PRINCIPLES

OF

THE MECHANICS

OF

MACHINERY AND ENGINEERING.

SECTION I.

THE APPLICATION OF MECHANICS IN BUILDING.

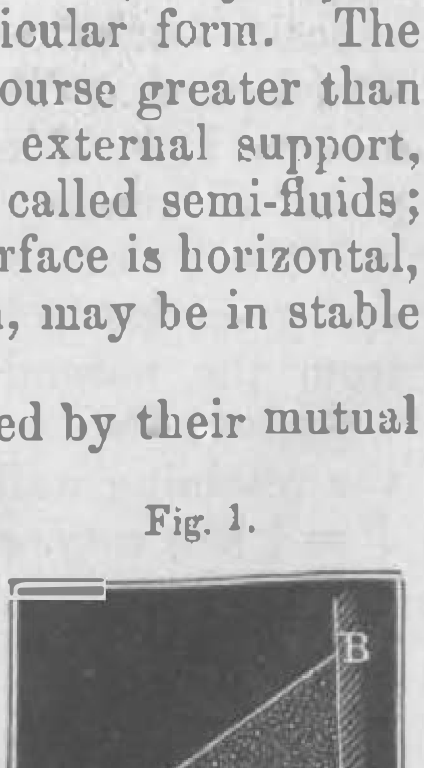
CHAPTER I.

OF THE EQUILIBRIUM AND PRESSURE OF SEMI-FLUIDS.

§ 1. *Sand, earth, corn seeds, shot, &c. &c.,* may be considered as *semi-fluids.*—They resemble fluids in so far as, like these, they require external support that they may preserve a particular form. The mutual adhesion of the parts of semi-fluids is of course greater than in the case of water. Water always requires external support, while this is only the more frequent case with so called semi-fluids; and whilst water is in equilibrium only when its surface is horizontal, the *disintegrated masses* or semi-fluids in question, may be in stable equilibrium, though their surface be inclined.

If the parts of a disintegrated mass be connected by their mutual friction alone, the mass will be in equilibrium when its surface is not inclined to the horizon at a greater angle than the *angle of repose* ρ (Vol. I. § 159). The *natural slope* of disintegrated masses is determined by the angle of repose. If by the *slope* of a declivity AB, Fig. 1, we understand the ratio $\frac{b}{a}$ of the base $AC = b$ to

Fig. 1.



the height $CB = a$, it is evidently $= cotang.\ \rho$, or, as $tang.\ \rho$ is equal to the *co-efficient* of friction f, $\frac{b}{a} = \frac{1}{f}$.

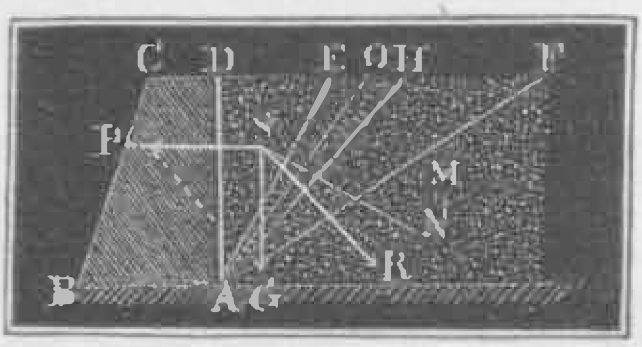
According to *Martony de Köszegh*, the natural slope of perfectly dry soil, for example, $= 1{,}243$, and for moist soil $= 1{,}083$. Hence, the angle of slope in the first case is $= 39°$, and in the second $43°$.

For very fine sand, the *slope* $= \frac{5}{3}$, therefore, the angle of slope $= 31°$. For rye seeds, the author found $\rho = 30°$, for fine shot $\rho = 25°$, and for the finest shot $\rho = 22\frac{1}{2}°$.

Remark. Experiments on the slope of disintegrated masses are made by heaping them up, and dressing them off from below upwards.

§ 2. *Pressure of Earth.*—If a disintegrated mass, such as earth, be supported by a retaining wall, it exerts a pressure (*poussée*) against it, a knowledge of which is of importance in practice. Suppose a body of earth M, Fig. 2, supported by a retaining wall AC, the *back* of which is vertical. Take as a first case that the earth and wall are the same height, and the earth's surface in no way extraneously loaded. Suppose that a wedge-shaped piece ADE separates from the general mass, and thus rests on the retaining wall on the one side, and on the earth on the other; put the height AD of the earth and wall $= h$, the density of the earth $M = \gamma$, and the angle AED which the surface of separation AE makes with the *horizontal* $= \phi$. Let us consider a length of the mass (at right angles to the plane of the figure) equal unity, then the weight of the wedge ADE:

Fig. 2.



$$G = \frac{AD \cdot DE}{2} \cdot 1 \cdot \gamma = \tfrac{1}{2}\, h \cdot h\, cotg.\ \phi \cdot \gamma = \tfrac{1}{2}\, h^2\, cotg.\ \phi \cdot \gamma.$$

The vertical back AD is acted upon by the pressure $SP = P$ at right angles to it; and, therefore, it may be assumed that an equal opposite horizontal pressure maintains the prism ADE on the inclined plane. We know also (Vol. I. § 159) that a force will be taken up by a body if its direction does not deviate from the normal to the plane of contact by more than the angle of repose, and we may, therefore, assume that the second component force R of G is taken up by the mass below AE, even supposing its direction to deviate from the normal SN by angle $RSN = \rho$. As $NSG = AED = \phi$, we have $RSG = \phi - \rho$, and, therefore, the horizontal pressure on the retaining wall $P = G\ tang.\ (\phi - \rho)$, (compare Vol. I. § 162), or $P = \frac{1}{2}\, h^2\, \gamma\ cotg.\ \phi\ tang.\ (\phi - \rho)$.

This force depends upon an unknown angle ϕ, or upon the dimensions of the prism of pressure, and is thus different for different values of ϕ, and a maximum for a certain value. If, now, ADE be the prism of greatest pressure, and ADO a prism exerting a less pressure, we have in AEO a prism which requires no force to maintain it on its basis, but which would rather require some force to pull it downwards. And so for other wedges AOH, &c., into which we might divide AEF, because these rest on still less inclinations; we may therefore assume, that by an opposite force equal to the maximum pressure P, not only the prism ADE, but also the prism below AE and AEF, is perfectly sustained, and that therefore this maximum pressure is that which the retaining wall is subjected to from the whole mass.

§ 3. *Prism of greatest Pressure.*—We must now determine the prism of maximum pressure. We have manifestly only to determine that value of ϕ for which *cotang.* ϕ *tang.* $(\phi - \rho)$ is a maximum.

Now *cotang.* ϕ *tang.* $(\phi - \rho) = \frac{\sin. (2\phi - \rho) - \sin. \rho}{\sin. (2\phi - \rho) + \sin. \rho}$, and as this fraction is greater, the greater $\sin. (2\phi - \rho)$ is, we shall have *cotang.* ϕ *tang.* $(\phi - \rho)$ a maximum when $\sin. (2\phi - \rho)$ is a maximum, that is $= 1$, or $2\phi - \rho = 90^\circ$, *i. e.* $\phi = 45^\circ + \frac{\rho}{2}$. Hence we name the pressure of the earth against the retaining wall:

$$P = \tfrac{1}{2} h^2 \gamma \, cotang. \left(45^\circ + \frac{\rho}{2}\right) tang. \left(45^\circ - \frac{\rho}{2}\right),$$

or since $cotang. \left(45^\circ + \frac{\rho}{2}\right) = tang. \left(45^\circ - \frac{\rho}{2}\right)$,

$$P = \tfrac{1}{2} h^2 \gamma \left[tang. \left(45^\circ - \frac{\rho}{2}\right)\right]^2.$$

The complement of $\phi = 45^\circ + \frac{\rho}{2}$, is $DAE = 45^\circ - \frac{\rho}{2}, = \frac{90^\circ - \rho}{2}$ $=$ one half of DAF the complement to 90° of the angle of friction ρ. Therefore the surface AE of the prism of pressure bisects the angle DAF which the natural slope AF makes with the vertical AD. We can now very easily compare the pressure of a disintegrated or semi-fluid mass with that of water. In the latter the pressure is $\frac{1}{2} h^2 \gamma_1$ (Vol. I. § 276), when $h =$ height, $1 =$ breadth of the pressed surface. In the case of earth, on the other hand, we have the pressure

$$P = \tfrac{1}{2} h^2 \varepsilon \gamma_1 \left[tang. \left(45^\circ - \frac{\rho}{2}\right)\right]^2,$$

where $\gamma_1 =$ the density of water, and ε the specific gravity of the semi-fluid. Hence the pressure of earth is always $\varepsilon \left[tang. \left(45^\circ - \frac{\rho}{2}\right)\right]^2$ times as great as the pressure of water, or the pressure of a semi-fluid may be set as equal to the pressure of perfect fluid of specific gravity $\varepsilon \left[tang. \left(45^\circ - \frac{\rho}{2}\right)\right]^2$.

Thus we see that the pressure of earth increases gradually from the surface downwards, or is proportional to the pressure height.

It follows, likewise, that the *centre of pressure* of earth-works, &c. &c., coincides with the centre of pressure of water, and that, therefore, in the case in question, where the surface is a rectangle, it is at one-third of the height h from the base (Vol. I. § 278).

Example. If the specific gravity of a mass of corn seeds, heaped 6 feet high, be 0,776 (Vol. I. § 291, remark 1), it exerts a pressure against each foot in length of a vertical wall: $P = \frac{1}{2} . 6^2 . 0{,}776 Λ 63Λ \left[\left(tang.\ 45^\circ - 15^\circ\right)\right]^2 = 18 . 63Λ 0{,}776\ (tang.\ 30^\circ)^2 =$ $880 \times 0{,}5773{,}5^2 = 293\frac{1}{2}$ lbs. (English.)

§ 4. *Cohesion of Semi-fluids.*—In the above investigations we have omitted to consider the *cohesion*, or that mutual union of the parts of the mass, increasing with the surface of contact. As this cohesion, however, in the case of the less disintegrated masses, as, for instance, in well compacted earth, is not unimportant, we shall now introduce it into the formula. Let us put the *modulus of cohesion*, or the force of union for the unit of surface of contact $= \varkappa$, we have for the case shown in Fig. 2, the force required to separate the prism ADE on the surface

$$AE, = 1 . AE g \varkappa = \frac{\varkappa h}{sin.\ \phi}.$$

The vertical component $\frac{\varkappa h}{sin.\ \phi}\ sin.\ \phi = \varkappa\ h$ counteracts gravity, and the horizontal component $\frac{\varkappa h}{sin.\ \phi} . cos.\ \phi = \varkappa\ h\ cotg.\ \phi$, counteracts the pressure. If, therefore, we introduce into the formula $P = G\ tang.\ (\phi - \rho)$, instead of P, $P + \varkappa\ h\ cotang.\ \phi$, and instead of G, $G - \varkappa\ h$, we then obtain the equation:

$$P = (G - \varkappa\ h)\ tang.\ (\phi - \rho) - \varkappa\ h\ cotang.\ \phi.$$

If again we substitute $G = \frac{1}{2}\ h^2\ \gamma\ cotang.\ \phi$, we have:

$$P = (\tfrac{1}{2}\ h^2\ \gamma\ cotang.\ \phi - \varkappa\ h)\ tang.\ (\phi - \rho) - \varkappa\ h\ cotang.\ \phi.$$

It is, however, convenient to make the following transformations in this formula.

$$P = h\ [(\tfrac{1}{2}\ h\ \gamma + \varkappa\ cotang.\ \rho)\ cotg.\ \phi\ tang.\ (\phi - \rho) - \varkappa\ cotang.\ \phi - \varkappa\ (1 + cotg.\ \phi\ cotg.\ \rho)\ tang.\ (\phi - \rho)],$$

or, as $tang.\ (\phi - \rho) = \frac{tang.\ \phi - tang.\ \rho}{1 + tang.\ \phi\ tang.\ \rho}$

$$= \frac{tang.\ \phi - tang.\ \rho}{1 + cotang.\ \phi\ cotang.\ \rho} . cotang.\ \phi . cotang.\ \rho,$$

we have $P = h\ [(\frac{1}{2}\ h\ \gamma + \varkappa\ cotang.\ \rho)\ cotg.\ \phi\ tang.\ (\phi - \rho)$ $- \varkappa\ (cotg.\ \phi + cotg.\ \rho - cotg.\ \phi)]$, hence

$$P = h\ [(\tfrac{1}{2}\ h\ \gamma + \varkappa\ cotg.\ \rho)\ cotg.\ \phi\ tang.\ (\phi - \rho) - \varkappa\ cotang.\ \rho].$$

This force becomes a maximum when the product $cotang.\ \phi\ tang.\ (\phi - \rho)$ is a maximum, and as we have seen this latter is so, when $\phi = 45^\circ + \frac{\rho}{2}$; therefore, the entire horizontal pressure of the earth against the wall:

$$P = h \left[(\tfrac{1}{2}\ h\ \gamma + \varkappa\ cotg.\ \rho) \left[tang.\left(45^\circ - \frac{\rho}{2}\right)\right]^2 - \varkappa\ cotang.\ \rho\right]$$

$$= \tfrac{1}{2}\ h^2\ \gamma \left[tang.\left(45^\circ - \frac{\rho}{2}\right)\right]^2$$

$$- \varkappa\ h\ cotang.\ \rho \left[1 - \left[tang.\left(45^\circ - \frac{\rho}{2}\right)\right]^2\right]$$

or as $cotang.\,\rho = \dfrac{2}{tang.\left(45^\circ + \frac{\rho}{2}\right) - tang.\left(45^\circ - \frac{\rho}{2}\right)}$, and

$$1 - \left[tang.\left(45^\circ - \frac{\rho}{2}\right)\right]^2$$

$$= \left[tang.\left(45^\circ + \frac{\rho}{2}\right) - tang.\left(45^\circ - \frac{\rho}{2}\right)\right] tang.\left(45^\circ - \frac{\rho}{2}\right),$$

$$P = \tfrac{1}{2}\, h^2\, \gamma \left[tang.\left(45^\circ - \frac{\rho}{2}\right)\right]^2 - 2\, h\, \varkappa\, tang.\left(45^\circ - \frac{\rho}{2}\right)$$

$$= h\, tang.\left(45^\circ - \frac{\rho}{2}\right)\left[\frac{h\gamma}{2}\, tang.\left(45^\circ - \frac{\rho}{2}\right) - 2\, \varkappa\right].$$

This force is 0 for $\frac{1}{2}\, h_1\, \gamma\, tang.\left(45^\circ - \frac{\rho}{2}\right) = 2\, \varkappa$, that is,

$$\text{for } h_1 = \frac{4\, \varkappa}{\gamma\, tang.\left(45^\circ - \frac{\rho}{2}\right)}.$$

For this height, therefore, a coherent mass may be cut vertically, and should continue so to stand. Inversely, from the height h_1 of the vertical face of any soil, we may deduce the modulus of cohesion, for

$$\varkappa = \tfrac{1}{4}\, h_1\, \gamma\, tang.\left(45^\circ - \frac{\rho}{2}\right).$$

Therefore, the cohesion of a mass is greater or less, according to the height h_1 for which it maintains a vertical face.

