the French measure and weight has this advantage, that in order to effect the multiplication of ε and γ, it is merely requisite to advance the decimal point; because a cubic centimetre of water weighs one gramme, and a cubic metre a million, or one thousand kilogrammes. The density of quicksilver, according to the French measure and weightt $= 13{,}598 \times 1000 = 13598$ kilog.; *i. e.* a cubic metre of quicksilver weighs 13598 kilogrammes.

§ 58. The following table contains the specific gravities of certain bodies constantly coming into applieation in mechanics:

Mean specific gravity of dry laurel wood . .	= 0,659
" saturated with water . .	= 1,110
Mean specific gravity of dry pine wood . .	= 0,453
" saturated with water . .	= 0,839*
Quicksilvee	= 13,598
Lead	= 11,33

* *See* "On the Absorption of Water by Wood."—*Polytechnisches Mittheilungen*, Part iv. 1844.

Copper, cast and compact	= 8,75
" forged	= 8,97
Brass	= 8,55
Iron, cast, white	= 7,50
" " gray	= 7,10
" " medium	= 7,06
" bar iron	= 7,60*
Zinc, fused	= 7,05
" rolled	= 7,54
Granite	2,50 to 3,05
Gneiss	2,39 to 2,71
Limestone	2,40 to 2,86
Sandstone	1,90 to 2,70
Brick	1,40 to 2,22
Masonry, with lime mortar of quarry stone: fresh	= 2,46
dry	= 2,40
" " " of sandstone: fresh	= 2,12
dry	= 2,05
" " " of brick: fresh	= 1,55 to 1,70
dry	= 1,47 to 1,59
Earth, loamy, hard stamped, fresh	= 2,06
dry	= 1,93
Garden earth, fresh	= 2,05
dry	= 1,63
Dry, poor earth	= 1,34

§ 59. *State of Aggregation.*—Bodies appear to us, according to the different cohesion of their parts, under three principal conditions, which we term states of aggregation. They are either *solid* or *fluid*, and in the latter case, either liquid or gaseous. Solid bodies are those whose parts adhere so strongly together that a certain force is required to change the form of these bodies, or to effect their division. Fluid bodies, on the other hand, are those whose parts may be displaced about each other by the smallest force. Elastic fluid bodies, whose representant is atmospheric air, are distinguished from the liquid represented by water, in as much as there is inherent in them an endeavor to dilate themselves more and more, which is not the case with water, &c.

While solid bodies have a proper form and determinate volume, liquid or aqueous bodies possess only a determinate volume without any proper form, and the elastic extensible fluid bodies have neither one nor the other.

§ 60. *Division of Forces.*—Forces are different according to their nature; we will here mention the principal:

1. *Gravity*, by means of which all bodies tend to approach towards the centre of the earth.

* Rolled *boiler plate* iron has a sp. gr. from 7,6013 to 7,7922, or a mean of 7,7344, the amount of variation being $\frac{1}{40.5}$th part of the mean density. By seventeen trials of *hammered bar* iron, its mean sp. gr. was found to be 7,7254. *See* "*Report on Strength of Materials for Steam Boilers*," p. 232. Also *Journal Franklin Inst.*, 1837.—Am. Ed.

2. The *force of inertia*, which manifests itself when changes in the velocity of inert masses occur.

3. The *muscular force* of animated beings; the force exerted by the muscles of men and animals.

4. *Elasticity* or *spring-force*, which bodies exhibit in a change of their form or volume.

5. The *force of heat* or *caloric*, in consequence of which bodies expand or contract by a change of temperature.

6. The *magnetic force*, or the attraction and repulsion of magnets.

7. The *cohesive force*, the force by which the parts of a body are kept together, and resist separation.

8. *Adhesion*, the force with which bodies brought into close contact attract each other.

The resistances of friction, rigidity, solidity, &c., arise mainly from the force of adhesion.

§ 61. *In reference to forces we have to distinguish:*

1. Its *point of application*, that point of a body on which the force immediately acts.

2. Its *direction*, the straight line in which a force moves forward its point of application, or strives to move it forward, or to impede its motion. The *direction of a force*, like every direction of motion, has *two senses*, it can take place from left to right, or from right to left, from above to below, and from below to above. The one is termed *positive, the other negative.* As we write from left to right, and from above to below, it would be most convenient were we to call these motions positive, and those in the opposite direction, negative.

3. The *absolute magnitude* or *intensity* of a force, which, as above stated, is measured by weights, as pounds, kilogrammes, &c.

§ 62. *Action and re-action.*—The first effect which a force produces in a body, is a change of form or volume combined with extension or contraction, which begins at the point of application, and from thence diffuses itself further and further. By this inward change of the body, its inherent elasticity is called into action, puts itself into equilibrium with the force, and, therefore, is equal and opposed to the force. Action and re-action are equal and opposed to each other. This law not only prevails in reference to forces produced by contact, but also in the so-called forces of attraction and repulsion amongst which the magnetic force and gravity itself may be ranked. The more strongly a magnet attracts a bar of iron, the more strongly is the magnet itself attracted by the iron. The force with which the moon is attracted towards the earth (gravitation) is equal to that with which the moon reacts upon the earth. The force with which a weight presses upon its support is given back in an opposite direction; the force with which a workman draws or pushes at a machine, &œ, reacts upon the workman and strives to move him in the opposite direction. When a body impinges against another, the pressures are reciprocally equal on each of the bodies.

§ 63. *Division of Mechanics.*—The whole subject of mechanics may be included under two principal divisions, according to the state of aggregation of bodies.

1. The *mechanics of solid bodies*, which is also well named *geo-mechanics*.

2. The *mechanics of fluid bodies*, hydromechanics or hydraulics; the last is subdivided into:

1. Into the mechanics of water and liquid bodies especially, *hydromechanics* or *hydraulics*.

2. Into the *mechanics* of air, and *other aëriform* bodies, especially, aëromechanics, the mechanics of elastic fluids.

If we now have regard to the division of mechanics into statics and dynamics, we have the following parts:

1. *Statics of solid bodies*, or *geostatics*.
2. *Dynamics of solid bodies*, or *geodynamics*.
3. *Statics of fluids*, or *hydrostatics*.
4. *Dynamics of fluids*, or *hydrodynamics*.
5. *Statics of aëriform bodies*, or *aërostatics*.
6. *Dynamics of aëriform*, *aërodynamics*, or *pneumatics*.

CHAPTER II.

THE MECHANICS OF A MATERIAL POINT.

§ 64. A material point is a material body, whose dimensions are indefinitely small in comparison with the space occupied by it. In order to simplify the representation, we will in the following consider only the motion and equilibrium of a material point. A finite body is a continuous union of an infinite number of material points. If the single points or elements are all perfectly equal, *i. e.* move equally quick, in parallel straight lines, we may then apply the theory of the motion of a material point to that of the whole body, because, in this case, we may assume that equal parts of the mass of the body are impelled by equal parts of the force.

§ 65. *Simple constant Force.*—If (p) be the acceleration with which a mass (M) is impelled by a force, we have, from § 53, the forcea

$P = Mp$, and inversely, the acceleration, $p = \frac{P}{M}$.

If, further, we put the mass $M = \frac{G}{g}$, where G is the weight of the body, and g the acceleration of gravity, we have the force:

1. $P = \frac{p}{g} G$, and the acceleration:

2. $p = \frac{P}{G} g$.

We find, therefore, the force (P) which impels a body with a certain acceleration (p) when we multiply the weight of the body (G) by the ratio $\left(\frac{p}{g}\right)$ of its acceleration, to that of gravity.

Inversely, the acceleration (p), with which a body is moved forward by a force (P) is given, when the acceleration (g) of gravity is multiplied by the ratio $\left(\frac{P}{G}\right)$ of the force and weight of the body.

Example. Let us suppose a body lying on an horizontal and perfectly smooth table, which presents no impediment to the body in its course, but counteracts the effect of gravity upon it. If this body be pressed upon by a force acting horizontally, the body will give way to this influence, and move forward in the direction of this force. If the weight of this body be $G = 50$ lbs., and if $P = 10$ lbs. presses uninterruptedly upon it, it will enter into a uniformly accelerated motion with the acceleration $p = \frac{P}{G} \cdot g = \frac{10}{50} \times 32{,}2 = 6{,}44$ feet. On the other hand, if the acceleration with which a 42 lb. heavy body becomes accelerated by a force (P) $= 9$ feet, then will this force $P = \frac{p}{g} \cdot G = \frac{9}{32{,}25} \times 42 = 0{,}031 \times 378 = 11{,}7$ lbs.

§ 66. If the force which acts upon a body is constant, there arises a uniformly variable motion, and indeed a uniformly accelerated one, if the direction of the force corresponds with the initial direction of the motion; and, on the other hand, a uniformly retarded one, if the direction of the force is opposite to that of the initial direction of motion. If we substitute in the formulæ (§ 13 and § 14) for p, the value $\frac{P}{M} = \frac{P}{G} g$, we obtain the following:

I. For uniformly accelerated motions:

1. $v = c + \frac{P}{G} gt$, or $v = c + 32{,}2 \frac{P}{G} t$.

2. $s = ct + \frac{P}{G} \frac{gt^2}{2}$, or $s = ct + 16{,}1 \frac{P}{G} t^2$.

II. For uniformly retarded motions:

1. $v = c - \frac{P}{G} gt = c - 32{,}2 \frac{P}{G} t$.

2. $s = ct - \frac{P}{G} \frac{gt^2}{2} = ct - 16{,}1 \frac{P}{G} t^2$.

With the help of these formulæ all those questions may be answered which can be proposed relative to the rectilinear motions of bodies by a constant force.

Example.—1. A carriage weighing 2000 lbs. goes with a 4 feet velocity upon a horizontal line, offering no impediments to it, and pushed forward by an invariable force of 25 lbs. during 15 seconds, with what velocity will it proceed after the action of this force? This velocity $v = c + 32{,}2 \frac{P}{G} t$, but $c = 4$, $P = 25$ lbs., $G = 2000$, and $t = 15$; hence it follows, $v = 4 + 32{,}2 \cdot \frac{25}{2000} \cdot 15 = 10{,}03$ feet.—2. Under similar circumstances a carriage, weighing 5500 lbs., which, setting out with a uniform velocity, has traversed 950 feet in 3 minutes, is so impelled forward by a force acting continuously for 30 seconds, that it afterwards passes over 1650 feet in 3 minutes; what is this force? Here the initial velocity $c = \frac{950}{3.60} = 5,277$ feet per second, and the terminal velocity, $v = \frac{1650}{3.60} = 9,166$ feet; therefore $\frac{P}{G}\, gt = v - c = 3,889$, and the force $P = \frac{3,889 \,.\, tG}{gt} = 0,031 \times 3,889 \times \frac{5500}{30} = 0,12056 \times \frac{550}{3} = 22,10$ lbs.—3. A sledge, weighing 1500 lbs., sliding forward with a 15 ft. velocity, loses, through friction, upon its horizontal support, its whole motion in 25 seconds; how great is this friction? Here the motion is uniformly retarded, and the terminal velocity $v = 0$; hence $c = 32,2\, \frac{Pt}{G}$, and $P = 0,031\, \frac{Gc}{t} = 0,031 \times \frac{1500 \times 15}{25} = 0,031 \times 900 = 27,9$ lbs. the friction demanded.—4. Another sledge, of 1200 lbs. and 12 feet initial velocity, has to overcome by its motion a friction of 45 lbs.; what velocity has it after 8 seconds, and how great is the distance described? The terminal velocity is $v = 12 - 32,2 \times \frac{45 \times 8}{1200} = 12 - 9,66 = 2.34$ feet, and the distance described $s = \left(\frac{c+v}{2}\right) t = \left(\frac{12 + 2.34}{2}\right) \times 8 = 57.36$ feet.