If we introduce h_1 into the expression for P, we obtainn

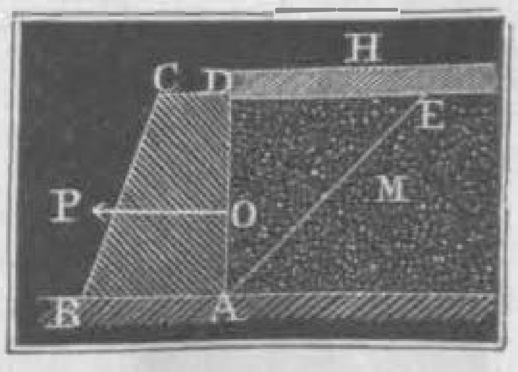
$$P = \frac{h\gamma}{2}\,(h - h_1)\left[tang.\left(45^\circ - \frac{\rho}{2}\right)\right]^2.$$

For sand, seeds, shot, and for newly turned soils, h_1 is very small; for compressed compact soils, it is sometimes considerable; for disintegrated, moist earth, Martony found $h_1 = 0{,}9$ feet, whilst, for the same material soaked with water, $h_1 = 0$. According to circumstances, a vertical face of from 3 to 12 feet maintains itself in different soils.

In most cases of practical application, it is advisable to omit the effects of cohesion.

§ 5. *Surcharged Masses of Earth.*—If the earth-work M, Fig. 3, be loaded on the surface, with buildings or otherwise, as DEH, the retaining wall undergoes an increased pressure. To determine this increased pressure, let us put the pressure on each square foot of the horizontal surface $= q$, then the pressure on the surface for $ADE = q\,.\,DE = q\, h\, cotang.\,\phi$, and, therefore, the horizontal pressure, without reference to cohesion

Fig. 3.



$$P = (G + q\, h\, cotang.\,\phi)\; tang.\,(\phi - \rho)$$

$= (\frac{1}{2} h^2 \gamma + q h) \cdot cotg. \phi \; tang. (\phi - \rho)$, or as $\phi = 45^\circ + \frac{\rho}{2}$

$$P = (\tfrac{1}{2} h^2 \gamma + qh) \left[tang. \left(45^\circ - \frac{\rho}{2}\right)\right]^2.$$

To find the point of application of this force, we must decompose it into its two parts $\frac{1}{2} h^2 \gamma \left[tang. \left(45^\circ - \frac{\rho}{2}\right)\right]^2$ and $q h \left[tang. \left(45^\circ - \frac{\rho}{2}\right)\right]^2$. The first part has its point of application at $\frac{1}{3}$ of the height h above the base A, and, therefore, its statical moment referred to this point:

$$= \frac{h}{3} \cdot \tfrac{1}{2} h^2 \gamma \left[tang. \left(45^\circ - \frac{\rho}{2}\right)\right]^2 = \frac{h^3 \gamma}{6} \left[tang. \left(45^\circ - \frac{\rho}{2}\right)\right]^2;$$

By the second part, however, equal portions of the vertical wall are equally pressed, and therefore the resultant pressure of this part passes through the centre of gravity of the wall, and acts at half the height $\frac{h}{2}$ from the base. Hence the statical moment of the second force

$$= \frac{h}{2} \cdot qh \left[tang. \left(45^\circ - \frac{\rho}{2}\right)\right]^2 = \frac{q h^2}{2} \left[tang. \left(45^\circ - \frac{\rho}{2}\right)\right]^2.$$

The moment of the entire pressure is thus:

$(\frac{1}{6} h^3 \gamma + \frac{1}{2} q h^2) \left[tang. \left(45^\circ - \frac{\rho}{2}\right)\right]^2$, and, therefore, the leverage of the force, or the distance $AO = a$ of its point of application O from the base,

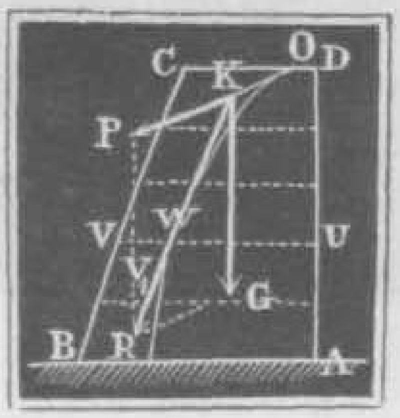
$$a = \frac{(\frac{1}{6} h^3 \gamma + \frac{1}{2} h^2 q) \left[tang. \left(45^\circ - \frac{\rho}{2}\right)\right]^2}{(\frac{1}{2} h^2 \gamma + h q) \left[tang. \left(45^\circ - \frac{\rho}{2}\right)\right]^2} = \frac{\frac{1}{6} h^2 \gamma + \frac{1}{2} h q}{\frac{1}{2} h \gamma + q}$$

$$= \left(\frac{h \gamma + 3 q}{3 h \gamma + 6 q}\right) . h.$$

Remark.—If the earth be carried above the cope of the wall, and form from it a natural slope, the formula of § 3 is still applicable, if h be put equal to the height of the earth, and not that of the wall.

§ 6. *Retaining Walls.*—The pressure of earth has often, in engineering, to be withheld by retaining walls (Fr. *revêtements*), or by *walling-timbers*, and *sheet-piling*. Retaining walls of masonry are most usual, and we shall, therefore, here treat of these in greater detail.

Fig. 4.



A wall AC, Fig. 4, may be either pushed forward, or turned over by a force $KP = P$. If we suppose this wall composed of horizontal courses of stone bedded on each other, we may assume that, should the wall give way, a horizontal crack will form, upon which the upper part CU slides forward or turns about. For security we shall neglect the effects of mortar, and take only the friction between the beds into consideration. From the force P, and the weight G of the part CU of the wall, there results a force $KR = R$ upon the magnitude and direction of which the possibility of an overturn or sliding forward of this part of the wall depends. If the angle RKG, by which this resultant deviates from the normal to the plane of separation UV, be less than the angle of friction ρ, the wall cannot slide forward (Vol. I. § 159); and if the direction of the resultant pass within the *joint* or plane of separation, then rotation about the axis V is not possible (Vol. I. § 130).

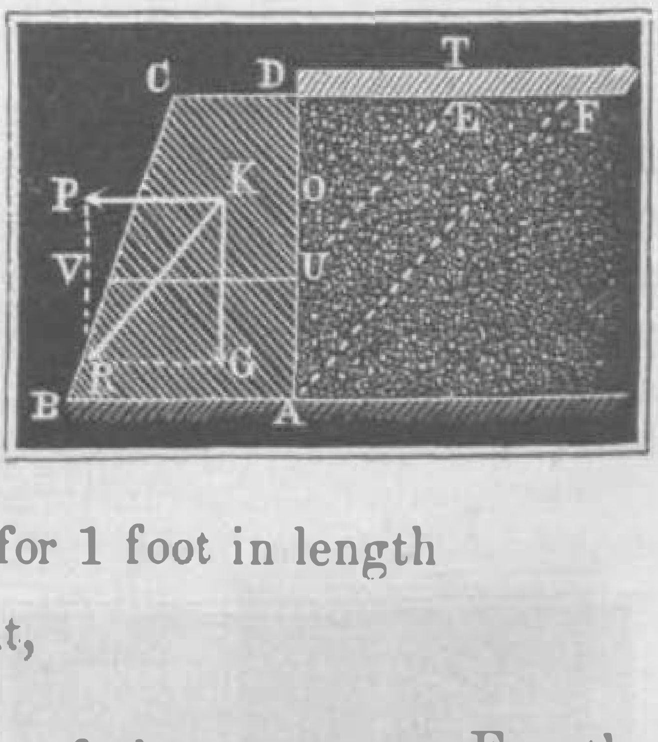
In most cases of application it will be found that rotation more readily takes place than sliding, and therefore, in building retaining walls, provision against the former has to be made. Rotation, or *heeling* is the more apt to occur, in as much as it not unfrequently takes place, not about the axis V, but about a point V_1 nearer the resultant R; because the pressure concentrated in V, compresses or breaks the stone near the point V.

If the points of intersection W, for a series of resultants R passing through the joints, be found, and a line drawn through these, we have what is termed the *line of resistance*, and it is easy to perceive that an overturn of the wall cannot take place, so long as this line does not pass beyond the joints of the wall.

If the force P, which the wall has to withstand, deviates in direction from the vertical more than the angle ρ, there can be no question of its sliding, because the resultant of P and G always makes a smaller angle with the vertical than P alone.

§ 7. *Slipping of Walls.*—If we substitute for P the pressure of earth found above, we can determine the thickness, having which, a wall will be sufficient to withstand this pressure. Let us consider, in the first place, the case of slipping for a wall AC, Fig. 5. Suppose that the earth-work pushes forward the part UC, on the joint UV. If we put the thickness at top of wall $CD = b$, the relative batter $= n$, and the height $DU = x$, we have the thickness:

Fig. 5.



$UV = b + n\,x$, and the contents of UC for 1 foot in length $= b\,x + \frac{n\,x^2}{2}$, and, therefore, the weight,

$G = \left(b + \frac{nx}{2}\right) x\,\gamma_1$; γ_1 being the density of the masonry. For the pressure of the earth on DU, we have generally

$$P = (\tfrac{1}{2}\,x^2\,\gamma + q x)\left[tang.\left(45^\circ - \frac{\rho}{2}\right)\right]^2;$$

and hence, for the angle $RKG = \phi$ made by the resultant R with the vertical:

$$tang.\,\phi = \frac{P}{G} = \frac{\frac{1}{2}x^2\gamma + q\,x}{\left(b + \frac{nx}{2}\right)x\,\gamma_1} \cdot \left[tang.\left(45^\circ - \frac{\rho}{2}\right)\right]^2;$$ or as ϕ must be less than ρ, therefore, $tang.\,\phi < f$,

$$\frac{\frac{1}{2}x\gamma + q}{\left(b + \frac{nx}{2}\right)\gamma_1} \cdot \left[tang.\left(45^\circ - \frac{\rho}{2}\right)\right]^2 < f,$$ from which we have the thickness of wall:

$$b > \frac{\frac{1}{2}x\gamma + q}{f\gamma_1}\left[tang.\left(45^\circ - \frac{\rho}{2}\right)\right]^2 - \frac{nx}{2}.$$

For $x = 0$ we have the thickness at the toph

$b > \frac{q}{f\gamma_1}\left[tang.\left(45^\circ - \frac{\rho}{2}\right)\right]^2$, therefore, for $q = 0$, we have $b = 0$; for $x = h$, the whole height of wall, the thickness is:

$$b > \frac{\frac{1}{2}h\gamma + q}{f\gamma_1}\left[tang.\left(45^\circ - \frac{\rho}{2}\right)\right]^2 - \frac{nh}{2}.$$

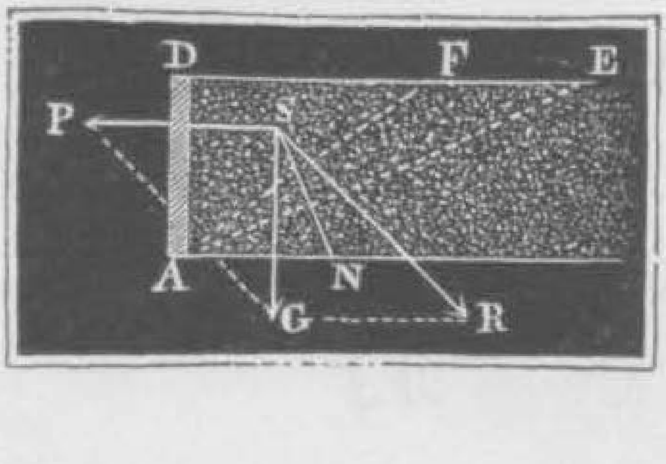
To apply this formula to a dyke or dam, we must put $\rho = 0$, and $q = 0$; then we get $b > \left(\frac{\gamma}{f\gamma_1} - n\right)\frac{x}{2}$, (Vol. I. § 280).

The formulas give for $q = 0$, (that is, when the surface of the fluid or semi-fluid reaches to the top of the wall), the breadth at top $= 0$; but experience has proved that the thickness here should rarely be less than 2 feet, and in positions liable to wear and tear, always above this dimension.

Remark. The co-efficient of friction for stones and bricks in contact with each other (Vol. I. § 161), is from 0,67 to 0,75. And when a bed of fresh mortar is interposed, only 0,60 to 0,70. Mortar once *set*, acts by cohesion or adhesion, and, according to Boistard, the cohesion of mortar is from 800 to 1500 lbs. per square foot. According to Morin, this amounts to from 2000 to 5000 lbs.

§ 8. *Abutting Resistance of Earth.*—We must distinguish between the *active* and *passive* pressure of the earth. In the cases hitherto considered, the pressure is active, pressing against a passive resistance. The pressure of earth, however, becomes passive when it opposes an active force as resistance, as when it resists the thrust of an arch, &c. &c. Poncelet has termed this effect of earth-works *butée des terres* (German, *Hebekraft der Erde*), and Moseley has termed it *resistance of earth.* The resistance which a body opposes to being pushed up an inclined plane, is greater than the force necessary to prevent the sliding of the body down the inclined plane, and just so, in the case of disintegrated masses, the resistance which they oppose to a vertical surface, moved horizontally, is

Fig. 6.



greater than the force with which they press against a vertical plane at rest. Whilst we have above (Vol. I. § 162) put the latter force, $P = G$ *tang.* $(\phi - \rho)$, the resistance of the latter must be set $P = G$ *tang.* $(\phi + \rho)$, or, as G is the weight $\frac{1}{2}\, h^2\, \gamma$ *cotang.* ϕ of the prism of pressure ADE, Fig. 6, $P = \frac{1}{2}\, h^2\, \gamma$ *cotang.* ϕ *tang.* $(\phi + \rho)$. This resistance P depends on the angle $AED = \phi$ at which the assumed plane of separation intersects the horizontal, and is a minimum for a certain value of ϕ. But in order to find this value, let us put:

$$\text{cotang.}\,\phi\ \text{tang.}\,(\phi + \rho) = \frac{\text{sin.}\,(2\,\phi + \rho) + \text{sin.}\,\rho}{\text{sin.}\,(2\,\phi + \rho) - \text{sin.}\,\rho},$$

and we see at once that this is a minimum, when *sin.* $(2\,\phi + \rho)$ is a maximum, that is when

$$2\,\phi + \rho = 90^\circ, \text{ therefore, } \phi = 45^\circ - \frac{\rho}{2}.$$

If we now introduce this value into the formula for P, we obtain the least resistance of the earth-work.

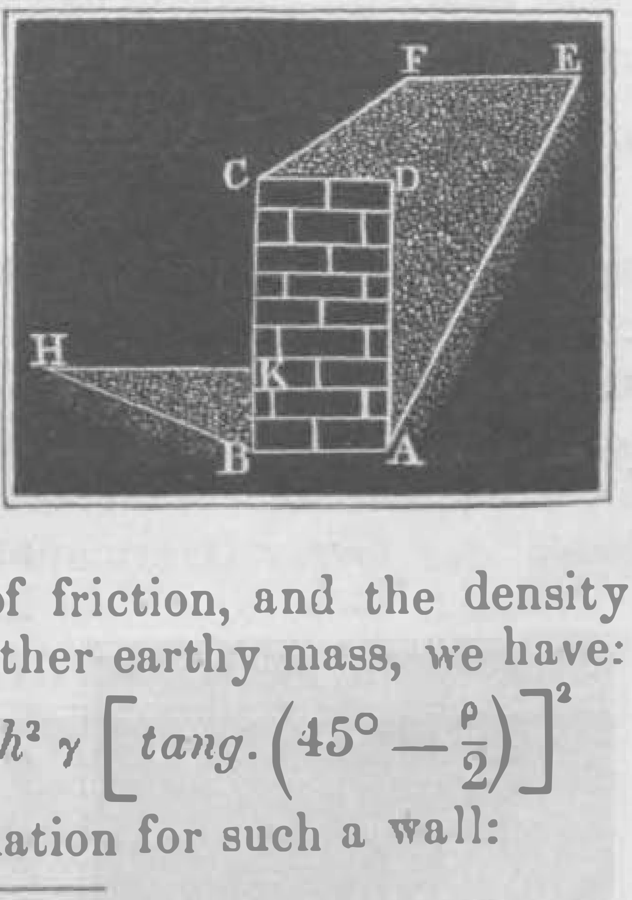
$$P = \tfrac{1}{2}\, h^2\, \gamma\ \text{cotang.}\left(45^\circ - \frac{\rho}{2}\right)\ \text{tang.}\left(45^\circ + \frac{\rho}{2}\right)$$

$$= \tfrac{1}{2}\, h^2\, \gamma \left[\text{tang.}\left(45^\circ + \frac{\rho}{2}\right)\right]^2.$$

This is, generally, the resistance with which earth or any other disintegrated mass withstands a moving force; for as soon as this force is equal to that resistance, a yielding of the mass takes place.

§ 9. *Depth of Foundations.*—An important application of the passive resistance of earth, arises in the founding of retaining and other walls. If the ground on which the retaining wall is to stand be clayey, or wet, the co-efficient of friction between the wall and the ground may fall as low as 0,3, and then a slipping of the wall may very easily occur. It is, therefore, necessary in such cases to dig the foundation to such a depth that the passive resistance on the outside, combined with the friction on the bottom, may counterbalance the active pressure on the inside. If G be the weight of a supporting wall AC, Fig. 7, therefore fG its friction on the bottom, AB, if h be the height of the earth at the back, and h_1 the height in front; if further, ρ and γ be the angle of friction, and the density for the one, and ρ_1 and γ_1 those for the other earthy mass, we have:

Fig. 7.



$$fG + \tfrac{1}{2}\, h_1^{\,2}\, \gamma_1 \left[\text{tang.}\left(45^\circ + \frac{\rho_1}{2}\right)\right]^2 = \tfrac{1}{2}\, h^2\, \gamma \left[\text{tang.}\left(45^\circ - \frac{\rho}{2}\right)\right]^2$$

and therefore the depth BK of the foundation for such a wall:

$$h_1 = \sqrt{\frac{h^2\, \gamma \left[\text{tang.}\left(45^\circ - \frac{\rho}{2}\right)\right]^2 - 2 f G}{\gamma_1}}\ .\ \text{tang.}\left(45^\circ - \frac{\rho_1}{2}\right).$$

For security, a *co-efficient* of stability 1,4 has been introduced (by French engineers for the *revêtement* walls of fortifications), and therefore the depth:

$$h_1 = 1{,}4\ tang.\left(45^\circ - \frac{\rho_1}{2}\right)\sqrt{\frac{h^2\,\gamma\left[tang.\left(45^\circ - \frac{\rho}{2}\right)\right]^2 - 2fG}{\gamma_1}}$$

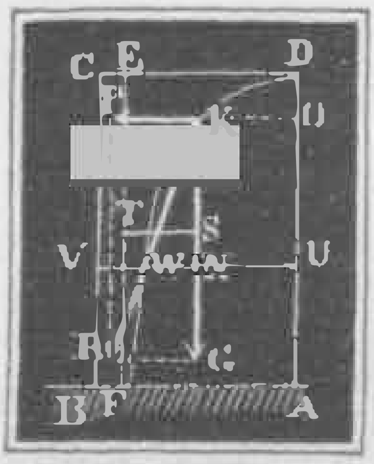
is given to such walls.

Example. To what depth must a parallel wall 8 feet thick, and 13 feet clear height, have its foundation sunk, that it may withstand the pressure of water standing level with the top of the wall? In this case $\varrho = 0, \gamma = 62{,}25$ lbs. (for which we take 63) $h = 13$ feet; also $f = 0{,}3$, $\varrho_1 = 30^\circ$, $\gamma = 1{,}6 \times 63 = 100{,}8$ lbs., and G (the density of the masonry) being $2 \times 63 = 126$ lbs., must be $= 8 \times 13 \times 126 = 13104$ lbs, therefore,

$$h_1 = 1{,}4\ tang.(45^\circ - 15^\circ)\sqrt{\frac{13^2 \times 63 - 2 \times 0{,}3 \times 13104}{100{,}8}} = 4{,}25 \text{ feet very nearly.}$$

§ 10. *Heeling of Retaining Walls.*—In order to appreciate a retaining wall in reference to stability, it is necessary to determine its *line of resistance.* For simplicity, we shall first take a parallel wall AC, Fig. 8. If we had only a horizontal force $KP = P$ to deal with, the point of application of which is at a distance $DO = a$ from the cope of the wall, the line of resistance would be a hyperbole, as the following simple view of the subject shows. Of the force P (whose point of application we assume in the line passing through the centre of gravity of the wall) and the weight G of $UVCD$, the resultant is R which intersects UV in W, a point in the line of resistance sought. If we now put the thickness of the wall $AB = CD = b$, its density $= \gamma_1$, the abscissa $KN = x$, and the ordinate $NW = y$, we have $G = (a + x)\ b\ \gamma_1$, and from similarity of triangles:

Fig. 8.

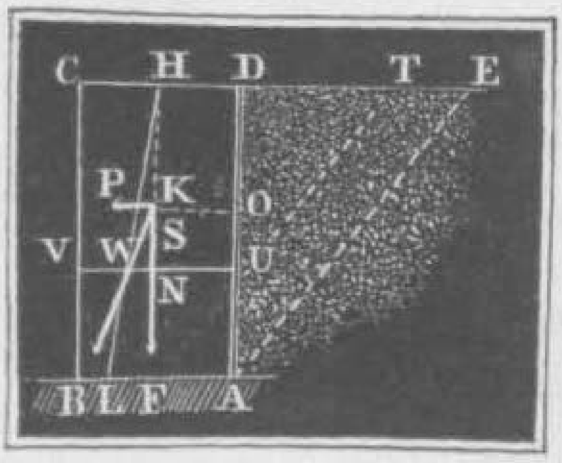


$$KWN \text{ and } KRG\text{: } \frac{WN}{KN} = \frac{RG}{KG}, \text{ that is } \frac{y}{x} = \frac{P}{(a + x)\ b\ \gamma_1},$$

and hence the equation of the line of resistance $y = \dfrac{P\,x}{(a + x)\ b\ \gamma_1}$.

From this we see that when $x = 0$, $y = 0$, and for $x = \infty$, $y = \dfrac{P}{b\,\gamma_1}$, and for $x = -a$, $y = -\infty$. The curved line of resistance, therefore, passes through K, and has not only the horizontal CD, but likewise a vertical EF for asymptote, distant $ST = \dfrac{P}{b\,\gamma_1}$ from the centre of gravity S of the wall.

Fig. 9.



It is otherwise, of course, for a wall to withstand pressure of earth or water as AC, Fig. 9, for here a is variable, because P is applied at a point U at $\frac{1}{3}$ of the height

DU from the base. If we draw the end of the vertical line through *S* as origin of the co-ordinates, that is, if we put $HN = x$, we have:

$$\frac{y}{\frac{1}{3}x} = \frac{P}{b\,x\,\gamma_1}, \text{ or as } P = \tfrac{1}{2}\,x^2\gamma\left[tang.\left(45^\circ - \frac{\rho}{2}\right)\right]^2,$$

$y = \dfrac{\gamma}{6\,b\,\gamma_1}\left[tang.\left(45^\circ - \dfrac{\rho}{2}\right)\right]^2 . \, x^2.$ This equation corresponds to the common parabola with absciss y and ordinate x.

If, however, we suppose the earth-work carried a height h_1 above the cope of the wall, we must adopt the proportion:

$\dfrac{y}{\frac{1}{3}\,(x + h_1)} = \dfrac{\frac{1}{2}\,\gamma}{b\,x\,\gamma_1}\left[tang.\left(45^\circ - \dfrac{\rho}{2}\right)\right]^2 (x + h_1)^2$, whence we have the equation $y = \dfrac{\gamma}{6\,b\,\gamma_1}\left[tang.\left(45^\circ - \dfrac{\rho}{2}\right)\right]^2 \dfrac{(x + h_1)^3}{x}$.

§ 11. The stability of a retaining wall requires not only that the line of resistance be within the wall, but also that it shall not come too near the outside of it. The famous Marshal Vauban gives the practical rule: that the line of resistance should intersect the basis of the wall in a point whose distance from the vertical passing through the centre of gravity of the wall is at most $\frac{4}{9}$ of the distance of the outer axis of the wall from this line. If, as Poncelet does, we call the reciprocal of this number, or the ratio $\dfrac{FB}{FL}$ between the distance of the outer axis from the vertical passing through the centre of gravity, and the distance of the point of intersection *L* of the line of resistance from this gravity line, the *co-efficient of stability*, and represent it generally by δ, we have for the stability of a parallel wall, withstanding the pressure of earth, (by introducing into the last formula instead of x, the height h of the wall, and instead of y $\dfrac{\frac{1}{2}b}{\delta}$).

$$\frac{b}{2\delta} = \frac{\gamma}{6\,b\,\gamma_1}\left[tang.\left(45^\circ - \frac{\rho}{2}\right)\right]^2 \frac{(h + h_1)^3}{h},$$

and, therefore, the requisite thickness of the wall:

$$b = (h + h_1)\; tang.\left(45^\circ - \frac{\rho}{2}\right)\sqrt{\frac{\delta\,\gamma}{3\,\gamma_1} \cdot \frac{h + h_1}{h}}.$$

If for δ we substitute $\frac{9}{4} = 2{,}25$, and for $\dfrac{\gamma}{\gamma_1}$, $\frac{2}{3}$ a mean value, we get:

$$b = 0{,}707\;(h + h_1)\sqrt{\frac{h + h_1}{h}} \,.\; tang.\left(45^\circ - \frac{\rho}{2}\right).$$

If we take $\rho = 30^\circ$, we obtain $b = 0{,}4\;(h + h_1)\sqrt{\dfrac{h + h_1}{h}}$.

Poncelet gives:

$b = 0{,}865\;(h + h_1)\; tang.\left(45^\circ - \dfrac{\rho}{2}\right)\sqrt{\dfrac{\gamma}{\gamma_1}}$, or approximately:

$b = 0{,}285\;(h + h_1)$, for cases in which h_1 varies from 0 to 2 h.

Example. What must be the thickness of a parallel wall of 28 feet in height to retain broken stones, mine rubbish for a height of 35 feet? Assuming that the density of the wall $= 2,4 \times 63 = 151,2$ lbs. The density of the rubbish $1,3 \times 63 = 81,3$ lbs., and $\rho = 50^\circ$. According to Poncelet's formula:

$$b = 0,865 \times 35 \; tang. \; (45^\circ - 25^\circ) \sqrt{\frac{13}{24}} = 30,3 \sqrt{\frac{13}{24}} \; . \; tang. \; 20^\circ = 8,11 \text{ feet.}$$

§ 12. *Poncelet's Tables.*—To facilitate applications of the formula, Poncelet has calculated the following table, which contains values of $\frac{b}{h}$ corresponding to given values of $\frac{h_1}{h}$, $\frac{\gamma}{\gamma_1}$, and ρ. There are two cases distinguished in the table, namely, the case when the earth-work is heaped, as is shown in Fig. 7, the coping being covered, and the case shown in Fig. 10, where a *berme* of the breadth 0,2 h, from the outer edge of the cope of the wall, is left before the natural slope of the embankment begins: so that, in short, a promenade is left of the width $CL = 0,2\ h$.

The headings of the table explain themselves.

Values of $\frac{h_1}{h}$	Values of $\frac{b}{h}$ for											
	$\frac{\gamma_1}{\gamma} = 1$; $f = 0{,}6$. Berme.		$\frac{\gamma_1}{\gamma} = 1$; $f = 1{,}4$. Berme.		$\frac{\gamma_1}{\gamma} = 1{,}5$; $f = 1$. Berme.			$\frac{\gamma_1}{\gamma} = \frac{5}{3}$; $f = 0{,}6$. Berme.		$\frac{\gamma_1}{\gamma} = \frac{5}{3}$; $f = 1{,}4$. Berme.	
	$= 0$	$= 0{,}2h$	$= 0$	$= 0{,}2h$	$= 0$	$= 0{,}2h$	$= b$	$= 0$	$= 0{,}2h$	$= 0$	$= 0{,}2h$
0,0	0,452	0,452	0,258	0,258	0,270	0,270	0,270	0,350	0,350	0,198	0,198
0,1	0,498	0,507	0,282	0,290	0,303	0,306	0,303	0,393	0,398	0,222	0,229
0,2	0,548	0,563	0,309	0,326	0,336	0,342	0,326	0,439	0,445	0,249	0,262
0,3	0,604	0,618	0,338	0,361	0,368	0,375	0,343	0,485	0,489	0,274	0,283
0,4	0,665	0,670	0,369	0,394	0,399	0,405	0,357	0,532	0,522	0,303	0,299
0,5	0,726	0,717	0,402	0,423	0,436	0,431	0,368	0,579	0,549	0,332	0,314
0,6	0,778	0,754	0,436	0,450	0,477	0,457	0,377	0,617	0,572	0,360	0,328
0,7	0,824	0,790	0,472	0,476	0,512	0,481	0,385	0,645	0,593	0,387	0,343
0,8	0,847	0,820	0,510	0,501	0,544	0,504	0,391	0,668	0,610	0,413	0,357
0,9	0,903	0,848	0,541	0,524	0,575	0,523	0,398	0,690	0,624	0,437	0,371
1,0	0,930	0,873	0,571	0,546	0,605	0,540	0,405	0,707	0,636	0,457	0,384
1,4	1,023	0,945	0,684	0,624	0,696	0,602	0,416	0,762	0,672	0,537	0,428
2,0	1,107	1,004	0,812	0,714	0,795	0,655	0,425	0,811	0,705	0,622	0,475
3,0	1,180	1,060	0,981	0,835	0,892	0,717	0,435	0,852	0,731	0,726	0,531
5,0	1,247	1,101	1,206	0,994	1,002	0,779	0,445	0,883	0,751	0,862	0,596
1,00	1,283	1,137	1,508	1,182	1,109	0,839	0,452	0,909	0,771	1,013	0,667
2,00	1,309	1,156	1,757	1,327	1,171	0,878	0,456	0,922	0,780	1,129	0,712
3,00	1,316	1,162	1,866	1,389	1,194	0,894	0,458	0,926	0,783	1,174	0,730
∞	1,337	1,175	2,144	1,541	1,243	0,927	0,461	0,934	0,789	1,279	0,769

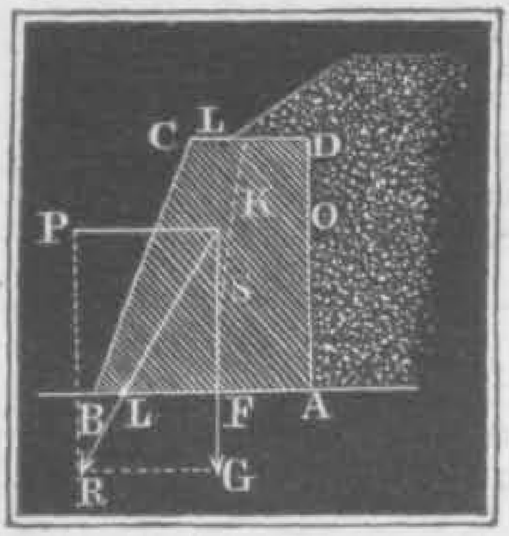
In this table the *limiting values* have been principally held in view. Thus $\frac{\gamma_1}{\gamma} = 1$ corresponds pretty nearly to one limit of the ratio of the densities of masonry and earth, while $\frac{\gamma_1}{\gamma} = \frac{5}{3}$ corresponds to the other. Again, f or $tang. \rho = 0{,}6$, is the value for the least coherent earths, and $f = 1{,}4$ the value for stiff compact earthwork. In many practical cases, the required proportion has to be deduced by interpolation.