§ 67. *Mechanical Effect.*—The *work done*, or mechanical effect, is that effect of a force which it produces in overcoming a *resistance:* as that of inertia, friction, gravity, &c. Work is performed when loads are lifted, a great velocity imparted to masses, bodies changed in their form or divided, &c. The work done, or the mechanical effect produced depends not only on the force, but also on the distance through which it is made to act or to overcome the resistance; it increases, of course, simultaneously with the force and the distance. If we lift a body slowly enough to allow of our neglecting its inertia, the labor expended is then proportional to its weight; for 1, the effect is the same whether m (3) times the weight ($m\, G$) is lifted to a certain height, or whether m (3) bodies of the single weight (G) are lifted to the same height; it is, namely, m times as great as the effort necessary for the lifting of a single weight to that height; and, again, 2, the work is the same, whether one and the same weight be raised to n (5) times the height ($n\, h$), or n (5) times through the height, and it is of course n (5) times as great as if the same weight were raised to a single height (h). The work again done by a slowly falling weight is proportional to the magnitude of this weight and the height from which it has descended. This proportionality also holds in every other kind of work done. In order to make a saw-cut of a given depth of double the length, there are twice as many particles to separate as from a cut of a single length; the work, therefore, is twice as great. The double length requires double the distance to be described by the force, consequently the work is proportional to the distance. In like manner the work of a pair of mill stones increases with the quantity of grains of a certain kind of corn, which they grind to a certain degree. This quantity, under otherwise similar circumstances, is proportional to the number of revolutions, or rather to the distance which the upper mill-stone, during the grinding of this quantity of corn, has gone through; consequently the work increases in proportion to the distance.

§ 68. The dependence above shown of the work produced by a force upon the magnitude of the force and distance described by it, allows us to take that amount of work which is expended in overcoming a resistance of the magnitude of the unit of weight (as a kilogramme, pound, &c.), along a path of the magnitude of the unit of length (metre or foot,) as a unit of the mechanical effect, or the dynamical unit, and then we may put the measure of this equal to the product of the force or resistance, and the distance described in the direction of the force whilst overcoming this resistance.

If we put the amount of the resistance itselfe $= P$, and the distance described by the force, or rather by its point of application, in overecoming this $= s$, the labor expended is:

$$L = P\,s \text{ units of work.}$$

In order to define more clearly the unit of work, for which the single name, *dynam*, may be used, both factors P and s are generally given; and, therefore, instead of units of work, we say kilograme metres, pounds-feet; and inversely, metrekilo. and feet-pounds according as the weight and distance are expressed in kilogrammes and metres, or in pounds and feet. These terms are usually expressed for simplicity by the abbreviations *mk*, or *km*, *lb. ft.*, or *ft. lb.*

Example.—1. In order to raise a stamper 210 lbs. 15 inches high, the mechanical effect $L = 210 \times \frac{15}{12} = 262{,}5$ ft. lbs. is necessary.—2. By a mechanical effect of 1500 ft. lbs., a sledge, which in its motion has to overcome a friction of 75 lbs., is driven forward a space $s = \frac{L}{P} = \frac{1500}{75} = 20$ feet.

§ 69. Not only in an invariable force or constant resistance is the labor a product of the force and distance, but also the labor may be expressed as a product of the distance and force, when the resistance whilst being overcome is variable, if a mean value of the continuous succession of forces be taken as *the force*. The relation is here the same as that of the time, the velocity, and the space; for the last may be regarded as a product of the time by the mean value of the velocities. The same graphical representations are here also applicable. The mechanical effect produced or expended may be considered as the area of a rectangular figure, *ABCD*, Fig. 27, whose

Fig. 27.

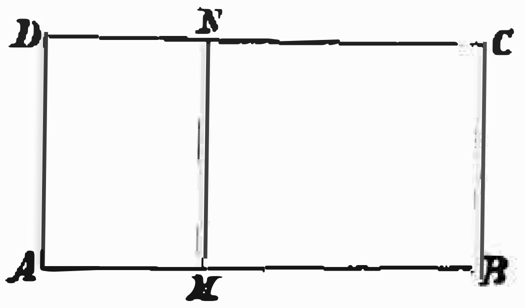
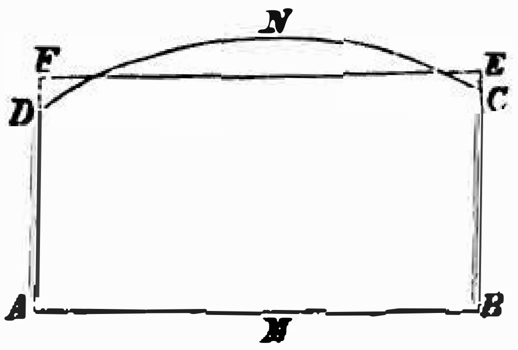


Fig. 28.

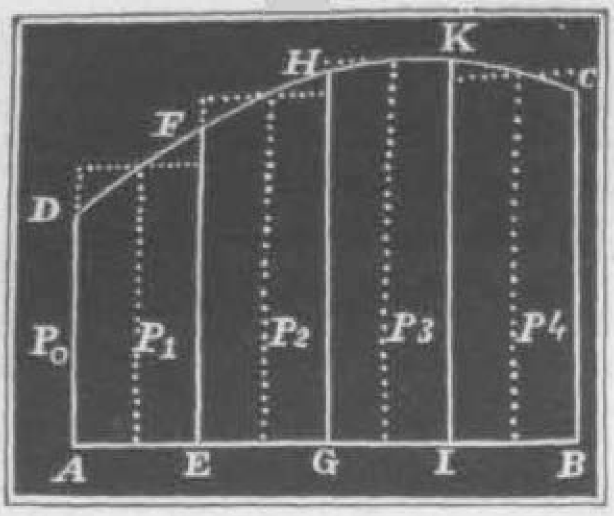


base *AB* is the space described (s), and whose height is either the invariable force (P) itself, or the mean of the different values of the forces. In general, the work may be represented by the area of a figure *ABCD*, Fig. 28, which has for its base the space (s), and whose height above each point of the base is equal to the force corresponding with each point of the path described. If the figure $ABCD$ be transformed into a rectangular one $ABEF$ of like area, we have the height $AF = BE$ for the mean value of the force—*the mean effort.*

§ 70. Arithmetic and geometry give different methods for finding a mean value from a constant succession of magnitudes. Amongst these, Simpson's rule is that which is the most frequently applied in practice, and it combines a high degree of accuracy with great simplicity.

In every case it is necessary to divide the space $AB=s$ (Fig. 29) into n (the more the better) equal parts, as $AE=EG=GI$, &c., and to measure the forces $EF=P_1$, $GH=P_2$, $IK=P_3$, &c., at the ends of these parts of the distance. If, then, we put the initial force $AD=P_0$ and the force at the other end $BC=P_n$, we have for the mean force:

Fig. 29.



$$P=(\tfrac{1}{2}P_0+P_1+P_2+P_3+\ldots+P_{n-1}+\tfrac{1}{2}P_n)\div n,$$

and, therefore, its work is:

$$Ps=(\tfrac{1}{2}P_0+P_1+P_2+\ldots+P_{n-1}+\tfrac{1}{2}P_n)\frac{s}{n}.$$

If the number of parts (n) be even, viz., 2, 4, 6, 8, &c., Simpson's rule gives still more accurately the mean force:

$$P=(P_0+4P_1+2P_2+4P_3+\ldots+4P_{n-1}+P_n)\div 3n,$$

and, therefore, the corresponding work:

$$Ps=(P_0+4P_1+2P_2+4P_3+\ldots+4P_{n-1}+P_n)\frac{s}{3n}.$$

Example. In order to find the mechanical work which a draught horse performs in drawing a carriage over a certain way, we make use of a dynamometer, or measurer of force, which is put into communication on one side with the carriage, and on the other with the traces of the horse, and the force is observed from time to time. If the initial force $P_0 = 110$ lbs., the force, after describing 25 feet = 122 lbs.; after 50 feet = 127 lbs.; after 75 feet = 120 lbs., and at the end of the whole distance of 100 feet = 114 lbs.; then the mean value, according to the first formula: $P=(\frac{1}{2}\,.\,110+122+127+120+\frac{1}{2}\,.\,114)\div 4 = 120{,}25$ lbs., and the mechanical work: $P\,s = 120{,}25 \times 100 = 12025$ ft. lbs.

from the second formula: $P = (110+4\,.\,122+2\,.\,127+4\,.\,120+114)\div 3\times 4$

$$=\frac{1446}{12}= 120{,}5 \text{ lbs., and the mechanical work}$$

$$P\,s = 120{,}5\times 100 = 12050 \text{ ft. lbs.}$$

§ 71. *Principle of the Vis Viva, or Living Forces.*—If, in the formula of (§ 13) $s = \dfrac{v^2-c^2}{2p}$ or $ps = \dfrac{v^2-c^2}{2}$ we substitute for the acceleration p, its value $\dfrac{P}{G}g$, we thus obtain $Ps=\left(\dfrac{v^2-c^2}{2g}\right)G$, or if we designate the heights due to the velocities $\dfrac{v^2}{2g}$ and $\dfrac{c^2}{2g}$ by h and h_1:

$$P\,s = (h-h_1)\,G.$$

If we interpret this equation, so useful in practical mechanics, we find that the work* ($P s$) which a mass either acquires when it passes from a lesser velocity (c) into a greater (v), or produces, when it is compelled to pass from a greater velocity into a less, is constantly equal to the product of the weight of this mass, and the difference of the heights due to the velocities $\left(\frac{v^2}{2g} - \frac{c^2}{2g}\right)$.