Remark. The formula $b = 0{,}865\,(h + h_1)\; tang. \left(45^\circ - \frac{\rho}{2}\right) \sqrt{\frac{\gamma_1}{\gamma}}$ gives results corresponding with those in the table to within $\frac{1}{12}$.

The values in the table refer to parallel walls, built with mortar. If the external batter of the wall does not exceed 0,2, that is 2,4 inches per foot, the breadth b found above, will be that of the wall at $\frac{1}{3}$ of its height from the base, and through this point the line of batter is to be drawn.

Remark. The dimensions resulting from Poncelet's rules or tables are applied in France for walls of fortifications, but give dimensions *nearly one-fourth greater* than the average practice of civil engineers in Britain for the same relative circumstances.—Tr.

Fig. 10.



§ 13. *Retaining Walls with Batter.*—If the wall has a *batter*, or if its profile be a trapezium AC, Fig. 10, the thickness necessary to insure resistance to rotation can only be determined by aid of a complicated expression. If we assume the face AB as the plane of separation, and put $KF = OA = x$, and $FL = y$, we then have againh $\frac{y}{x} = \frac{P}{G}$, and

$FB = \delta\, y = \frac{\delta\, P x}{G}$. But as $x = \frac{1}{3}\,(h + h_1)$, and

$$P = \frac{1}{2}\,(h + h_1)^2\, \gamma \left[tang. \left(45^\circ - \frac{\rho}{2}\right)\right]^2,$$

and if b the thickness at top, and n the relative batter, therefore, $n\,h$ the absolute batter, $G = (bh + \frac{1}{2}\, nh^2)\, \gamma_1$, and, therefore,

$$FB = \frac{\delta\gamma}{\gamma_1} \cdot \frac{(h + h_1)^3 h \left[tang. \left(45^\circ - \frac{\rho}{2}\right)\right]^2}{6\, h\, (b + \frac{1}{2}\, nh)}$$

The distance BF of the outer edge of the wall from the vertical passing through the centre of gravity, is:

$$= \frac{b + nh}{2} + \frac{3\,b + nh}{2\,b + nh} \cdot \frac{nh}{6} = \frac{3\,b^2 + 6\,n\,h\,b + 2\,n^2\,h^2}{3\,(2\,b + n\,h)},$$

and hence we may put:

$$3\,b^2 + 6\,n\,h\,b + 2\,n^2\,h^2 = \frac{\delta\,\gamma}{\gamma_1} \frac{(h + h_1)^3}{h} \left[tang. \left(45^\circ - \frac{\rho}{2}\right)\right]^2,$$

and hence the thickness of wall at top:

$$b = -nh + \sqrt{\frac{\delta\gamma}{3\gamma_1} \cdot \frac{(h+h_1)^3}{h}\left[tang.\left(45^\circ - \frac{\rho}{2}\right)\right]^2 + \frac{1}{3}n^2h^2}.$$

Remark 1. If the back of the wall have a batter likewise, we have a different prism of greatest pressure to deal with, because the force applied to the wall is no longer horizontal. The investigation becomes complicate, and we forbear to enter upon it, but shall refer to works treating of the subject.

Remark 2. Coulomb was the first to propound a good theory of the pressure of earth. See "Théorie des machines simples." Prony, in his "Leçons sur la poussée des terres, (1802,)" extended Coulomb's theory. Navier pursues the same notions, with much elegance and precision, in hist"Leçons sur l'application de la mécanique, tome I." Mayniel, in 1808, published a special treatise on the pressure of earth, in which the observations and theories of his predecessors are reviewed, "Traité expérimental, &c., de la poussée des terres." C. Martony de Köszegh made experiments on a large scale for the Austrian government, which were published in 1828, under the following titlet "Versuche über den Seitendruck der Erde, ausgeführt auf höchsten Befehl, &c., und verbunden mit den theoretischen Abhandlungen von Coulomb und Français, Wien, 1828." The most complete work on the pressure of earth is that of Poncelet in the "Mémorial de l'officier du génie, 1838," and which has been translated into German by Lahmeyer, Braunschweig, 1844. In Moseley's "Engineering and Architecture," this subject is handled with great elegance and success. Hagen has a chapter on this subject in the second part of his admirable "Wasserbaukunst," in which he takes a peculiar view of it.

CHAPTER II.

THEORY OF ARCHES.

§ 14. *Arches.*—An arch (Fr. *voûte*, Ger. *Gewölbe*), is a system of bodies resting upon each other, and supported by two fixed points, in such manner that they are in equilibrium not only among themselves, but with certain external forces. The material of these bodies is usually stone, and hence are termed *arch-stones* (Fr. *voussoirs*, Gr. *Gewölbesteine*). The planes of contact of the stones are the *beds* or *joints*. The fixed points upon which the arch rests are termed *abutments* (Fr. *Pieds-droits*, Ger. *Widerlager*), and in certain cases *piers* (Fr. *culées*, *piliers*, Ger. *Pfeiler*). Of the arch-stones, the highest is termed the *key-stone* (Fr. *clef*, Ger. *Schlussstein*), and those which rest on the abutments or piers, are termed *imposts* or *springers*, (Fr. *coussinets*, Ger. *Kämpfer*.) An arch is included between two more or less curved surfaces, the *intrados* and *extrados*, which are sometimes termed *the soffit*, and *the back of the arch.*

As regards the intrados and extrados, arches are very various. Cylindrical surfaces are most usual, but conical surfaces occur, and we have domes, and variously proportioned groinings. We shall treat of cylindrical arches only, and limit ourselves still further, to the consideration of those having a horizontal axis. Such arches are bounded by two vertical parallel planes, the *faces* of the arch, (Fr. *parements*, Ger. *Stirnflächen.*) According as the faces are perpendicular or inclined to the geometrical axis of the arch, the arch is *direct*, or *oblique*, or *skewed* (Fr. *droites* or *biaises*); groined arches or vaults (Fr. *voûtes d'arète*, Ger. *Kreuz*, or *Kloster gewölbe*), are merely intersecting cylindrical arches. Domes or cupolas (Fr. *voûtes en dome*, Ger. *Kuppel* or *Kesselgewölbe*), are arches generated by the revolution of a curve about a vertical axis.

As regards the curvature of arches, it is very various. The section is sometimes circular, sometimes elliptical, catenarian, or formed of several circular arcs, and plate bands, or straight arches are sometimes built.

Remark.—As experience has abundantly proved that arches fail or give way by a rotation of determinate parts round the edges where certain joints meet the extrados or intrados, and not by sliding dislocation, we need here only consider the conditions of equilibrium in reference to the former circumstance, omitting our author's investigation of the latter, which show, as is usually done in elementary treatises of mechanics, that for the case of equilibrium without friction, *the weight of the arch-stones must be to each other as the differences of the cotangents of the angles of inclination of the joints to the horizon*.—Tr.

Remarks—The dislocation of an arch by slipping of voussoirs might occur in two ways: according as the joint of maximum pressure lies above or below the joint of minimum pressure. In the former case, Fig. 11, the hauches of the arch slide out, and

Fig. 11.

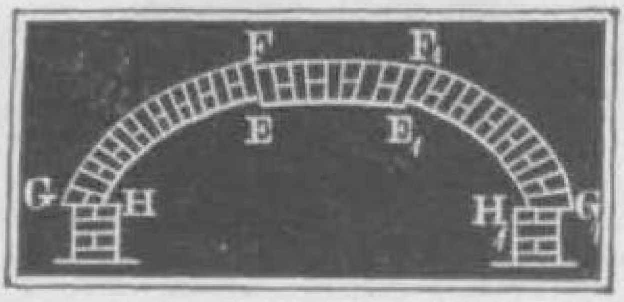
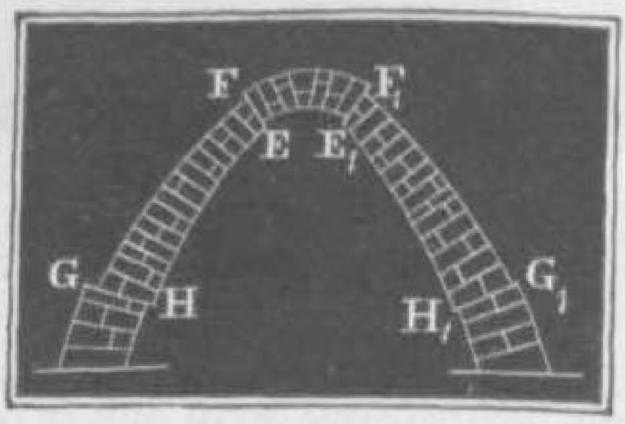
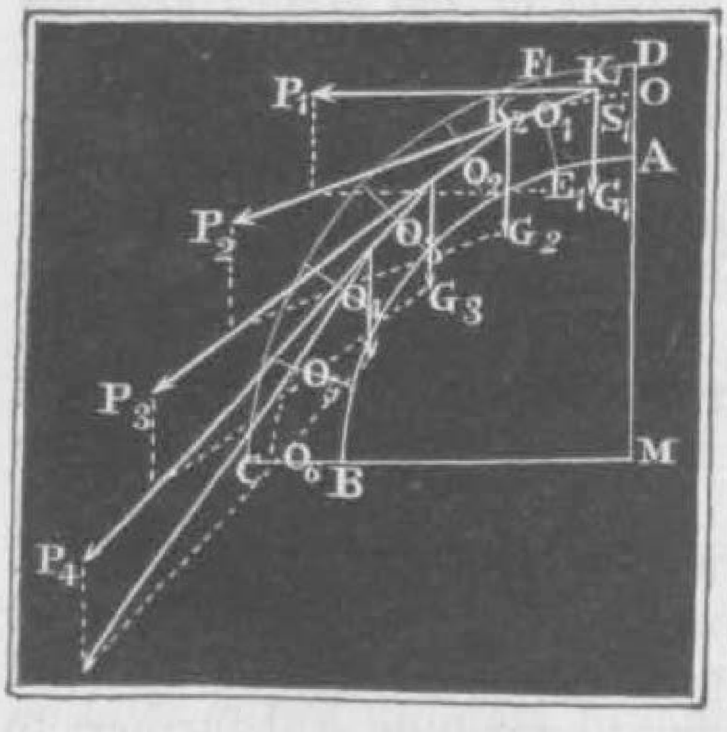


Fig. 12.



the crown slips down. In the other case, the reverse happens, Fig. 12. This second case scarcely ever occurs, so that we shall not farther recur to it.

Fig. 13.



§ 15. *Line of Pressure and Resistance.*

—An arch is so much more likely to fall in by rotation round the outer or inner edge of a joint, than by slipping, that the former may be considered as the usual accident. The stability of an arch in reference to rotation may be considered exactly in the same manner as the stability of a pier or wall (Vol. II. § 6). From the horizontal force P_1 applied at any point O, Fig. 13, in the crown of the arch and the weight of the first arch-stone acting in its centre of gravity S_1, there results the force P_2 acting on the first joint, and the intersection O_1 of the direction of this force with the joints $E_1 F_1$.

Again, from the pressure P_2, and the weight G_2 of the second arch-stone, acting in its centre of gravity S_2, there results the pressure P_3 in the second joint, and the intersection O_2 of the direction of this force with the second joint. Proceeding in this manner, we obtain the remaining normal pressures, and the intersections O_3, O_4, &c., in the other joints. But the lines O, O_1, O_2, O_3 . . . , which unite the intersections or points of application of the pressures P_1, P_2, P_3 . . . , is the *line of resistance* (Fr. *ligne de Pression*, Ger. *Widerstandslinie*), (Vol. II. § 6). So long as at least one line of resistance can be found in the face of an arch, which neither passes beyond the intrados nor the extrados at any point, so long dislocation of the arch by rotation cannot occur. If, on the other hand, the line of resistance intersects the intrados, the arch will fall inwards, and if it goes beyond the extrados, the arch will rise upwards, and so fall to pieces. Fig. 14 represents the former case, and Fig. 15 the latter.

Fig. 14.

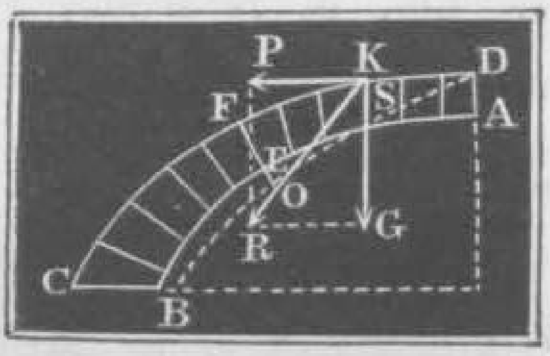
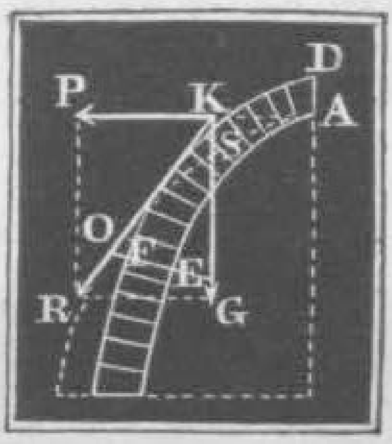


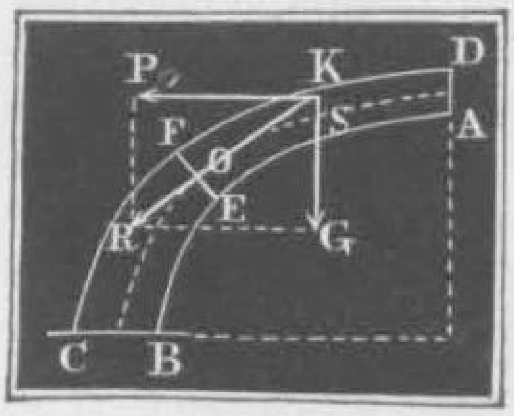
Fig. 15.



The dislocation becomes inevitable, however, from the circumstance that the incompressibility of the stones opposes resistance to the forces *RR*, acting with the leverages *EO*, *FO*. The cohesion of the mortar alone resists this force; but as a very slight concussion is sufficient to destroy this cohesion, its effects should not be relied upon as available.

It is easy to perceive that arches are so much the more stable (in reference to rotation) the greater the number of lines of resistance that can be drawn within them; the less, therefore, the number of lines of resistance that intersect the intrados or extrados. The arch of greatest stability, Fig. 16, is necessarily that in which a line of resistance may be drawn, which passes through the centre of all the arch-stones, or *bisects their depth.*

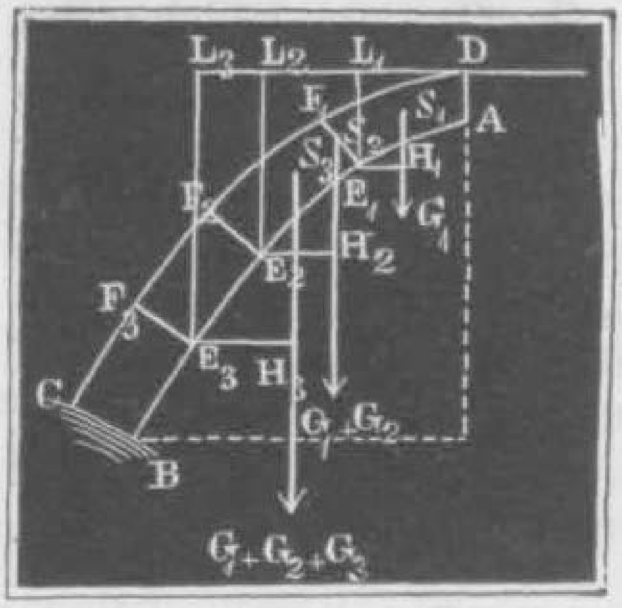
Fig. 16.