Example 1. In order to impart to a carriage of 4000 lbs. weight, upon a perfectly smooth railroad, a velocity of 30 feet, a mechanical work $P s = \frac{v^2}{2g} G = 0{,}0155\, v^2 G = 0{,}0155 \times 900 \times 4000 = 55800$ ft. lbs. is required; and just so much work will this carriage perform if a resistance be opposed to it, and it be gradually brought to rest.—2. Another carriage of 6000 lbs. goes forward with a velocity of 15 feet, which is transformed by a force acting upon it into a velocity of 24 feet, how great is the work acquired by this carriage, or done by the force? To the velocities 15 and 24 feet correspond the heights due to velocity $h_1 = \frac{c^2}{2g} = 3{,}49$ ft., and $h = \frac{v^2}{2g} = 8{,}928$ ft.; from this the mechanical work $P s = (h - h_1)\, G = 5{,}441 \times 6000 = 32646$ ft. lbs. If, now, the distance be known in which this change of velocity goes on, the force may be found; and when this is known, the distance may be determined. In this last case, for example, let the distance of the carriage amount to 100 feet, and, whilst describing this, the velocity passes from 15 into 24 feet; we have the force $P = (h - h_1)\frac{G}{s} = \frac{32646}{100} = 326{,}46$ lbs. Were the force itself 2000 lbs., the space s would be $= (h - h_1)\frac{G}{P} = \frac{32646}{2000} = 16{,}323$ feet.—3. If a 500 lbs. sledge has entirely lost, through friction on its path, its velocity of 16 feet, after describing a space of 100 feet, then is the resistance of friction $P = \frac{h G}{s} = 0{,}0155 \times 16^2 \times \frac{500}{100} = 0{,}0155 \times 256 \times 5 = 19{,}84$ lbs.

§ 72. The formula found for the work in the foregoing paragraph:

$$P s = (h - h_1)\, G$$

is not only good for constant, but also for variable forces, if, instead of P, the mean value of the force (from § 70) be introduced; for if the whole space (s) of motion be considered as consisting of equal and uniformly accelerated parts described $\left(\frac{s}{n}\right)$, then we have the amount of work for these:

$$P_1\left(\frac{s}{n}\right) = \frac{v_1^2 - c^2}{2g} G,$$

$$P_2\left(\frac{s}{n}\right) = \frac{v_2^2 - v_1^2}{2g} G,$$

$$P_3\left(\frac{s}{n}\right) = \frac{v_3^2 - v_2^2}{2g} G,$$

&c., in so far as v_1, v_2, v_3, &c., stand for the velocities acquired at the end of these parts of space; and by the addition of all these works we have the whole work required for the transformation of the velocity c into v:

$P s = (P_1 + P_2 + P_3 + \ldots)\frac{s}{n} = \frac{v^2 - c^2}{2g} G$, because for an infinite num-

* i. e. Working power.

ber (n) of forces $(P_1+P_2+P_3+\ldots)\div n$, it transforms itself into a mean force, and because the members on the right hand of the equation $\frac{v_1^2}{2g}G$ and $-\frac{v_1^2}{2g}G$, as also $\frac{v_2^2}{2g}G$ and $-\frac{v_2^2}{2g}G$, &c. are opposed to each other, so that the members $\frac{v^2}{2g}G$ and $\frac{c^2}{2g}G$, determined by the terminal velocity v and the initial velocity c, only remain.

The formula $Ps=\left(\frac{v^2-c^2}{2g}\right)G=(h-h_1)G$ is not used merely for the determination of the work, but not unfrequently, also, for the measurement of the terminal velocity. In the last case h is put $=h_1+\frac{Ps}{G}$ or $v=\sqrt{c^2+2g\frac{Ps}{G}}$. If by the constant motion of a body, the terminal velocity $v=$ the initial velocity c, the work done $=$ zero, *i. e.* as much work is performed by the accelerated, as is expended by the retarded part of the motion.

Example.—A carriage of 2500 lbs. proceeding upon a railroad without friction, has acquired by an augmentation of its velocity, which at the commencement amounted to 10 ft., a mechanical work of 8000 lbs., its velocity after this work will be:

$$v=\sqrt{10^2+64{,}4\cdot\frac{8000}{2500}}=\sqrt{100+206}=17{,}49 \text{ feet.}$$

Remark. The product of the mass $M=\frac{G}{g}$ and the square of the velocity (v^2): Mv^2 is called, without attaching to it any definite idea, the living force (*vis viva*) of the moved mass: and hereafter, the mechanical work which a moved mass acquires, may be put equal to half of the *vis viva* of the same. If a mass enters from a velocity c into another v, the work performed is equal to half the difference of the *vis viva* at the commencement and end of the change of velocity. This law of the mechanical performance of bodies by means of their inertia, is called the *principle of living forces*, or the *vis viva*.

§ 73. *Composition of Forces.*—Two forces P_1 and P_2 act upon one and the same body, in the same or in an opposite direction, the effect is the same as if only one force acted upon the body, which is the sum or difference of these forces; for these forces impart to the mass M the acceleration, $p_1=\frac{P_1}{M}$ and $p_2=\frac{P_2}{M}$, consequently from § 28, the acceleration resulting from both, is

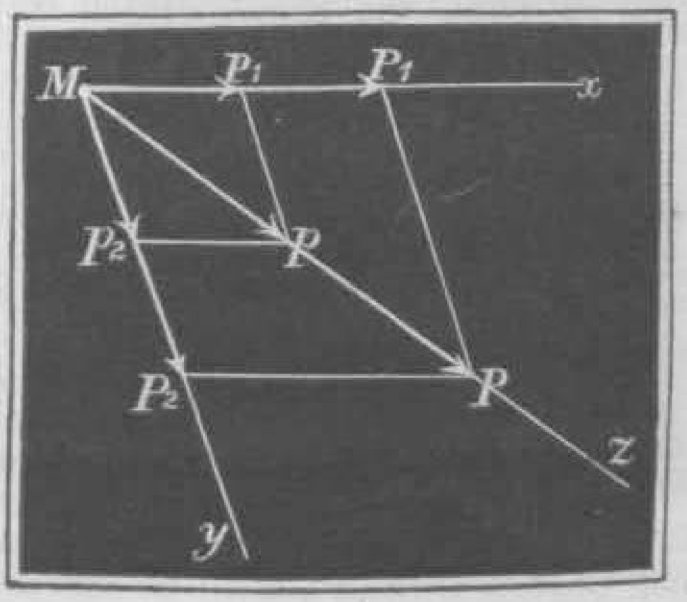
$p=p_1\pm p_2=\frac{P_1\pm P_2}{M}$, and accordingly the force corresponding to this, is: $P=Mp=P_1\pm P_2$.

The equivalent force P derived from these two is called the *resultant*; its constituents P_1 and P_2 the *components*.

Example.—1. A body lying flat upon the hand presses so long only upon it with its absolute weight as the hand is at rest, or is moved up and down uniformly with the body; but if the hand be raised quickly, it suffers a greater pressure; on the other hand, if it be suddenly dropped, the pressure is then less than the weight; it becomes null if the hand be drawn back with the acceleration of gravity. If the pressure on the hand $=P$, the body falls with a force $G-P$, whilst its mass $M=\frac{G}{g}$; if we put the acceleration with which the hand falls $= p$, $G - P = \frac{G}{g} p$, and therefore the pressure $P = G - \frac{p}{g} G = \left(1 - \frac{p}{g}\right) G$. If the body on the hand be raised with the acceleration p, $-p$ is then opposed to the acceleration g, therefore the pressure upon the hand $P = \left(1 + \frac{p}{g}\right) G$. According as a body ascends or descends with a 20 feet acceleration, the pressure upon the hand $= \left(1 - \frac{20}{32,2}\right) G = (1 - 0,62)\, G = 0,38$, of the weight of the body, or $= 1 + 0,62 = 1,62$.—2. If with the flat hand I throw a body of 3 lbs. 14 feet perpendicularly upwards, whilst I urge it on with the hand for the first 2 feet, the mechanical work performed is $P\, s = G\, h = 3 \times 14 = 42$ ft. lbs., and the pressure upon the hand, $P = \frac{42}{2} = 21$ lbs. Whilst the resting body presses with 3 lbs., it reacts upon the hand during the projection with 21 lbs.

§ 74. *Parallelogram of Forces.*—When a material point M, Fig. 30, is acted upon by two forces, P_1, P_2, whose directions MX and MY make, with each other, the angle $XMY = \alpha$, these lines generate the accelerations in these directions, $p_1 = \frac{P_1}{M}$ and $p_2 = \frac{P_2}{M}$, and from their union, there arises a mean acceleration (§ 34) in the direction MZ, both of which are given by the diagonal of a parallelogram formed from p_1, p_2, and the angle α; this mean or resultant acceleration $p = \sqrt{p_1^2 + p_2^2 + 2\, p_1 p_2 \cos. \alpha}$, and for the angle ϕ which its direction makes with $M\,X$ of the one acceleration p_1:

Fig. 30.



$$\sin. \phi = \frac{p_2 \sin. \alpha}{p}.$$

If we substitute in these formulæ the above values of p_1 and p_2:

$$p = \sqrt{\left(\frac{P_1}{M}\right)^2 + \left(\frac{P_2}{M}\right)^2 + 2\left(\frac{P_1}{M}\right)\left(\frac{P_2}{M}\right) \cos. \alpha} \text{ and}$$

$$\sin. \phi = \left(\frac{P_2}{M}\right) \frac{\sin. \alpha}{p}.$$

If we multiply the first equation by M,

$$M\, p = \sqrt{P_1^2 + P_2^2 + 2\, P_1\, P_2 \cos. \alpha}, \text{ or,}$$

since $M\,p$ is the force corresponding to the acceleration:

$$1.\ P = \sqrt{P_1^2 + P_2^2 + 2\, P_1\, P_2 \cos. \alpha}.$$

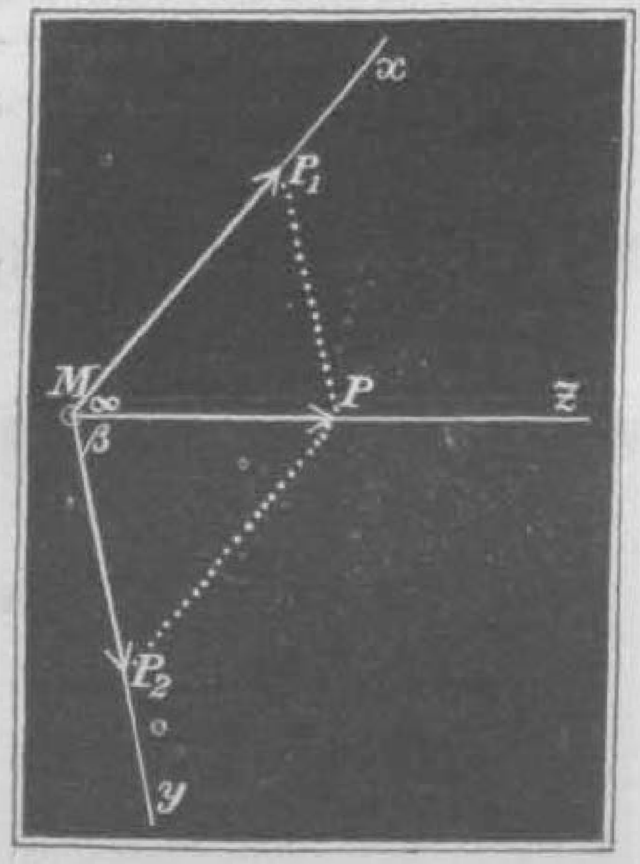
$$2.\ \sin. \phi = \frac{P_2 \sin. \alpha}{P}.$$

Thus, the resultant force is determined in magnitude and direction from the component forces exactly as the resultant acceleration from the component accelerations.

If we represent the forces by straight lines, and these lines be drawn, bearing the same proportions to each other as do weights, as pounds, &c., the mean force may be represented by the diagonal of the parallelogram whose sides are formed by the lateral forces, and one of whose angles is equal to that made by the directions of these lateral forces. The parallelogram which is constructed from the lateral forces, and whose diagonal is the mean force, is called the parallelogram of forces.

Example. When a body of 150 lbs. weight, resting upon a perfectly smooth table (Fig. 31) is acted upon by two forces $P_1 = 30$ lbs., and $P_2 = 24$ lbs, which make with each other an angle $P_1 M P_2 = \alpha + \beta = 105^\circ$: in what direction, and with what acceleration, will the motion take place? Since $cos.\ (\alpha + \beta) = cos.\ 105^\circ = -cos.\ 75^\circ$, the mean force:

Fig. 31.



$$P = \sqrt{30^2 + 24^2 - 2 \times 30 \times 24\ cos.\ 75^\circ}$$
$$= \sqrt{900 + 576 - 1440\ cos.\ 75^\circ}$$
$$= \sqrt{1476 - 372{,}7} = 33{,}21 \text{ lbs.},$$ the acceleration corresponding with it is:

$$p = \frac{P}{M} = \frac{Pg}{Gn} = \frac{33{,}21 \times 32{,}2}{150} = 7{,}1291 \text{ ft.}$$ The direction of motion makes with the direction of the first force an angle α, which is determined by:

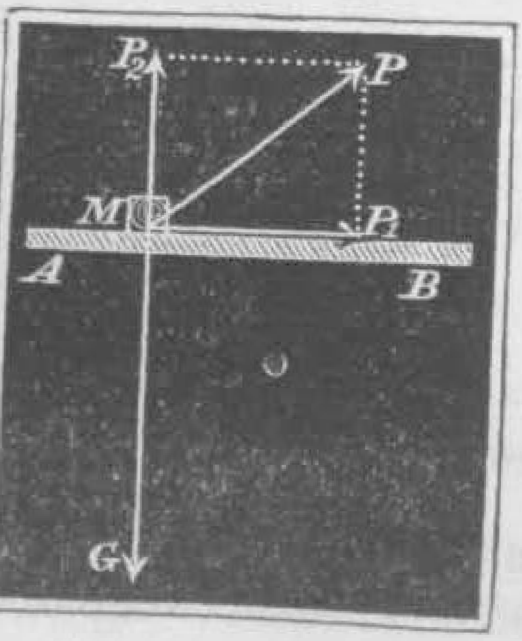
$$sin.\ \alpha = \frac{24}{33{,}22}\ sin.\ 105^\circ = 0{,}7224\ sin.\ 75^\circ = 0{,}6978, \text{ or } \alpha = 44^\circ, 15'.$$

Remark. The mean force P depends, from the formulæ found, only on the component forces, and not on the mass of the body upon which the forces act. For this reason, we find in many works on mechanics, the correctness of the parallelogram of forces proved without regard to the mass, but with the assumption of some fundamental law.

§ 75. *Resolution of Forces.*—By help of the parallelogram of forces, not only two or more forces may be reduced to a single one, but also given forces under given relations may be resolved into two or more forces. If the angles α and β are given, which the components $M P_1 = P_1$, and $M P_2 = P_2$, make with the given force $MP = P$, the components may be found from the formulæ:

$$P_1 = \frac{P\ sin.\ \beta}{sin.\ (\alpha + \beta)}, \quad P_2 = \frac{P\ sin.\ \alpha}{sin.\ (\alpha + \beta)}.$$

Fig. 32.



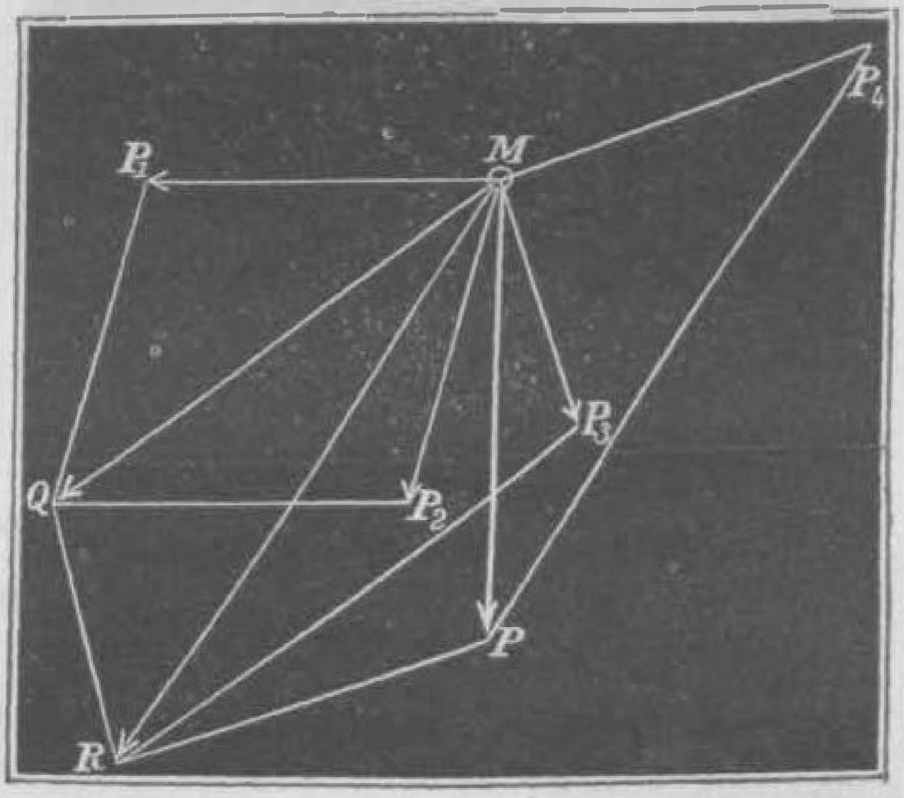
If the components are at right angles to each other, $\alpha + \beta = 90^\circ$, and $sin.\ (\alpha + \beta) = 1$, and $P_1 = P\ cos.\ \alpha$ and $P_2 = P\ sin.\ \alpha$. If β and α be equal to one other, $P_2 = P_1$, viza

$$P_2 = \frac{P\ sin.\ \alpha}{sin.\ 2\alpha} = \frac{P}{2\ cos.\ \alpha} = P_1.$$

Example 1. What is the pressure of a body M upon a table $A\ B$, Fig. 32, whose weight $G = 70$ lbs. and upon which a force $P = 50$ lbs. acts, and whose direction is inclined to the horizon at an angle $P\ M\ P_1 = \alpha = 40^\circ$? The horizontal component of P is $P_1 = P\ cos.\ \alpha = 50\ cos.\ 40^\circ = 38{,}30$ lbs, and the vertical component $P_2 = P\ sin.\ \alpha = 50\ sin.\ 40^\circ = 32{,}14$ lbs.; the latter strives to draw the body from the table, there remains then for the pressure $G - P_2 = 70 - 32{,}14 = 37{,}86$ lbs.—2. If a body of 110 lbs. is so moved along an horizontal way, by two forces, that it describes in the first second a space of 6,5 feet, in a direction which deviates from the two directions of force by an angle $\alpha = 52^\circ$ and $\beta = 77^\circ$, the forces themselves are given as follows. The acceleration is twice the space in the first second, so that $p = 2 \times 6{,}5 = 13$ ft. Now the mean force is $P = \frac{pG}{g} = 0{,}031 \times 13 \times 110 =$ 44,33 lbs., therefore the one component $P_1 = \frac{P \sin. 77^\circ}{\sin. (52^\circ + 77^\circ)} = \frac{44{,}33 \sin. 77^\circ}{\sin. 51^\circ} = 55{,}58$ lbs., and the other $P_2 = \frac{44{,}33 \sin. 52^\circ}{\sin. 51^\circ} = 45{,}59$ lbs.

§ 76. *Forces in a Plane.*—In order to find the mean force P for a system of forces P_1, P_2, P_3, &c., we may adopt exactly the same method (§ 33) as that followed in the composition of velocities, viz: by the repeated application of the parallelogram of forces, we may resolve them two and two and so on, till but a single force remains. The forces P_1 and P_2, for example, give from the parallelogram $M P_1 Q P_2$, the mean force $MQ = Q$, if this be joined to P_3, we have from the parallelogram $MQRP_3$, $MR = R$; and this last again forms a parallelogram with P_4 and gives the force $MP = P$ the last, and the resultant of the four forces P_1, P_2, P_3, P_4.

Fig. 33.



It is not necessary, in this way of composing forces, to complete the parallelogram, and draw its diagonal. We may form a polygon $MP_1 QRP$, whose sides MP_1, P_1Q, QR, RP, are parallel and equaleto the given components P_1, P_2, P_3, P_4, the last side MP completing the polygon will be the mean force sought, or rather its measure.