For the usual construction of arches, that is, for circular arches, a rotation or rising upwards, that is, an intersection of all lines of resistance with the extrados, cannot possibly occur; we may, therefore, limit ourselves in the investigation of stability to the rotation from which the arch falls inwards. That we may be certain that at least one line of resistance passes beyond neither intrados nor extrados, we may start to draw it from the point D in the crown, and try whether it intersects the intrados.

§ 16. *Equilibrium in Reference to Rotation.*—The conditions of stability in reference to rotation may be considered in another point of view, and one more adapted for calculation. We may eliminate the forces P_1, P_2, P_3 . . ., acting in the crown D, Fig. 17, which are necessary to hinder a rotation of the arch-stones round the inner edges E_1, E_2, E_3, &c., and then investigate which is the greatest of these forces. If we designate the leverages E_1L_1, E_2L_2, E_3L_3, . . . of the force P referred to the points E_1, E_2, E_3, &c., as axis of rotation by a_1, a_2, a_3, &c., and the leverages E_1H_1, E_2H_2, E_3H_3, &c., of the weights G_1, $G_1 + G_2$, $G_1 + G_2 + G_3$, &c., in reference to these same axis by b_1, b_2, b_3, &c., we have for the force P acting at the crown:

Fig. 17.



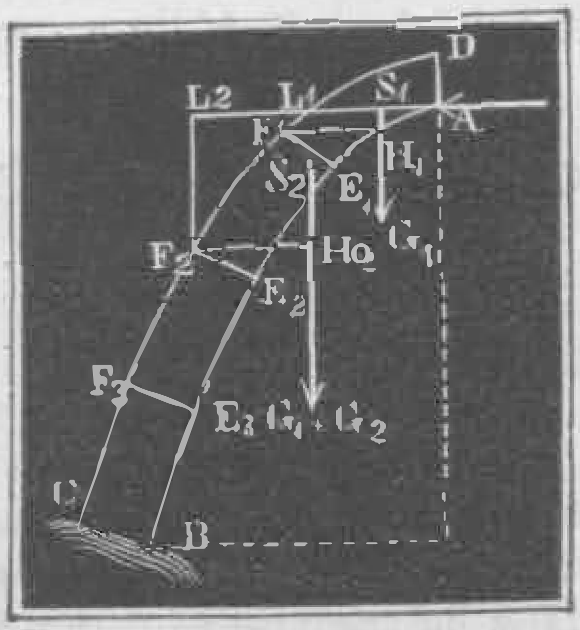
$$P_1 = \frac{b_1}{a_1} G_1,\ P_2 = \frac{b_2}{a_2}(G_1 + G_2),\ P_3 = \frac{b_3}{a_3}(G_1 + G_2 + G_3),\ \&c.$$

But not only the factors b_n, and $G_1 + G_2 + \ldots + G_n$ of the numerator increase from the crown towards abutments, but the denominator A_n increases also; hence one of the values of P_1, P_2, P_3, &c., is a maximum; and it is necessary for equilibrium, that the effective force P_m, acting in the crown, should he equal to it. That joint which corresponds to the maximum pressure, or the pressure on the crown, is termed the *joint of rupture* (Fr. *joint de rupture*, Ger. *Bruchfuge*), because dislocation by rotation first begins round its lower edge, if the force P_m at the crown diminishes. It is determined by the *angle of rupture*, which its plane makes with the horizon (or with the vertical). It is also easy to perceive, that the angle of rupture gives that point in the arches, in which the line of resistance, starting in D in the crown of the arch, touches the intrados.

If we compare the maximum effort required to hinder rotation inwards with the maximum effort required to resist slipping, we find that in most cases the force required to resist rotation is greater than that totresist slipping, and, therefore, *the pressure in the crown of an arch is equal to the greatest of all the forces P_1, P_2, P_3, &c., which oppose the rotation of the parts of the arch G_1, $G_1 + G_2$, $G_1 + G_2 + G_3$, &c., round the inner edges.* If, therefore, we have once determined this pressure at the crown of the arch, it is easy to find the pressure in any other part of the arch.

Arches falling by rotation *outwards* are exceptional cases. To discriminate by calculation as to the possibility of such an accident occurring, the point of application of the force P is taken at the lower edge A, Fig. 18, of the joint of the key-stone, because the leverage, in reference to rotation about F_1, F_2, F_3, &c., is here the least. If now we again designate the leverages: F_1L_1, F_2L_2, F_3L_3, &c., by a_1, a_2, a_3, &c., and the leverages F_1H_1, F_2H_2, F_3H_3, &c., of the weights G_1, $G_1 + G_2$, $G_1 + G_2 + G_3$, &c., we have the values of P:

Fig. 18.



$$P_1 = \frac{b_1}{a_1} G, \; P_2 = \frac{b_2}{a_2}(G_1 + G_2),$$

$$P_3 = \frac{b_3}{a_3}(G_1 + G_2 + G_3),$$

and if the *least* of these values be greater than the pressure in the crown, or the greatest of the forces which prevent a falling inwards, the arch is stable; unless this be the case, dislocation takes place.

Remark. The falling to pieces of an arch by rotation may likewise happen in two ways: according as the joint of rupture of the maximum value is above or below the joint of rupture of the minimum value. Fig. 19 represents the first, and Fig. 20 the second case.

Fig. 19.

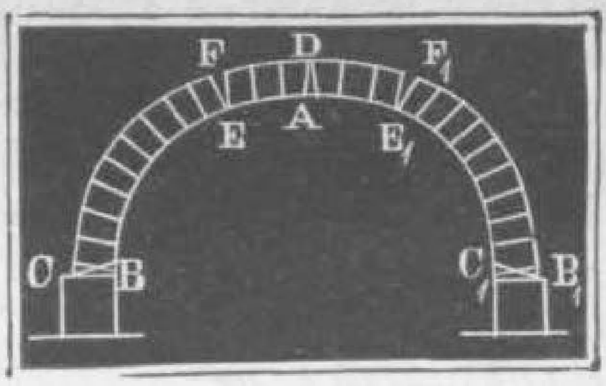
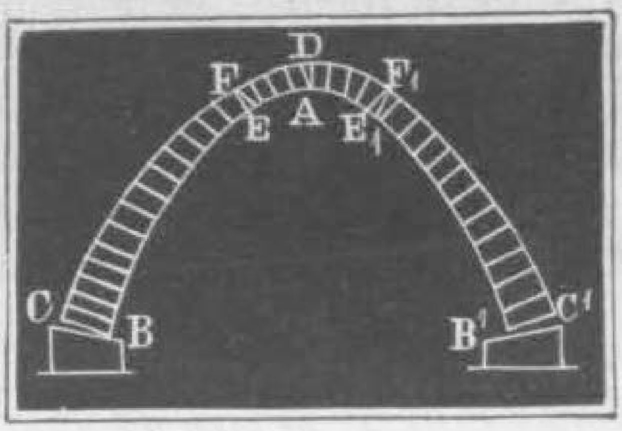
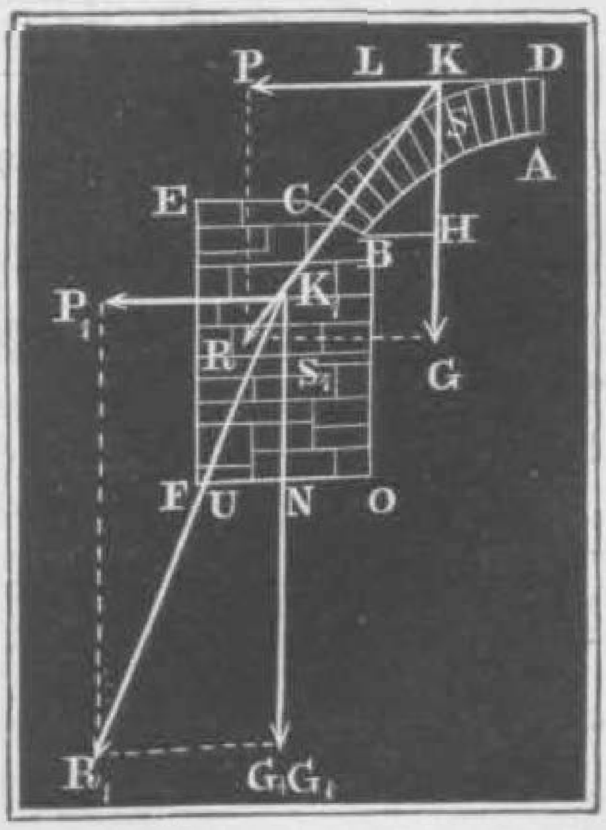


Fig. 20.



§ 17. *Stability of Abutments.*—If we have satisfied ourselves by the calculations indicated in the foregoing paragraphs, that an arch is stable, and have also determined the pressure in the key-stone, we have still to investigate the stability of the abutment walls; that is, to determine the thickness of abutment wall necessary to resist a detrusion or an overturn. This investigation is the more important, as it is not unfrequently in consequence of insufficient resistance of these, that arches, in themselves stable, fall in.

Fig. 21.



It is evident that a retaining wall FB, Fig. 21, is stable when the direction of the resultant force $K_1 R_1 = R_1$ of the weight of the one semi-arch acting at its centre of gravity S, the horizontal thrust P acting at the crown, and S_1, the weight of the retaining wall, passing through its centre of gravity, passes through the base FO of the retaining or *abutment* wall, and deviates from the vertical K_1N by an angle less than the angle of repose ρ.

For the angle ϕ, which the resultant R_1 of the forces $P = \overline{KP_1}$ and $G + G_1 = \overline{K_1 G_1}$, makes with the vertical, we have *tang.* $\phi = \frac{P}{G + G_1}$; but *tang.* $\rho =$ the co-efficient of friction f, and hence to insure stability in reference to sliding, we should have $\frac{P}{G+G_1} < f$.

In order, further, that the resultant may pass through the outer edge F of the abutment, let us put the moment of P, referred to this edge, equal tohthe sum of the moments of the weights G and G_1. If a be the rise of the arch BL, and h the height of the abutment, then the moment of the force P referred to the edge F as an axis $= P\,(a + h)$; if, again, b be the horizontal distance BH of the vertical passing through the centre of gravity of the semi-arch AC from the inner edge B of the springing point, c the thickness of the abutment wall, and e the distance FN of the vertical gravity-line of the abutment wall from the edge F, we have the moment of the weights G and $G_1 = G\,(b + c) + G_1 e$, and thus we get the equation: $P\,(a + h) = G\,(b + c) + G_1 e$.

In order to insure permanence, experience dictates, according to Audoy's deductions, the employment of 1,9 P instead of P, so that the equation for determining the thickness of the abutment becomes: $1{,}9\,P\,(a + h) = G\,(b + c) + G_1 e$. If h_1 be the mean height of the abutment or pier, and γ the density of its masonry, we have for each foot in length of the pier the weight $G_1 = h_1\, c\, \gamma$, and if we put $e = \frac{1}{2} c$, the moment $G_1 e = \frac{1}{2} h_1\, c^2 \gamma$, and hence:

$$\tfrac{1}{2} h_1\, c^2 \gamma + G\,c = 1{,}9\,P\,(a + h) - G\,b, \text{ or,}$$

$$c^2 + \frac{2\,G\,c}{h_1 \gamma} = \frac{1{,}9\,P\,(a + h) - G\,b}{\frac{1}{2} h_1 \gamma},$$

and hence the thickness of the abutment in question:

$$c = -\frac{G}{h_1 \gamma} + \sqrt{\frac{1{,}9\,P\,(a + h) - G\,b}{\frac{1}{2} h_1 \gamma} + \left(\frac{G}{h_1 \gamma}\right)^2}.$$

In order to secure this wall against sliding, we must have:

$$G_1 > \frac{P}{f} - G, \text{ i. e. } c > \frac{P - f\,G}{f\,h_1 \gamma}.$$

It will usually be found that the former value of c is greater than the latter; and that, therefore, the thickness of the abutment must be regulated by the former condition of stability.

For very high piers, as $G\,c$, $1{,}9\,P\,a$ and $G\,b$, arc very small compared with $1{,}9\,P h$ and $\frac{1}{2} h_1\, c^2 \gamma$ (which may be put $\frac{1}{2} h\, c^2 \gamma$), we have: $\frac{1}{2} h\, c^2 \gamma = 1{,}9\,P\,h$, i.e. $\frac{1}{2} c^2 \gamma = 1{,}9\,P$, and hence the greatest strength:

$$c = \sqrt{\frac{3{,}8\,P}{\gamma}}.$$

§ 18. *Loaded Arches.*—We have hitherto neglected to consider the influence of the *backing* on the arch; which, however, it is essential to examine. That the stability of an arch, such as a bridge, may not be altered by the passage of heavy weights upon it, it is necessary that the arch should in itself possess such weight, or be permanently loaded with backing, that any weight arising from traffic, such as heavy wagons, locomotives and the like, can only occasion a slight change in the entire load, or forces in action.

The backing consists usually of a system of walling (*spandril walls*), supporting the road-way, and carried up either to form a horizontal line *EF*, Fig. 22, or an inclined line, Fig. 23. In very

Fig. 22.

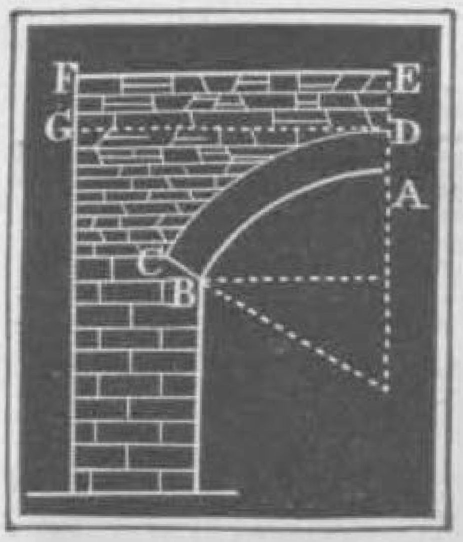
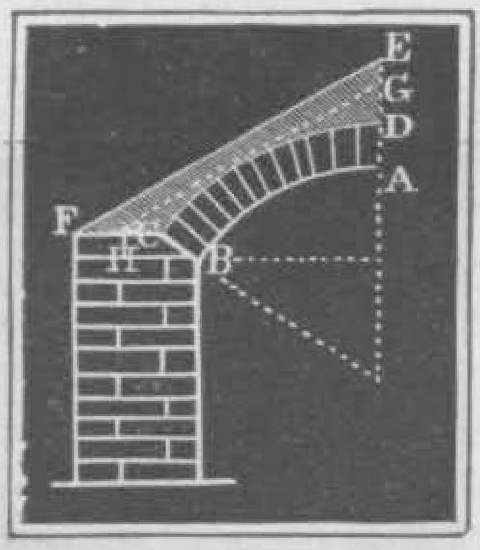


Fig. 23.



many cases, the spandril walls or backing of arches consist of the same materials as the arches; and if it be uniformly built, we may assume a common density for the whole, and thus considerably abbreviate calculation. If, according to Vol. I. § 58, we take the specific gravity of masonry at from 1,6 to 2,4, we have for the density of the masonry 100 to 150 lbs. per cubic foot, the former answering to brick-work, the latter to ashlaring. The loading of arches generally increases their thrust, and also their stability. That the voussoirs may resist crushing, they must have a certain depth proportioned to the pressure of the arch; and as this increases from the crown towards the springing, the depth of the voussoirs should likewise increase from the crown to the springing. Perronet has given as a rule for the depth at the crown, the formula $d = 0,0694r + 0,325$ metres, or, in English measure $d = 0,0694r + 1$ foot, in which formula r is the greatest radius of curvature of the intrados.

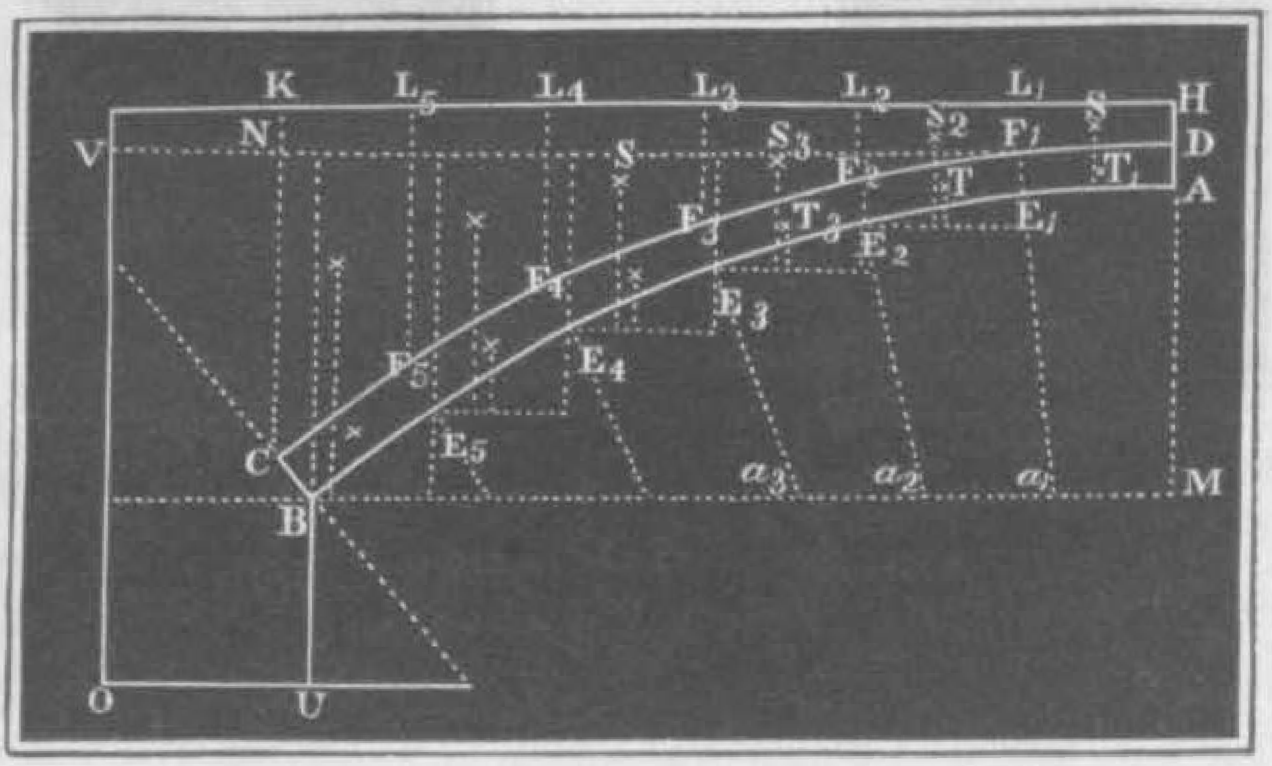
For arches whose radius is above 48 feet, or 15 metres, this formula gives greater dimensions than is given in ordinary practice. The depth of voussoirs must be regulated by the strength of the materials, and the position of the line of resistance in the arch. The joints being kept very thin, so that the mortar serves rather to distribute the pressure uniformly over the bed of the stone, it will be found that a thickness which reduces the strain to 225 lbs. per square inch of surface, allows of ample security for the average of materials. One-half this thickness must, however, exist on *each side* of the line of resistance.

Remark 1. 225 lbs. per square inch is only $\frac{1}{20}$ of the absolute strength of sound sandstone and limestone. In the celebrated bridge at Neuilly, near Paris, built in 1768 to 1780, by Perronet, the estimated pressure per square inch is 280 lbs.

Remark 2. When, as in the sequel we always do, we take the thrust or pressure at the top of the crown of the arch, and in like manner, only consider a rotation round the lowest point of the angle of rupture, it is the more necessary to assume this high degree of security, and to give the arch corresponding depth of voussoirs, as in these assumptions we only get the least value of the pressure. Besides, it is chiefly the upper edges of the voussoirs at the crown of the arch, and the lower edges of those at the joints of rupture that have to withstand the pressure, and, therefore, soonest give way; the depth we have indicated on each side the line of pressure is, therefore, necessary to insure stability.

§ 19. *Test of the Equilibration of Arches.*—The investigation of the stability of an arch may be gone through as follows: let $ABCD$, Fig. 24, be the one-half of the arch to be examined, and $CDHK$ the

Fig. 24.



spandril wall, which for simplicity's sake we shall assume to be of the same density as the arch. First, divide the arch in any convenient number, in this case six, equal or unequal parts, by lines E_1F_1, E_2F_2, E_3F_3, &c., in the direction of the joints, and determine, not only the area and the centres of gravity T_1, T_2, T_3 . . . of these parts, but also the areas and centres of gravity S_1, S_2, S_3 . . . of the superincumbent parts F_1H, F_2L_1, F_3L_2 . . . This done, take the statical moment of the first part AF_1 and F_1H referred to the first point of division E_1, and divide their sum by the vertical distance of this point of division from the horizontal DN drawn through the crown. In like manner take the moments of AF_1, E_1F_2, F_1H and F_2L_1, referred to the second point of division E_2, and divide the sum of these moments by the vertical distance of this second point from the horizontal DN. Again, determine the moments of the parts of the arch AF_1, E_1F_2, E_2F_3, and the parts of the spandril F_1H, F_2L_1, F_3L_2, referred to the edge E_3, and divide that sum by the vertical distance of the point E_3, from the horizontal DN, &c. By going through this process for all the parts, from A to B, we arrive at the forces that must be applied at D to prevent rotation round the points

E_1, E_2, E_3, &c., and the greatest of these forces is that which has to be taken as acting at the crown.

Having done this, multiply the sum of the areas $AF_1 + F_1H$ by the *tang.* $(a_1 - \rho)$, and again $AF_1 + E_1F_2 + F_1H + F_2L_1$ by *tang.* $(a_2 - \rho)$, &c., (where a_1, a_2 ... are the several angles of inclination of the joints with the horizon), and find the greatest value of these products. If the greatest of these values be *less* than that necessary to prevent rotation round E_1, E_2, E_3 ..., there need be no further consideration of these forces; but if it be *greater*, then must this value be introduced as the pressure in the crown, and not that first found.

Lastly, it has to be determined whether the horizontal force so found is not sufficient to dislocate the arch by *pushing* or *turning* out a part of it.

With the horizontal thrust, determined as above, we can examine, as shown in § 16, the conditions of stability of the abutment.

Example. The relative stability of the arch in Fig. 24, may be calculated as followso area of the part $AF_1 = 6{,}89$ square feet; area of the piece F_1H above $d = 8{,}48$ square feet, the lever of the former referred to $E_1 = 2{,}50$, and of the latter $= 2{,}45$; *i. e*, the moment of botho$= 6{,}89 \,.\, 2{,}5 + 8{,}48 \,.\, 2{,}45 = 38{,}001$. The distance of E_1 from DN, or leverage of horizontal force in $D = 1{,}50$; and, therefore, the first value of this force $= \frac{38 \,.\, \gamma}{1{,}50} = 25{,}33 \,.\, \gamma$ lbs. Area of second part $E_1F_2 = 7{,}15$, and the part of spandril above it $F_2L_1 = 11{,}02$ square feet; the moment of both referred to $E_2 =$d$7{,}52 + 23{,}69 = 41{,}21$, adding to this the moment of $AL_1 = 38 + 15{,}37 \,.\, 5{,}10 = 38 + 78{,}39 = 116{,}39$, and hence the moment of the whole piece AL_2o$= 157{,}60$; the distance of E_2 from $DN = 2{,}35$, and hence the second value of the horizontal force in $D = \frac{157{,}60 \,.\, \gamma}{2{,}35} = 67{,}05$o γ lbs. Again, the area of the third piece $E_1F_3 = 7{,}68$, and of the part of spandril above it $F_3L_3 = 16{,}51$ square feet; the moment of both $= 46{,}61$, adding to this the moment of the piece $E_2H = 157{,}60 + 16602$o$= 323{,}62$; we find the moment of the whole $= 370{,}23$; and as the distance of the point E_3 from $HN = 3{,}90$, the value of the force in $D = \frac{370{,}23 \,.\, \gamma}{3{,}90} = 94{,}93 \,.\, \gamma$ lbs. Proceeding in this manner, a value of the force that has to counteract the tendency to rotation round $E_4 = \frac{701{,}92 \,.\, \gamma}{5{,}9} = 118{,}97 \,.\, \gamma$ lbs.; and a fifth force in reference to rotation round $E_5 = \frac{1163{,}43 \,.\, \gamma}{8{,}45} = 137{,}68 \,.\, \gamma$ lbs.; and, lastly, in reference to rotation round B, a force $= \frac{1760{,}21 \,.\, \gamma}{11{,}6} = 157{,}74o\gamma$ lbs. As this is the greatest value found, we put the pressure or thrust at the crown, $P = 151{,}74 \,.\, \gamma$, or, taking the weight of masonry as 150 lbs. per cubic foot, $P = 22761$ lbs. The depth of arch at crown is 1,3 feet; and, therefore, the area for each foot of length of the arch $= 144 \,.\, 1{,}3 = 187{,}2$ square inches; and hence the pressure on each square inch $\frac{22761}{187{,}2} = 122$ lbs., supposing the line of resistance to bisect the voussoirs.

If, with M. Petit, we take the angle of reposeo$= 30°$, we obtain for the force to prevent dislocation of the arch by sliding, the following values. The joints E_1F_1, E_2F_2, E_3F_3 ... are inclined to the horizon at the angles 83° 40′, 77° 20′, 71°, 64° 40′, 58° 20′, 52°, respectively, therefore,

$P_1 = (6{,}89 + 8{,}48)$ *tang.* $(83° 40' - 30°) \,.\, \gamma = 15{,}37$. *tang.* $53° 40' \,.\, \gamma = 20{,}9 \,.\, \gamma$ lbs.;
$P_2 = (15{,}37 + 18{,}17)$ *tang.* $(77° 20' - 30°) \,.\, \gamma = 33{,}54$. *tang.* $47° 20' \,.\, \gamma = 36{,}4 \,.\, \gamma$ lbs.;
$P_3 = 57{,}73$. *tang.* $41° \,.\, \gamma = 50{,}1 \,.\, \gamma$ lbs.
$P_4 = 90{,}56$ *tang.* $34° 40' \,.\, \gamma = 62{,}6 \,.\, \gamma$ lbs.
$P_5 = 134{,}13$ *tang.* $28° 20' \,.\, \gamma = 72{,}3 \,.\, \gamma$ lbs.
$P_6 = 188{,}53$ *tang.* $22° \,.\, \gamma = 76{,}2 \,.\, \gamma$ lbs.

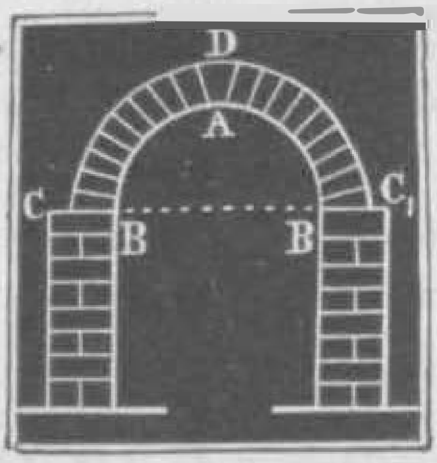
and, therefore, the greatest horizontal pressure counteracting sliding $= 76,2 \,.\, \gamma$ lbs. As, however, this pressure, in its tendency to cause rotation round an inner joint, amounts to $151,7 \,.\, \gamma$, it is evident that a sliding cannot take place. And in like manner it is easy to convince ourselves that neither rotation nor sliding outwards is possible. As to the stability of the abutment OUK, the moment of the force P referred to O as an axis of rotation $= 151,74 \,.\, \gamma \,.\, \overline{OV} = 151,74 \,.\, 18 \,.\, \gamma = 2731 \,.\, \gamma$ lbs.; the moment of the loaded arch $ABKH$, is:

$$1760,2 \,.\, \gamma + 188,53 \,.\, \overline{OU} \,.\, \gamma = (1760,2 + 188,53 \,.\, 6,8)\, \gamma = 3042 \,.\, \gamma \text{ lbs.},$$

and that of the piern $= 343 \,.\, \gamma$ lbs.; hence the moment resisting rotation round $O = (3042 + 343) \,.\, \gamma = 3385 \,.\, \gamma$ lbs., and, therefore, *heeling* cannot possibly take place. If, however, more ample security be desired, we must substitute for P, $1,9\,P$, as above explained, and, therefore, take the moment of the force to produce heeling $= 5189 \,.\, \gamma$, and thus we see that our abutment would be too thin; instead of 5,15 feet thickness, it would require from 11 to 12 feet. For a thickness of 11 feet, the moment of stability $= 1760,2 \,.\, \gamma + 188,53 n\ 11\, \gamma + 1281 n\, \gamma = 5115 n\, \gamma$, which would prove a *sufficient* stability.

§ 20. *Tables for Arches.*—In order to facilitate investigations on the stability of arches of the more usual forms, M. Petit calculated a series of tables of which we shall here give a short abstract. The first of these tables refers to semi-circular vaults, as in Fig. 25, the second refers to semi-circular arches with spandril walls at an angle of 45° as shown by the dotted line in Fig. 23. The third table refers to semi-circular arches with horizontal spandrils, as shown by the dotted line in Fig. 22, and the fourth table refers to segmental arched vaults.

Fig. 25.



In the first three tables, the two first vertical columns contain the proportions of the archese the third column contains the angle of rupturee the fourth and fifth co-efficients of *horizontal thrust*, in terms of the radius or half span, and the weight of the materials (see example 1 following); and in the sixth, the co-efficient of the maximum thickness of abutment in terms of the half span.

To apply these tables, we have to look in column 1 for the ratio $k = \frac{r_2}{r_1}$ of the radius of the extrados to that of the intrados, and pass along horizontally to the fourth and fifth columns, and the greater of the numbers found in these two columns is to be taken as a co-efficient by which to multiply the square of r_1, the radius of intrados, and the weight per cubic foot γ of the masonry, the product of which gives the horizontal thrust in question. The sixth column gives the thickness of abutment, supposing the height infinite, by multiplying the co-efficients there found by the radius r_1. For low abutments, the thickness is less, and should be calculated according to § 17.

The fourth table contains in its first column the ratio $k = \frac{r_1}{r_2}$, and in the other columns the thrust of the arch for various proportions of the span s to the versed sine or height h. This latter table is only applicable when the angle of rupture, given in the first table, is *less* than the half of the central angle α, and of the arc of the vault.

TABLE I.

SEMICIRCULAR ARCH WITH PARALLEL VAULTED SURFACES.

Ratio of the radii. $k = \frac{r_2}{r_1}$	Ratio of radius of intrados to depth of vous soir.	Angle of rupture.	Co-efficient p of the thrust of arch; for rotation.	Co-efficient p of the thrust of arch; for sliding.	Co-efficient for greatest thickness of abutments.
2,732	1,154	0° 00′	0,00000	0,98923	
2,70	1,176	13 42	0,00211	0,96262	
2,50	1,333	35 52	0,02283	0,80346	
2,20	1,666	51 4	0,08648	0,58767	
2,00	2,000	57 17	0,13017	0,45912	1,3223
1,80	2,500	61 24	0,16373	0,34281	1,1414
1,60	3,333	63 49	0,17517	0,23874	0,9525
1,55	3,636	64 3	0,17478	0,21464	0,9031
1,50	4,000	64 9	0,17254	0,19130	0,8527
1,45	4,444	64 5	0,16798	0,16872	0,8007
1,40	5,000	63 48	0,16167	0,14691	0,7838
1,35	5,714	63 19	0,15287	0,12587	0,7622
1,30	6,666	62 14	0,14330	0,10559	0,7370
1,25	8,000	61 15	0,12847	0,08608	0,6987
1,20	10,000	59 41	0,11140	0,06733	0,6504
1,15	13,333	57 1	0,09176	0,04935	0,5905
1,10	20,000	53 15	0,06754	0,03213	0,5066
1,05	40,000	46 22	0,03813	0,01568	
1,02	100,000	38 12	0,01691	0,00618	
1,00	∞	0 00	0,00000	0,00000	

TABLE II.