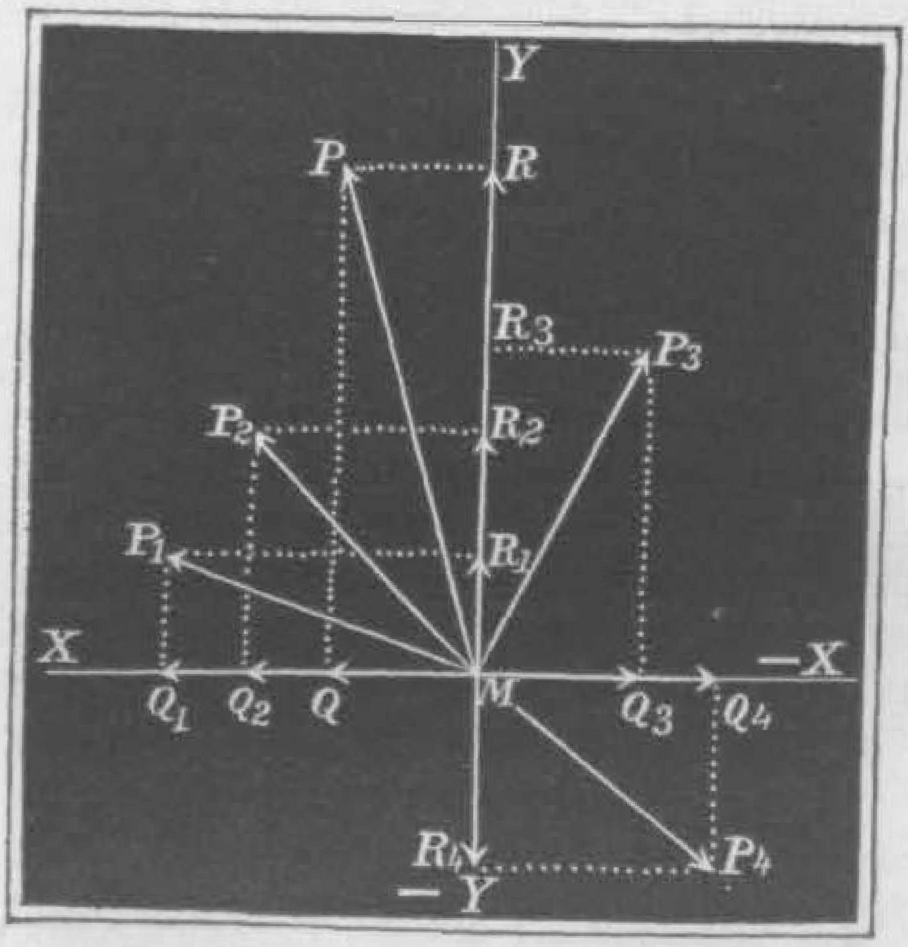
Remark. It is very nseful to solve mechanical problems by construction also: though this method does not admit of such accuracy as that of calculation, it is free on the other hand from great errors, and may therefore serve as proof of the calculation. In Fig. 33 the forces meet each other under the given angles $P_1 M P_2 = 72^\circ, 30'$; $P_2 M P_3 = 33^\circ, 20'$, and $P_3 M P_4 = 92^\circ, 40'$, and are so drawn that a pound is represented by a *line* or $\frac{1}{12}$ of a (Prussian*) inch. The forces $P_1 = 11{,}5$ lb., $P_2 = 10{,}8$ lbs., $P_3 = 8{,}5$ lbs., $P_4 = 12{,}2$ lbs, are therefore expressed by sides of 11,5 lines $= 0{,}958\ldots$ inches, 10,8 lines $= 0{,}900\ldots$ inches, 8,5 lines $= 0{,}708\ldots$ inches, 12,2 lines $= 1{,}016\ldots$ inches in length. A careful construction of the polygon of forces gives the magnitude of the mean force $P = 14{,}6$ lbs. and the variation of its direction MP from the direction MP_1 of the first force $= 86\frac{1}{2}^\circ$.

§ 77. The resultant P is determined more simply and clearly if each of the given components P_1, P_2, P_3, &c., be resolved according to two axial directions $\overline{XX}$ and $\overline{YY}$, Fig. 34, at right angles to each other, into component forces as Q_1 and R_1, Q_2 and R_2, Q_3 and

* The Prussian inch (see § 15) is equal 1.031 English inches.—Am. Ed.

R_3, &c., the forces lying in the same direction of axis, added together, and the resultants in magnitude and direction of these two rectangular forces be then sought for. If the angles $P_1 MX$, $P_2 MX$, $P_3 MX$, &c., which the directions of the forces P_1, P_2, P_3, make with the axis $\overline{X}X = \alpha_1$, α_2, α_3, &c., we have the components $Q_1 = P_1 \cos. \alpha_1$, $R_1 = P_1 \sin. \alpha_1$, $Q_2 = P_2 \cos. \alpha_2$, $R_2 = P_2 \sin. \alpha_2$, whence it follows from $Q = Q_1 + Q_2 + Q_3 + \ldots$,

Fig. 34.



1. $Q = P_1 \cos. \alpha_1 + P_2 \cos. \alpha_2 + P_3 \cos. \alpha_3 + \ldots$, and

from $R = R_1 + R_2 + R_3 + \ldots$,

2. $R = P_1 \sin. \alpha_1 + P_2 \sin. \alpha_2 + P_3 \sin. \alpha_3 + \ldots$

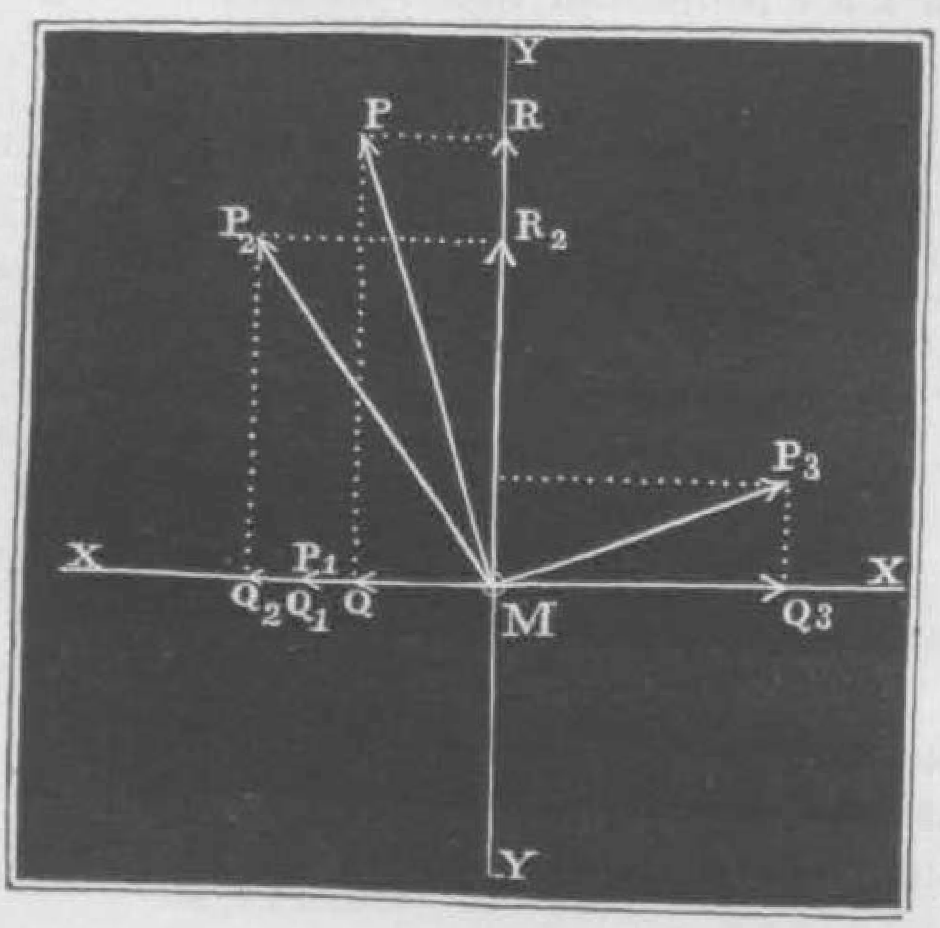
From the two compenents Q and R so found, the magnitude of the resultant sought, ise

3. $P = \sqrt{Q^2 + R^2}$ and the angle $PMX = \phi$, whose direction with $\overline{X}X$ is given by

4. $tang. \phi = \frac{R}{Q}$.

In the algebraical addition of the forces, regard must be had to the sign, for if it be different in two forces, *i. e.* if the directions of these be upon opposite sides of the point of application M, this addition then becomes arithmetical subtraction (§ 73). The angle ϕ is acute, as long as Q and R are positive; it is between one and two right angles, when Q is negative and R positive; between two and three, when Q and R are both negative, and lastly, between three and foue, when R only is negative.

Fig. 35.



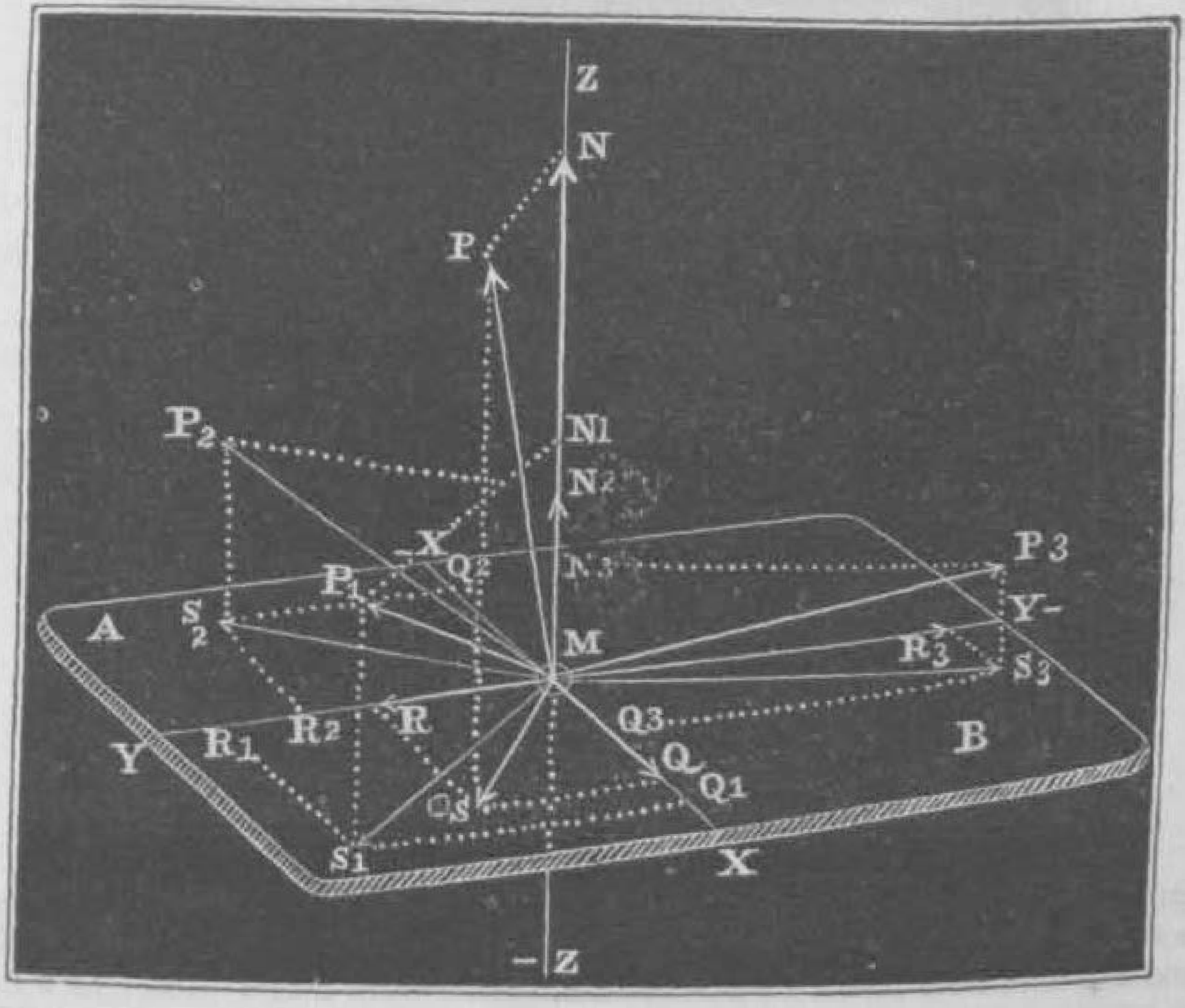
Example. What is the magnitude and direction of the resultant of the three components $P_1 = 30$ lbs., $P_2 = 70$ lbs., $P_3 = 50$ lbs., whose directions, lying in a plane, make between them the angles $P_1 M P_2 = 56°$ and $P_2 M P_3 = 104°$. If we draw the axis $\overline{X}X$

in the direction of the first force, we have $\alpha_1 = 0$, $\alpha_2 = 56°$, and $\alpha_3 = 56° + 104° = 160°$; hence, 1. $Q = 30 \times cos.\ 0° + 70 \times cos.\ 56° + 50 \times cos.\ 160° = 30 + 39{,}14 - 46{,}98 = 22{,}16$ lbs.; and 2. $R = 30 \times sin.\ 0° + 70 \times sin.\ 56° + 50\ sin.\ 160° = 0 + 58{,}03 + 17{,}10 = 75{,}13$ lbs. Hence, 3. $tang.\ \phi = \frac{75{,}13}{22{,}16} = 3{,}3903$; therefore, the angle which the resultant makes with the positive part of the axis MX or the force P_1 is $\phi = 73°\ 34'$; lastly, the force itself $P = \sqrt{Q^2 + R^2} = \frac{Q}{cos.\ \phi} = \frac{R}{sin.\ \phi} = \frac{75{,}13}{sin.\ 73°\ 34'} = \frac{75{,}13}{0{,}9591} = 78{,}33$ lbs.

§ 78. *Forces in Space.*—If the directions of the forces do not lie in one and the same plane, we must draw through the point of application a plane, and resolve each of the forces into two others, one lying in the plane, and the other at right angles to the plane; we must then find the resultant of the components so obtained in the plane, from the rule in the foregoing paragraph, and add together the components at right angles to the plane, and from the two rectangular components thus obtained, their resultant may be found according to the known rule (§ 74).

Fig. 36 puts the above mode of proceeding more clearly before us; let $MP_1 = P_1$, $MP_2 = P_2$, $MP_3 = P_3$ be the separate forces, AB the plane (of projection) and $Z\overline{Z}$ the axis at right angles to it. From the resolution of the forces P_1, P_2, &c., the forces S_1, S_2 are given in the plane, and those of N_1, N_2, &c., in the normal to it $Z\overline{Z}$. These are again resolved according to two axes $X\overline{X}$ and $Y\overline{Y}$ into the lateral

Fig. 36.



forces Q_1, Q_2, &c., R_1, R_2, &c., and give the components Q and R, of which the resultant S consists, which, joined to the sum of all the normal forces, N_1, N_2, &c., gives P the resultant required.

If we put β_1, β_2, for the angles at which the directions of force are inclined to the plane AB or to the horizon, the forces in the plane are given, $S_1 = P_1 \cos. \beta_1$, $S_2 = P_2 \cos. \beta_2$, &c., and the normal forces, $N_1 = P_1 \text{sine}\, \beta_1$, $N_2 = P_2 \sin. \beta_2$, &c.; lastly, if we designate the angles which the projections of the directions of the forces lying in the plane AB, make with the axis $X\overline{X}$, by α_1, α_2, we obtain the three following forces, forming the sides of a rectangular parallelopiped.

$$Q = S_1 \cos. \alpha_1 + S_2 \cos. \alpha_2 + S_3, \cos. \alpha_3, \text{ or}$$

1. $Q = P_1 \cos. \beta_1 \cos. \alpha_1 + P_2 \cos. \beta_2 \cos. \alpha_2 + \ldots,$
2. $R = P_1 \cos. \beta_1 \sin. \alpha_1 + P_2 \cos. \beta_2 \sin. \alpha_2 + \ldots$
3. $N = P_1 \sin. \beta_1 + P_2 \sin. \beta_2 + \ldots$

From these three follows the final resultant:

4. $P = \sqrt{Q^2 + R^2 + N^2}$, further

the angle of inclination to the plane of projection $PMS = \psi$, from

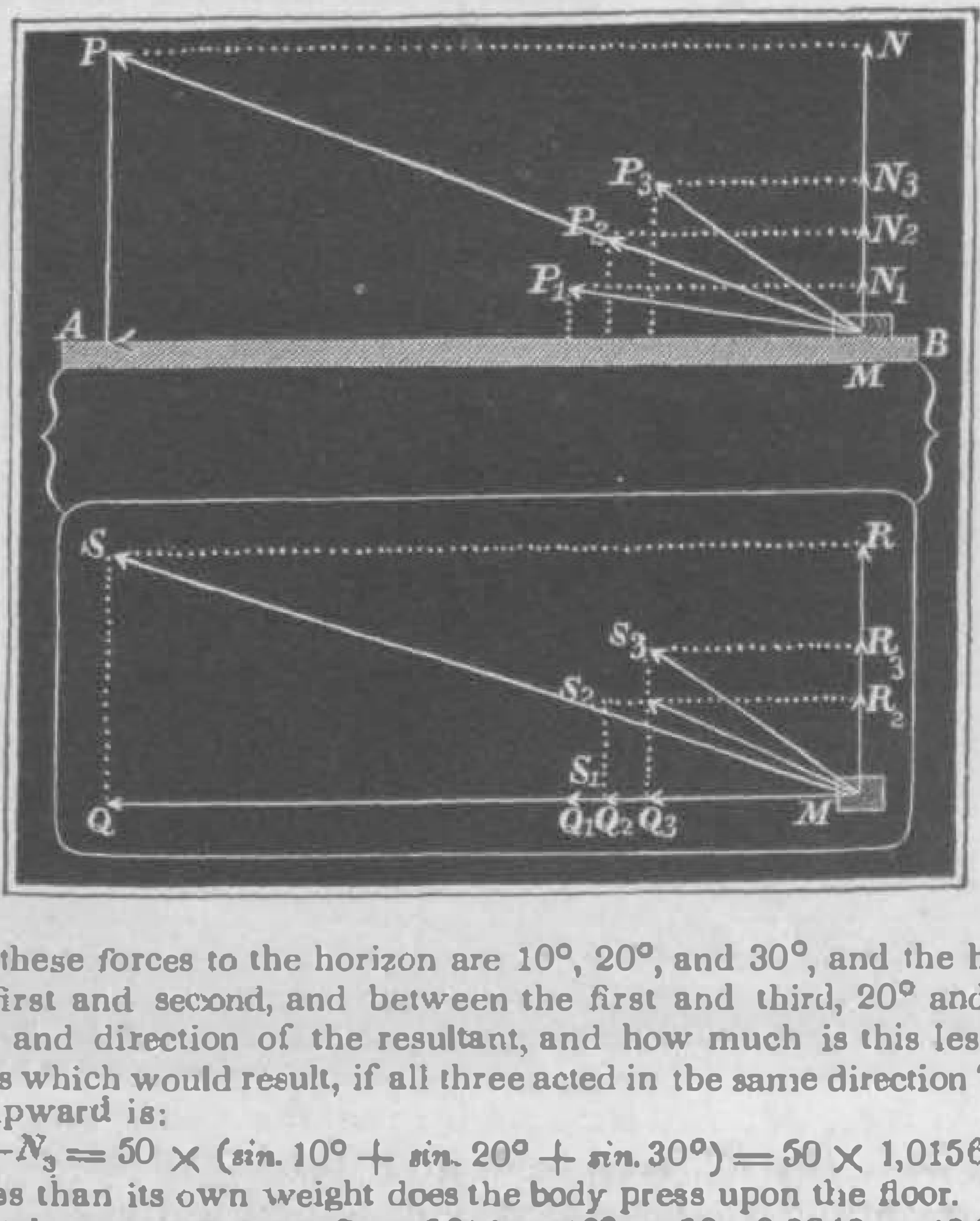
5. $\tang. \psi = \frac{N}{S} = \frac{N}{\sqrt{Q^2 + R^2}}$, lastly

the angle $SMX = \phi$, which the projection of the resultant in the plane AB makes with the first axis $X\overline{X}$, by

6. $\tang. \phi = \frac{R}{Q}$.

Example. Three workmen pull at the end of three ropes, which are attached to a load M lying upon a horizontal floor AB, Fig. 37, each with a force of 50 lbs.; the angles of

Fig. 37.



inclination of these forces to the horizon are 10°, 20°, and 30°, and the horizontal angle between the first and second, and between the first and third, 20° and 35°; what is the magnitude and direction of the resultant, and how much is this less than the sum of all the forces which would result, if all three acted in the same direction? The vertical force pulling upward is:

$N = N_1 + N_2 + N_3 = 50 \times (\sin. 10° + \sin. 20° + \sin. 30°) = 50 \times 1{,}01567 = 50{,}78$ lbs.; by so much less than its own weight does the body press upon the floor.

The horizontal components are $S_1 = 50 \times \cos. 10° = 50 \times 0{,}9849 = 49{,}24$ lbs.; $S_2 = 50$

$\times \cos. 20° = 46,98$ lbs.; $S_3 = 50 \times \cos. 30° = 43,30$ lbs. If we draw the axis $X\overline{X}$ in the direction of the first force S_1, we obtain the lateral force in this axis $X\overline{X}$, $Q = Q_1 + Q_2 + Q_3 = S_1 \cos. \alpha_1 + S_2 \cos. \alpha_2 + S_3 \cos. \alpha_3 = 49,24 \times \cos. 0° + 46,98 \times \cos. 20° + 43,30 \times \cos. 35° = 49,24 + 44,15 + 35,47 = 128,86$ lbs.; on the other hand, the lateral force in the second axis $Y\overline{Y}$: $R = R_1 + R_2 + R_3 = 49,24 \times \sin. 0° + 46,98 \times \sin. 20° + 43,30 \times \sin. 35° = 0 + 16,07 + 24,84 = 40,91$ lbs.