SEMICIRCULAR ARCHES, MASONRY AT THE BACK, OF 45° INCLINATION.

Ratio of the radii. $k = \frac{r_2}{r_1}$	Ratio of radius of intrados to depth of vous-soir.	Angle of rupture.	Co-efficient p of the thrust of arch; for rotation.	Co-efficient p of the thrust of arch; for sliding.	Co-efficient for greatest thickness of abutments.
2,00	2,000	60°	0,26424	0,74361	1,7264
1,80	2,500	60	0,29907	0,57383	1,5147
1,60	3,333	60	0,31245	0,42191	1,2990
1,55	3,636	61	0,31222	0,38673	1,2437
1,50	4,000	61	0,30996	0,35266	1,1877
1,45	4,444	60	0,30587	0,31971	1,1308
1,40	5,000	59	0,30001	0,28787	1,0954
1,35	5,714	58	0,29285		1,0823
1,30	6,666	57	0,28231	0,22756	1,0626
1,25	8,000	54	0,27102		1,0412
1,20	10,000	50	0,25806	0,17171	1,0160
1,15	13,333	47	0,24477		0,9894
1,10	20,000	42	0,23292	0,12032	0,9652
1,05	40,000	36	0,22902		0,9571

TABLE III.

SEMICIRCULAR ARCHES, WITH HORIZONTAL MASONRY ABOVE.

Ratio of the radii $k = \frac{r_2}{r_1}$.	Ratio of radius of intrados to depth of voussoir.	Angle of rupture.	Co-efficient p of the thrust of arch; for rotation.	Co-efficient p of the thrust of arch; for sliding.	Co-efficient for greatest thickness of abutments.
2,00	2,000	36°	0,05486	0,50358	1,3834
1,80	2,500	44	0,08508	0,37901	1,2001
1,60	3,333	52	0,12300	0,26755	1,0082
1,55	3,636	54	0,13027	0,24173	0,9584
1,50	4,000	56	0,13648	0,21673	0,9075
1,45	4,444	57	0,14122	0,19256	0,8554
1,40	5,000	59	0,14421	0,16920	0,8018
1,35	5,714	60	0,14504	0,14666	0,7465
1,30	6,666	61	0,14332	0,12495	0,7379
1,25	8,000	62	0,13872	0,10405	0,7260
1,20	10,000	63	0,13073	0,08397	0,7048
1,15	13,333	64	0,11895	0,06471	0,6723
1,10	20,000	65	0,10279	0,04627	0,6249
1,05	40,000	69	0,081755	0,02865	0,5573
1,00	∞	75	0,055472	0,01185	

TABLE IV.

VAULTED ARCHES, WITH PARALLEL ARCHED SURFACES.

Ratio of the radii $k = \frac{r_2}{r_1}$.	Co-efficient p of the thrust of the arch. $s = 4h$	$s = 5h$	$s = 6h$	$s = 7h$	$s = 8h$	$s = 10h$	$s = 16h$
1,40	0,15445	0,14691	0,14691	0,14691	0,14691	0,14478	
1,35	0,14771	0,13030	0,12587	0,12587	0,12587	0,12405	
1,30	0,13764	0,12331	0,10682	0,10559	0,10559	0,10406	
1,25	0,12547	0,11402	0,10009	0,08668	0,08608	0,08483	0,07180
1,20	0,11023	0,10196	0,09102	0,07999	0,06981	0,06636	0,05616
1,15	0,09123	0,08634	0,07866	0,07050	0,06259	0,04904	0,04116
1,10	0,06737	0,06563	0,06158	0,05666	0,05160	0,02414	0,02681
1,05	0,03776	0,03804	0,03709	0,03550	0,03357	0,02944	0,01882
1,01	0,00834	0,00871	0,00886	0,00889	0,00885	0,00862	0,00747

The following table contains a synopsis of the relative dimensions of segmental arches.

Ratio of the span to height $\frac{s}{h}$	Half central angle α.	$sin. \alpha$.	Ratio of radius of intrados r_1 to height h $\frac{r_1}{h}$.
4	53° 7′ 30″	0,8000	2,500
5	43 36 10	0,6897	3,625
6	36 52 10	0,6000	5,000
7	31 53 26	0,5283	6,625
8	28 4 20	0,4706	8,500
10	22 37 10	0,3846	13,000
16	14 15 0	0,2462	32,500

Example 1. A semicircular arch with horizontal road-way over it, having radius of intrados $r_1 = 10$ feet. What should be the dimensions? What will be the thrust? According to Perronet's formula, $d = 0{,}0694 \,.\, 10 + 1 = 1{,}694$ feet, for which take 1,7 feet. We have now $r_2 = 11{,}7$ and $k = \frac{r_2}{r_1} = 1{,}17$. From Table 3, the angle of rupture is $63\frac{4}{5}°$, the co-efficient of horizontal thrust $= 0{,}1190 + \frac{4}{5} \,.\, 0{,}0118 = 0{,}1237$ (0,0118 being the difference between .119, and the number next above it). Taking 150 lbs. per cubic foot as weight of masonry, the thrust at crown $= 0{,}1237 \,.\, 150 \,.\, 10^2 = 1855$ lbs. For the extreme thickness of abutment, we have from the same table the co-efficient $0{,}6723 + \frac{4}{5} \,.\, 0{,}0325 = 0{,}6855$, and, therefore, the thickness $= 0{,}6855 \,.\, 10 = 6{,}85$ feet. For low abutments, the formula of § 17 gives smaller dimensions.

Example 2. What dimensions and forces correspond to a vault of 10 feet span, and 2 feet rise? Here we have $\frac{h}{s} = \frac{1}{5}$, therefore, the half central angle $\alpha = 43° \; 36' \; 10''$, and $sin. \alpha = 0{,}6897$, and the radius $r = 3{,}625 \,.\, 2 = 7{,}25$ feet. Table 4 gives the co-efficient of horizontal thrust, (as $s = 5\,h$, and according to Perronet's formula: $d = 1{,}5$, so that $k = 1{,}2) = 0{,}10196$, and hence the thrust $= 0{,}102 \,.\, 150 \,.\, 7{,}25^2 = 804$ lbs.

Remark 1. That the part of the abutment on which the arch rests may not be thrust away, it is necessary that the horizontal thrust $P = pr^2\gamma$ should be *less* than $\frac{1}{2} f \alpha \, (r_2^2 - r_1^2) \, \gamma$ the friction on the bed. If this be not the case, as, for example, in very flat arches, this sliding out of the upper part of the abutment must be prevented by artifices, such as iron tie rods. The co-efficient of friction $f = 0{,}76$, therefore, $\frac{1}{2} f = 0{,}38$, and, therefore, the strength of the ties must be such as to resist a force $= p - 0{,}38 \, \alpha \, (k^2 - 1) \, r_1^2 \, \gamma$. This is the state of the case when $s = 4\,h$ and k is less than 1,06; when $s = 5\,h$ to 10 h, and k less than 1,15, and when $s = 16\,h$, this sliding is sure to take place.

Remark 2. The literature on the subject of arches is very extensive; but the theories treated therein are not always admissible, because the assumptions are inconsistent with experience. We shall here only mention the authors whose theories and investigations are generally accepted as the best approximations. We refer, therefore, to Coulomb, "Théorie des machines simples," who first gave a rational theory of the arch, and such as is in substance given in the foregoing paragraphs. This theory is given with greater completeness by Navier, "Résumé des Leçons sur l'application de la Mécanique," t. I. There are papers by Audoy, Garidel, Poncelet and Petit, in the "Mémorial de l'officier du génie." The substance of the papers of Garidel and Petit, and their tables, are given by Mr. Hann in his Treatise on Bridges, published by Weale, 1839. Moseley's paper on the "Theory of the Arch," is, perhaps, the most elegant exposition of this interesting and important subject. The works of Robison, Whewell, Eytelwein, Gerstner, and others, contain particular expositions of Coulomb's theory. Hagen has published an interesting essay, entitled "Über Form und Stärke gewölbter Bogen," Berlin, 1844.

CHAPTER III.

THEORY OF FRAMINGS OF WOOD AND IRON.

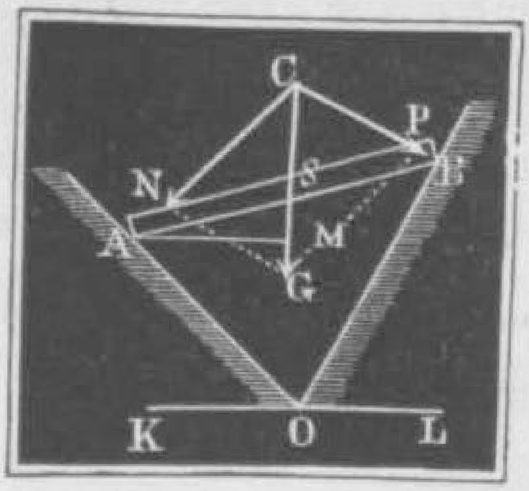
§ 21. *Wooden Structures.*—Structures of wood and of iron differ essentially from those in stone, in that these materials are subjected to what have been termed tensile and transverse, as well as compressive strains, to which latter alone masonry is exposed.n Hence, in carpentry and iron-work, the *pieces* of which the framings are composed are not only laid one upon the other, but are *morticed*, *tenoned*, *fished*, *bolted*, *strapped*, &c., to unite them together. The principal axis of the pieces of any framing may be *horizontal*, *inclined*, or *vertical*. In the first case, they are termed *beams* or *joists*; in the second, *rafters*, *braces*, or *spears*, &c.; in the other, *posts*, *pillars*, *uprights*, &c. According to the function they fulfil, some pieces are termed *struts* or *spears* (viz: those resisting compression), and others, *ties* or *braces* (*i.e.* those resisting tension).

To investigate the *stability* or *equilibrium* of a framing, it is essential, in the first place, to know the forces and weights which the framing has to counteract. From these we determine, not only the forces which individual pieces have to withstand, but the forces acting at the points of connection, and the strains or pressures upon the points of support. Each part should have such form, position and dimensions, as to completely withstand every force acting on it.

As to the connection of the pieces of a framing with each other, we have principally to distinguish *bolts* and *pins*, *tenons* and *mortices*, *scarfs* and *shoulders*. Bolts and pins counteract, or take up all forces passing through their axes. Tenons and mortices counteract only forces acting in certain directions, and shoulders or scarfs counteract such forces as are directed at right angles to the plane of the shoulder.

§ 21*. A beam AB, Fig. 26, lying on inclined planes, is in an instable condition, unless friction or some artificial fastening, as *bolts* or *mortices* retain it. To establish equilibrium, it is a necessary condition that the vertical SG passing through the centre of gravity of the beam, should pass through the point C, in which the normals to the ends A and B of the planes intersect each other, for only then are the two components N and P, into which the weight G of the beam may be decomposed, taken up or

Fig. 26.



counteracted by the planes. If α and β be the angle AOK and BOL of the planes to the horizon, these forces are:

$$N = \frac{G \sin.\beta}{(\sin.\ \alpha+\beta)}, \text{ and } P = \frac{G \sin.\alpha}{\sin.(\alpha+\beta)}.$$

If, again, l be the length AB of the beam, s the distance AS of its centre of gravity S from the end A, and δ the angle of inclination BAM of the beam to the horizon, then the horizontal projection of $AS = s$ is $AM = s \cos.\delta$, or $= AC \sin.\alpha$, but as

$$AC = \frac{AB \sin. ABC}{\sin. ACB} = \frac{l \sin.(90^\circ - \beta + \delta)}{\sin.(\alpha+\beta)} = \frac{l \cos.(\beta-\delta)}{\sin.(\alpha+\beta)},$$

we have $AM = \dfrac{l \sin.\alpha \cos.(\beta-\delta)}{\sin.(\alpha+\beta)}$, and, therefore, we have the equation of condition:

$$s \sin.(\alpha+\beta) \cos.\delta = l \sin.\alpha \cos.(\beta-\delta).$$

If one of the planes be horizontal as AO in Fig. 27, then as $\alpha = 0$, we have $s \sin.\beta \cos.\delta = 0$, *i.e.* $\beta = 0$, or the other plane must likewise be horizontal. In order to prevent slipping of the beam in every other position, we must, Fig. 28, mortice one end of the beam

Fig. 27.

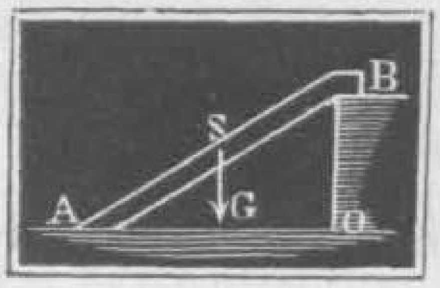
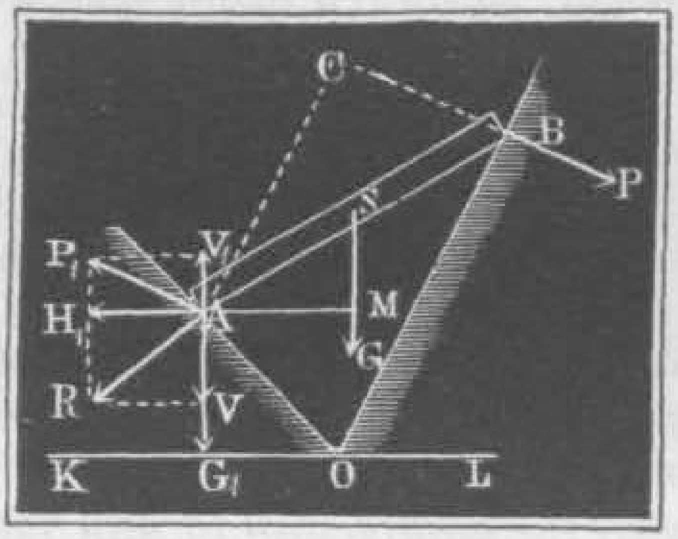


Fig. 28.



as A, or fasten it in some way. The pressure which the end of the beam there exerts on the inclined plane OB may be deduced from the theory of the bent lever MAC, whose arm $AM = AS \cos. SAM = s \cos.\delta$, and $AC = AB \cos. BAC = l \cos.(\beta-\delta)$, and hence P the pressure required

$$= \frac{G s \cos.\delta}{l \cos.(\beta-\delta)}$$

As the pressure on the point of support A is equal to the mean of all the forces acting on AB, we may assume that the vertical pressure $G_1 = G$, and its counter pressure $P_1 = P$, acts at this point. If, therefore, we decompose this latter into the horizontal force $H_1 = P_1 \sin.\beta$, and the vertical force $V_1 = P_1 \cos.\beta$, we obtain for the total pressure in A the horizontal component or *thrust* $H_1 = \dfrac{G s \sin.\beta \cos.\delta}{l \cos.(\beta-\delta)}$, and the vertical component, or vertical pressure:

$$V = G - V_1 = G\left(1 - \frac{s \cos. \beta \cos. \delta}{l \cos. (\beta - \delta)}\right),$$

from which we can easily calculate the magnitude and direction of the total pressure or strain.