The horizontal mean force with which the body is drawn forward is from this:

$$S = \sqrt{Q^2 + R^2} = \sqrt{(128,86)^2 + (40,91)^2} = \sqrt{18278,7} = 135,2 \text{ lbs.}$$

The angle ϕ which this force makes with the axis $X\overline{X}$ is determined by the $\tang. \phi = \frac{R}{Q} = \frac{40,91}{128,86} = 0,3175$; $\phi = 17°,37'$; the entire resultant is:

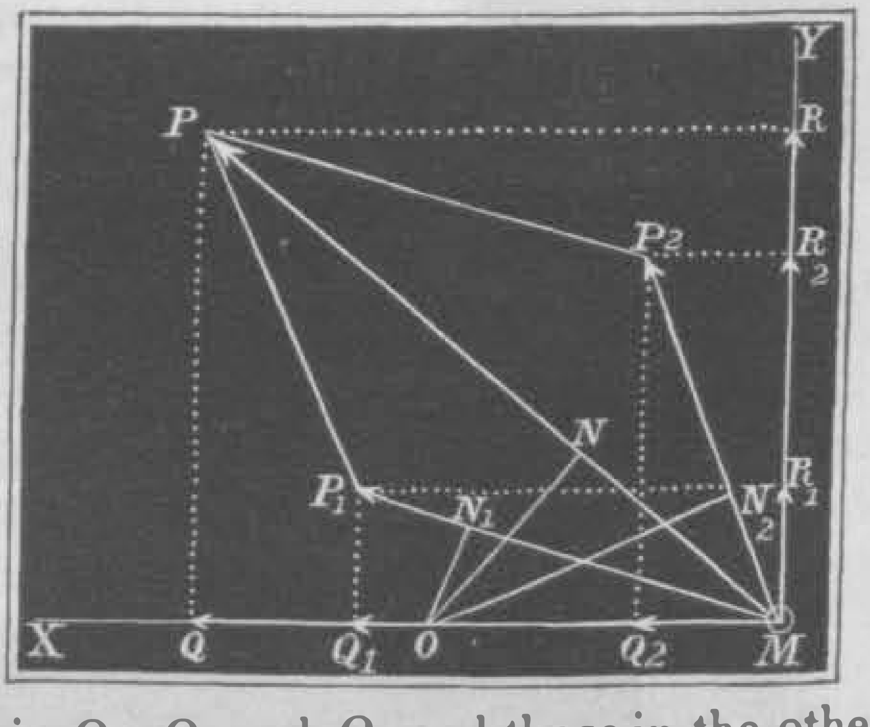
$$P = \sqrt{(135,2)^2 + (50,78)^2} = \sqrt{20856,6} = 144,42 \text{ lbs.}$$

If the forces act in the same direction, the resultant ist $= 3 \times 50 = 150$ lbs., and the loss of forcet $= 150 - 144,42 = 5,58$ lbs.; further, because the horizontal force drawing the body forwards amounts only to 135,20 lbs., we have, with reference to the horizontal motion, the loss of force $150 - 135,20 = 14,80$ lbs.

The angle of inclination ψ of the mean force to the horizon is determined by the $\tang. \psi = \frac{N}{S} = \frac{50,78}{135,20} = 0,3756$, wherefore ψ comes out $= 20°, 35'$.

§ 79.—From the rules found in the foregoing upon the composition of forces, two others of essential service for practical use may be deduced. In Fig. 37, let M be a material point, $MP_1 = P_1$ and $MP_2 = P_2$, the forces acting upon it; lastly, let $MP = P$, the resultant of P_1 and P_2. If we draw through M two axes, MX and MY, at right angles to each other, and resolve the forces P_1 and P_2, as well as their resultant P, into components in the direction of these axes, viz: P_1 into Q_1 and R_1, P_2 into Q_2 and R_2, and P into Q and R, we then obtain the forces in the one axis Q_1, Q_2 and Q, and those in the other R_1, R_2, R, and $Q = Q_1 + Q_2$, and $R = R_1 + R_2$.

Fig. 37.



If now we take in the axis MX any point O, and let fall from the same perpendiculars ON_1, ON_2 and ON on the directions of the forces P_1, P_2 and P we obtain rectangular triangles MON_1, MON_2, MON, which are similar to the triangles formed by the three forces, viz:

$$\triangle MON_1 \backsim \triangle MP_1Q_1$$
$$\triangle MON_2 \backsim \triangle MP_2Q_2$$
$$\triangle MON \backsim \triangle MPQ.$$

Principle of Virtual Velocities.—But from these similarities $\frac{MQ_1}{MP_1}$ i. e. $\frac{Q_1}{P_1} = \frac{MN_1}{MO}$, also $\frac{Q_2}{P_2} = \frac{MN_2}{MO}$ and $\frac{Q}{P} = \frac{MN}{MO}$; if we put the values hence derived of Q_1, Q_2, and Q into the equation $Q = Q_1 + Q_2$, we then obtain

$$P \,.\, MN = P_1 \,.\, MN_1 + P_2 \,.\, MN_2.$$

Likewise also $\frac{R_1}{P_1} = \frac{ON_1}{MO}$, $\frac{R_2}{P_2} = \frac{ON_2}{MO}$ and e$\frac{R}{P} = \frac{ON}{MO}$, therefore

$$P \,.\, ON = P_1 \,.\, ON_1 + P_2 \,.\, ON_2.$$

These equations still hold good, if P the mean force be made up of three or more forces P_1, P_2, P_3, because generally

$$Q = Q_1 + Q_2 + Q_3 + \ldots$$
$$R = R_1 + R_2 + R_3 + \ldots$$

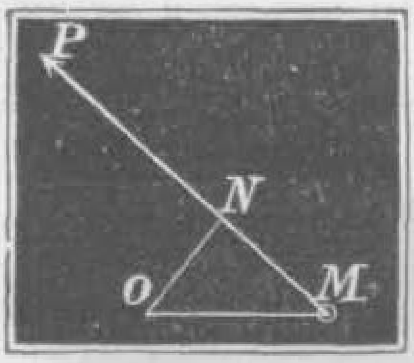
and, therefore, generally we may pute

1. $P \,.\, MN = P_1 \,.\, MN_1 + P_2 \,.\, MN_2 + P_3 \,.\, MN_3 + \ldots,$
2. $P \,.\, ON = P_1 \,.\, ON_1 + P_2 \,.\, ON_2 + P_3 \,.\, ON_3 + \ldots$

In both equations the mean force P must correspond to the forces P_1, P_2, P_3, and from these equations, not only the magnitude, but also the direction of this force may be determined.

§ 80. If the point of application M move in a straight line towards O, or if we imagine this point to have described the space $MO = s$, then the projection of this space $MN = s_1$ in the direction of the force MP is called the *space of the force P*, and the product Ps_1 of the force and its space, *the work or efficiency of* the force. If we substitute in the equation (1) of the last (§) these designations, we have

Fig. 38.



$$Ps = P_1 s_1 + P_2 s_2 + P_3 s_3 + \ldots,$$

or the *work, or mechanical effect, of the resultant is equivalent to the sum of the works, or mechanical effects, of the components.*

In the summation of the mechanical effects, as in that of the forces, we must have regard to their signs. If a force (Q_3) of the forces Q_1, Q_2, &c., of the last § acts in an opposite direction to the rest, we must

Fig. 39.

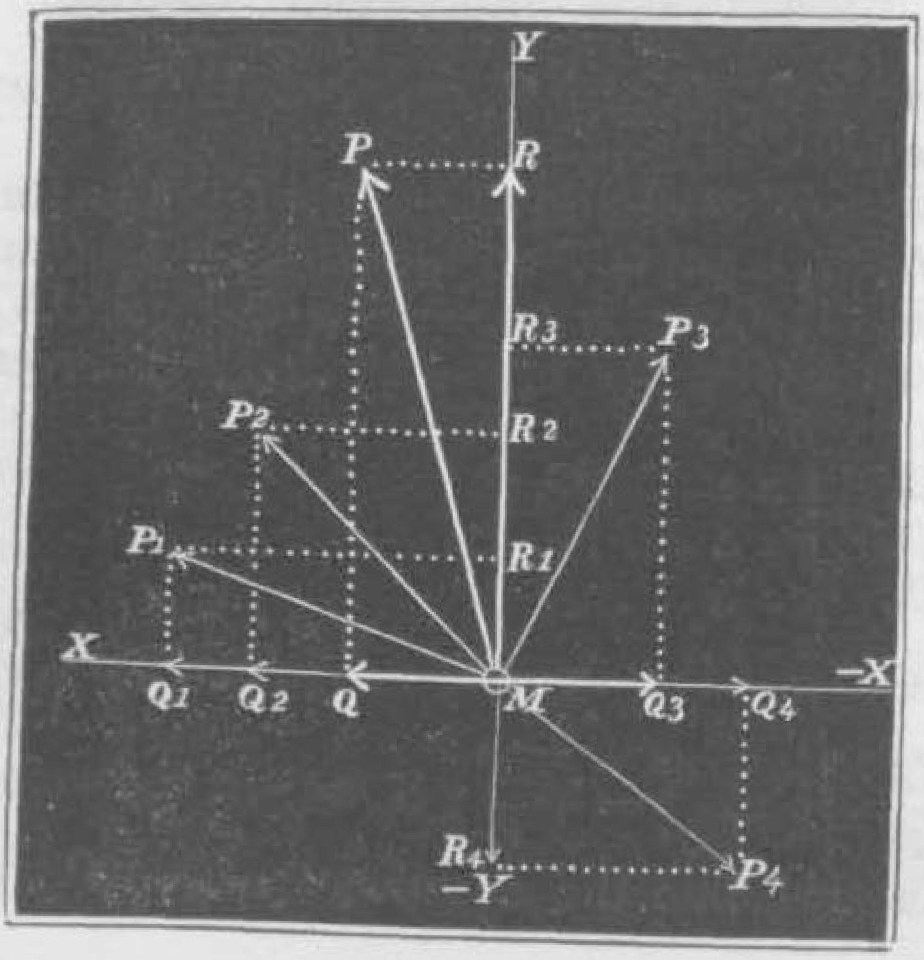


Fig. 40.

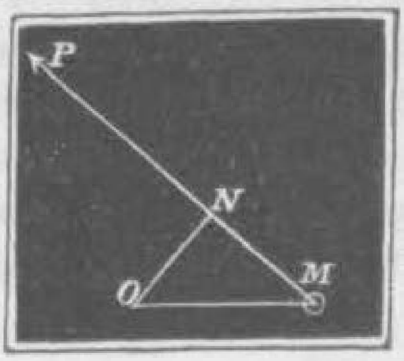
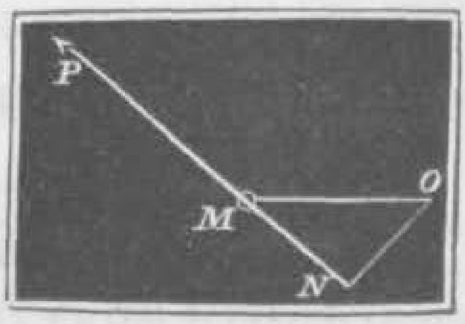


Fig. 41.



introduce it as negative, but this force Q_3, Fig. 39, is the component of a force P_3, which, acting in the circumstances set forth in the former §, opposed to their proper motion MN_3, we are, therefore, obliged to consider that force opposed to the motion MN, Fig. 40, as negative, and that one P, Fig. 41, acting in the direction of motion MN as positive.

If the forces are variable in magnitude or direction, the formula $Ps = P_1 s_1 + P_2 s_2 + P_3 s_3 + \ldots$ is only correct for infinitely small spaces s, s_1, s_2, &c.

The spaces of the forces $\sigma_1, \sigma_2, \sigma_3$, corresponding to an infinitely small displacement σ of a material point, are called their *virtual velocities;* and the law corresponding to the formula $P\sigma = P_1\sigma_1 + P_2\sigma_2 + P_3\sigma_3$, *the principle of virtual velocities.*

§ 81. *Transmission of Mechanical Effect.*—From the principle of vis viva, the mechanical effect (Ps) in rectilinear motion, which a force (P) generates in changing the velocity c of a mass M into another v is

$$Ps = \left(\frac{v^2 - c^2}{2}\right) M.$$

If P be now the mean force arising from other forces, P_1, P_2, &c., acting upon the mass M, and the spaces which these describe s_1, s_2, whilst the mass itself M describes s, we then have from the foregoing:

$$Ps = P_1 s_1 + P_2 s_2 + \ldots$$

and, therefore, the following general formula:

$$P_1 s_1 + P_2 s_2 + \ldots = \left(\frac{v^2 - c^2}{2}\right) M,$$

which expresses that the sum of the mechanical effects of the single forces is equal to half the gain of vis viva of the mass taking up these forces.

If the velocity during the motion be invariable, that is $v = c$, and the motion itself be uniform, we have then $v^2 - c^2 = 0$, consequently neither loss nor gain of vis viva, and, therefore:

$$P_1 s_1 + P_2 s_2 + P_3 s_3 + \ldots = 0,$$

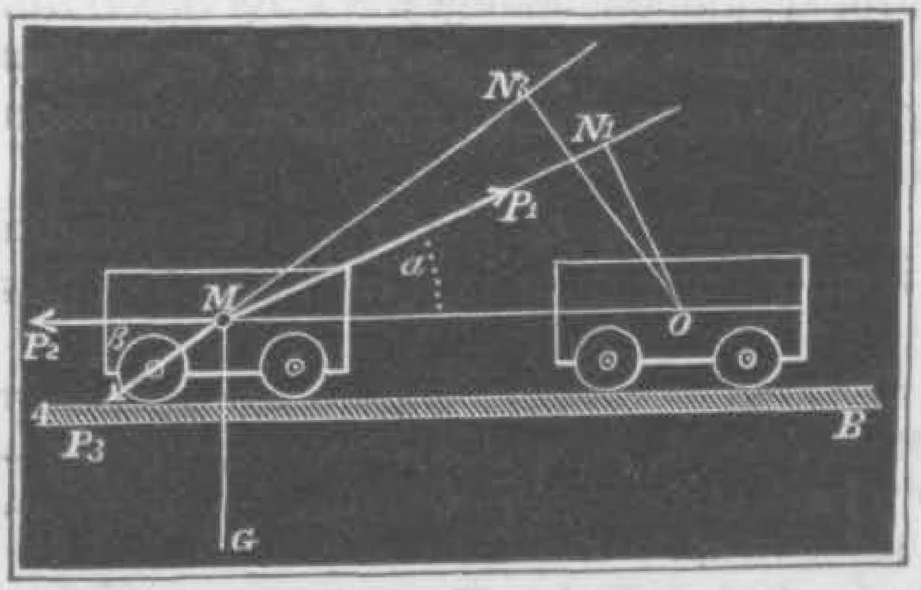
i. e. the sum of the mechanical effects of the single forces $= 0$.

If inversely the sum of the mechanical effects $= 0$, then the forces do not change the motion of the body in the given direction, nor impart to it in the given direction any motion which it had not before.

If the forces are variable, the variable velocity v after a certain time again passes into its initial velocity c, which takes place in all periodic motions as they present themselves in many machines. Now $v = c$ gives the effect $\left(\frac{v^2 - c^2}{2}\right) M = 0$; therefore within a period of the motion the loss or gain in mechanical effect is null.

Example. A carriage, of the weight $G = 6000$ lbs., Fig. 42, is moved forward upon a horizontal surface by means of a force $P_1 = 660$ lbs., ascending under an angle $\alpha = 24°$, and has during its motion two resistances to overcome; one, horizontal $P_2 = 350$ lbs., corresponding to the friction; and a resistance $P_3 = 230$ lbs., acting downwards, and inclined to the horizon at an angle $\beta = 35^\circ$. What work will the force (P_1) perform, in order to convert the two feet initial velocity of the carriage into a velocity of 5 feet?

Fig. 42.



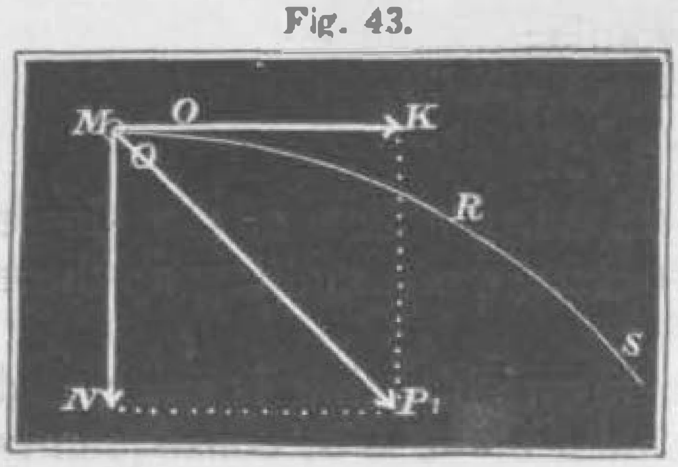
If we put the distance of the carriage $MO = s$, we then have for the work of the force $P_1 = P_1 . MN_1 = P_1\, s \cos.\, \alpha = 660 \times s \cos.\, 24^\circ = 602{,}94 \,.\, s$; further, the work of the resisting force $= (-P_2) \,.\, s = -350 \,.\, s$; lastly, the work of $P_3 = (-P_3 . MN_3 = -P_3\, s \cos.\, \beta = -230 \times s \cos.\, 35^\circ = -188{,}40 \,.\, s$. There then remains for the work of the effective force: $Ps = P_1\, s \cos.\alpha - P_2\, s \cos.\, 0 - P_3\, s \cos.\, \beta = (602{,}94 - 350 - 188{,}40) \,.\, s = 64{,}54 \,.\, s$ ft. lbs.

The mass, however, requires for the change of its velocity, the mechanical effect:
$\left(\frac{v^2-c^2}{2g}\right) G = \left(\frac{5^2-2^2}{2g}\right) \times 5000 = 0{,}0155 \times (25-4) \times 5000 = 1627$ ft. lbs.

If now we equate both mechanical effects, we then obtain $64{,}54 \,.\, s = 1627$, consequently the distance of the carriagee $s = \frac{1627}{64{,}54} = 25{,}26$ feet; and lastly, the mechanical effect of the force $P : P_1\, s \cos.\, \alpha = 602{,}94 \times 25{,}26 = 15230{,}2$ ft. lbs.

§ 82. *Curvilinear Motion.*—Provided that the spaces σ, σ_1, &c., be infinitely small, we may also apply the formula last found to curved paths. Let *MORS*, Fig. 43, be the path of a material point, and $MP_1 = P_1$ the resultant of all the forces acting upon it; if we resolve this force into two others, of which the one $MK = K$ is tangential, and the other $MN = N$ normal to the curve, we then term the one a *tangential*, and the other a *normal* force.

Fig. 43.



Whilst the material point describes the element $MO = \sigma$ of its curved path *MS*, and its velocity c is transformed into v_1, its mass M lays claim to the work $\left(\frac{v_1^2-c^2}{2}\right) M$, but the tangential force K performs at the same time the work $K\sigma$, and the normal force the work $N \,.\, 0 = 0$; consequently $K\sigma = \left(\frac{v_1^2-c^2}{2}\right) M$.

If the projection MQ of the elementary space MO in the direction of force be put $= \sigma_1$, then also $P_1\sigma_1 = K\sigma$; and, therefore,

$$P_1\sigma_1 = \left(\frac{v_1^2-c^2}{2}\right) M.$$

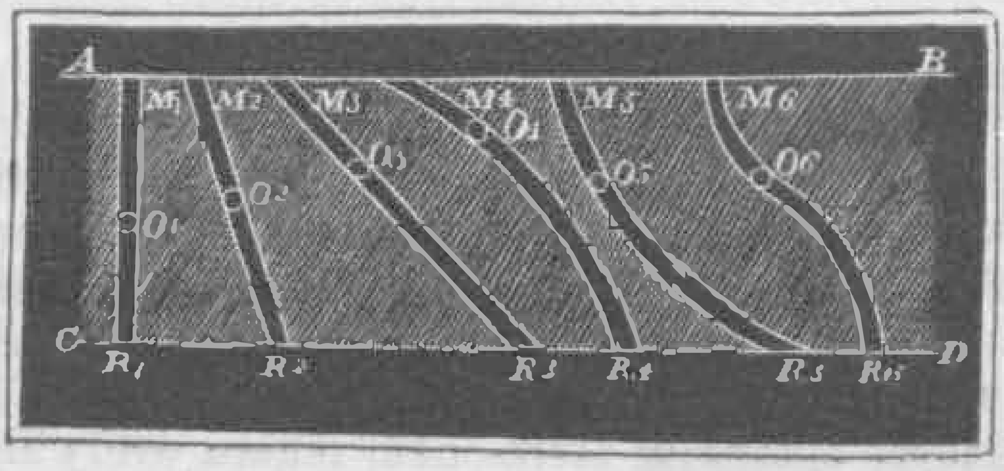
If the whole space described by the material point MR be decomposed into infinitely small parts, and each part be projected upon the direction of force at each moment, we then obtain the elementary space of the force at each moment, and the work at each moment by the multiplication of the space and force, and if we add together all these mechanical effects, we then havee $P_1\sigma_1 + P_2\sigma_2 + P_3\sigma_3 + \ldots = \left(\frac{v_1^2 - c^2}{2}\right) M + \left(\frac{v_2^2 - v_1^2}{2}\right) M + \left(\frac{v_3^2 - v_2^2}{2}\right) M + \ldots = \left(\frac{v^2 - c^2}{2}\right) M = (h - h_1) M$, if h_1 be the height due to the initial velocity c, and h that due to the terminal velocity v. Thus, *in curvilinear motion, the whole effect of the moving force is equal to half the gain of vis viva*, or equal to the product of the mass into the difference of the heights due to the velocities.

Remark and Example. The formula obtained which is derived from combining the principle of the vis viva with that of the virtual velocities, is especially applicable in cases where bodies are constrained by a fixed track or by suspension to describe a determinate path. If gravity alone act upon such a body, the work which it generates in a body of the weight G falling from a height corresponding to the vertical projection $M_1 R_1 = s$, is $= G s$, and therefore:

$$G s = (h - h_1) G, \text{ i.e. } s = h - h_1.$$

This is also the space which a body describes in falling from a horizontal plane AB, Fig. 44, to another CD; the difference of the heights due to the velocity is always equal

Fig. 44.



to the perpendicular height of fall; bodies which begin to describe the paths $M_1 O_1 R_1$, $M_2 O_2 R_2$, $M_3 O_3 R_3$, &c., with equal velocity (c), acquire at the end of these paths, as well as at different times, equal velocities (v). If the initial velocity $c = 10$ feet, and the vertical height of fall $s = 20$ feet, then $h = s + h_1 = 20 + 0{,}0155o\,10^2 = 21{,}55$ feet, andthe terminal velocity $v = \sqrt{2\,gh} = 8{,}02o\sqrt{21{,}5} = 37{,}18$ feet, in whatever curved or right line the descent may take place.