For the case of a beam leaning on a wall, Fig. 29, $\beta = 90°$, hence:

$$H = \frac{G\, s \cos. \delta}{l \sin. \delta} = G \frac{s}{l} \operatorname{cotg.} \delta = P, \text{ and } V = G =$$

the weight of the beam.

Fig. 29.

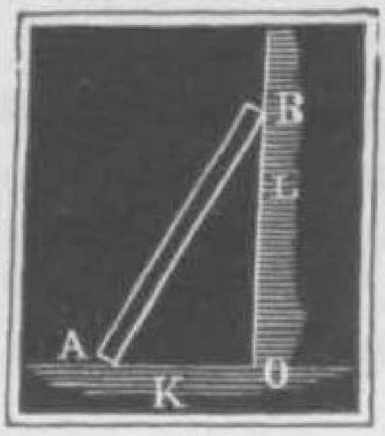
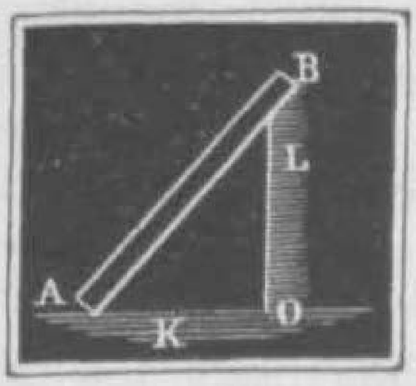


Fig. 30.



For the case of a beam leaning on a wall inclined at the same angles as the beam, as in Fig. 30 at B, $\beta = \delta$, hence:

$$P = G \frac{s}{l} \cos. \delta,$$

$$H = G \frac{s}{l} \sin. \delta \cos. \delta, \text{ and } V = G\left(1 - \frac{s}{l} \cos. \delta^2\right).$$

§ 22. *Thrust of Roofs.*—The formulas found in the preceding paragraphs are immediately applicable to calculating the thrust of rafters or "couples" for roofs (Fr. *fermes*). According to these, we have in the case of simple *lean-to* and coupled roofs, as in Figs. 31 and 32,

Fig. 31.

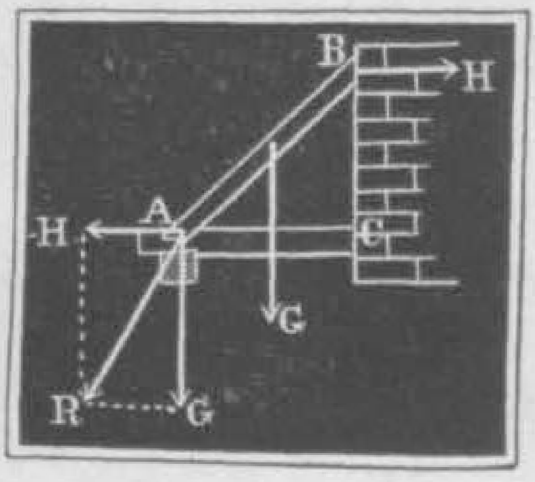
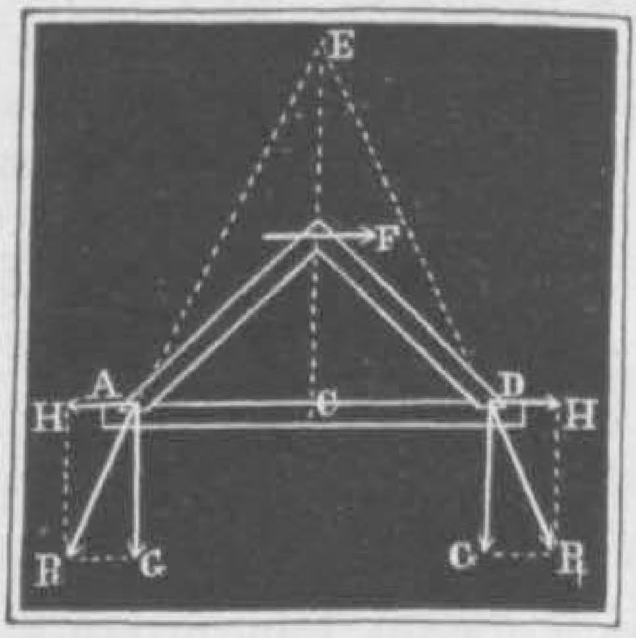


Fig. 32.



for the horizontal thrust acting at the lower and upper end: $H = \frac{Gs}{l} \operatorname{cotg.} \delta$, or, as in this case $s = \frac{1}{2}\, l$, $H = \frac{1}{2}\, G \operatorname{cotg.} \delta$; again the vertical pressure at the upper end $= 0$, and $= G$ the weight of the couple and its load at the lower end. If we put the height of roof $BC = h$, and the span or width $AC = DC = b$, then $cotg.\ \delta = \frac{b}{h}$, and hence the thrust of the couple $= \frac{1}{2} G \frac{b}{h}$; *and thus we see that the horizontal thrust increases directly as the span, and inversely as the height or pitch of the roof.* The usual limits of h are between $2b$ and $\frac{1}{2} b$. The former ratio is that of church roofs of the Saxon and Norman period, and the latter that of the flat Italian roofs of modern houses. In the former, $\delta = 26^\circ\ 34'$, and in the latter $63^\circ\ 26'$. The thrust of the couples is very great in flat roofs; in the Italian roof, for instance, as above specified, the thrust equals the whole weight of the couple and load; in the Saxon roof the thrust is not above one-fourth of this. The feet of the couple must be morticed, or otherwise fastened into the beam (tie-beam) to prevent sliding. The entire pressure of a rafter at its foot A isn

$$R = \sqrt{H^2 + V^2} = \sqrt{1 + \tfrac{1}{4}\,(cotg.\ \delta)^2}\ .\ G = \sqrt{1 + \left(\frac{b}{2h}\right)^2}\ .\ G,$$

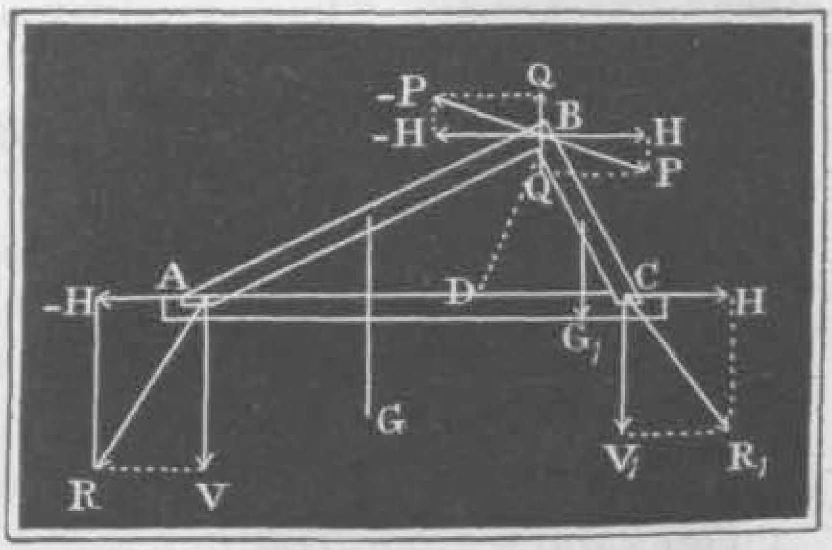
and for $R.AH = \phi$, the angle made by the line of pressure with the horizon, we have

$$tang.\ \phi = \frac{G}{H} = \frac{G}{\frac{1}{2}\, G\, \frac{b}{h}} = \frac{2h}{b} = 2\ tang.\ \delta.$$

Thus we may find the direction of the total thrust at the foot, by doubling the height of the couple; or, by making $CE = 2\ .\ CB$, and drawing a line through the foot A, from the point E, and producing it to R.

For the pair of rafters, Fig. 32, in which the rafters are of equal length, these exert on each other only a horizontal thrust; but if the rafters be of unequal length, as in Fig. 33, the force P with which one rafter presses upon the other, deviates by a certain angle from the horizontal. If G be the weight of one rafter AB, and G_1 that of the other CB, and if δ and δ_1 be the angle of inclination of these rafters to the horizon, and if β be the angle of inclination BDC of the plane in which we may conceive the rafters to abut on each other, and against which the force P acts at right angles, we haven

Fig. 33.



$$P = \tfrac{1}{2}\,\frac{G\ cos.\ \delta}{cos.\ (\beta - \delta)},\ \text{and} = \tfrac{1}{2}\,\frac{G_1\ cos.\ \delta_1}{cos.\ (180^\circ - \beta - \delta_1)},\ \text{hence}$$

$$-\,G\ cos.\ \delta\ cos.\ (\beta + \delta_1) = G_1\ cos.\ \delta_1\ cos.\ (\beta - \delta),\ \text{or}$$

$$\frac{G\,(sin.\ \beta\ sin.\ \delta_1 - cos.\ \beta\ cos.\ \delta_1)}{sin.\ \beta\ cos.\ \delta_1} = \frac{G_1\,(sin.\ \beta\ sin.\ \delta + cos.\ \beta\ cos.\ \delta)}{sin.\ \beta\ cos.\ \delta};$$

dividing, we have:

$$G\,(tang.\ \delta_1 - cotg.\ \beta) = G_1\,tang.\ \delta + cotg.\ \beta),\ \text{thus}$$

$$cotang.\ \beta = \frac{G\ tang.\ \delta_1 - G_1\ tang.\ \delta}{G + G_1}.$$

And from this we have the horizontal thrust of both rafters:

$$H = P\ sin.\ \beta = \tfrac{1}{2}\,\frac{G\ sin.\ \beta\ cos.\ \delta}{cos.\ (\beta - \delta)} = \frac{\tfrac{1}{2}\,G}{cotg.\ \beta + tang.\ \delta}$$

$$= \frac{\tfrac{1}{2}\,(G + G_1)}{tang.\ \delta + tang.\ \delta_1}.$$

As to the vertical pressures V and V_1 at the rafter feet, the one is equal to the weight G, minus the vertical component $Q = P\ cos.\ \beta$, and the other is equal to the weight G_1 plus this component; or,

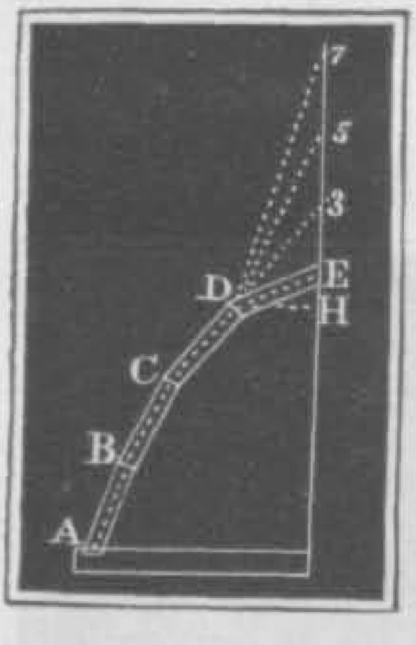
$$V = G - H\ cotg.\ \beta = G - \tfrac{1}{2}\,\frac{(G\ tang.\ \delta_1 - G_1\ tang.\ \delta)}{tang.\ \delta + tang.\ \delta_1},$$

$$\text{and } V_1 = G_1 + \tfrac{1}{2}\left(\frac{G\ tang.\ \delta_1 - G_1\ tang.\ \delta}{tang.\ \delta + tang.\ \delta_1}\right).$$

Example. The roof ABD (Fig. 32), is 40 feet span, and 30 feet height, and consists of couples 4 feet from centre to centre, 6×8 inch scantlings — required the thrust. Assuming each square foot of roofing to weigh 15 lbs., we have for the load on each rafter $15 \times 4\sqrt{20^2 + 30^2} = 60\sqrt{1300} = 2163$ lbs. The rafter itself weighs $\frac{1}{2} \times \frac{2}{3} \times 44\sqrt{20^2 + 30^2} = \frac{44}{3}\sqrt{1300} = 529$ lbs., and, therefore, the vertical pressure of a rafter $V = G = 2163 + 529 = 2692$ lbs., and the thrust $= \frac{1}{2}\,G\,\frac{b}{h} = \frac{1}{2} \cdot 2692\,\frac{20}{30} = 897$ lbs.

§ 23. *Compound Roofs.*—In many framings, as in mansard roofs, the rafter DE, Fig. 34, does not rest on a tie-beam, but on a second rafter CD, and this again on a third, and fourth, and so on. That the pressure of one beam may be completely transferred to the next in this case, it is necessary that they should have certain relative positions. These positions are determined by the conditions that any two beams abutting against each other should undergo equal horizontal pressures. The horizontal pressure of the rafter DE, is $H = \frac{1}{2}\,G\ cotg.\ \delta$, when $G =$ the weight, and δ its inclination. For the second beam or rafter DC: $H = \dfrac{\frac{1}{2}\,(G + G_1)}{tang.\ \delta_1 - tang.\ \delta}$, when G_1 and δ_1 denote weight and inclination of this second beam. Hence by equating the two values, we have:

Fig. 34.



$$G\ cotg.\ \delta = \frac{G + G_1}{tang.\ \delta_1 - tang.\ \delta},\ \text{i. e.}$$

$$tang.\ \delta_1 = tang.\ \delta + \frac{(G + G_1)}{G}\,tang.\ \delta = \left(2 + \frac{G_1}{G}\right)tang.\ \delta;$$

and, in like manner, for the inclination δ_2 of a third beam, seeing *that the horizontal thrust is everywhere the same.*

$$G \; cotg. \; \delta = \frac{G_1 + G_2}{tang. \; \delta_2 - tang. \; \delta_1}, \text{ hence}$$

$$tang. \; \delta_2 = tang. \; \delta_1 + \frac{G_1 + G_2}{G} tang. \; \delta$$

$$= \left(2 + \frac{G_1}{G} + \frac{G_1}{G} + \frac{G_2}{G}\right) tang. \; \delta = \left[2\left(1 + \frac{G_1}{G}\right) + \frac{G_2}{G}\right] tang. \; \delta,$$

and in like manner for a fourth:

$$tang. \; \delta_2 = tang. \; \delta_2 + \frac{G_2 + G_3}{G} tang. \; \delta$$

$$= \left[2\left(1 + \frac{G_1}{G} + \frac{G_2}{G}\right) + \frac{G_3}{G}\right] tang. \; \delta, \text{ \&c.}$$

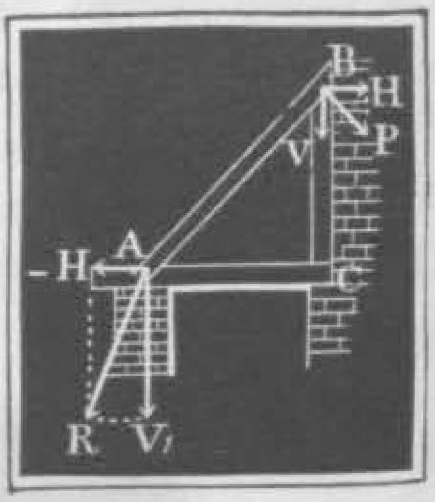
If each beam be of the same weight G, then $tang. \; \delta_1 = 3 \; tang. \; \delta$, $tang. \; \delta_2 = 5 \; tang. \; \delta$, $tang. \; \delta_3 = 7 \; tang. \; \delta$, $tang. \; \delta_4 = 9 \; tang. \; \delta$, &c.

If, therefore, in this form of roof, the height EH, Fig. 34, corresponding to the first beam or rafter DE, be set off upwards repeatedly, and through the divisions 1, 3, 5, 7, &c., lines $D1$, $D3$, $D5$, $D7$, &c., be drawn, these lines give the inclinations of the other rafters. It is also evident, that the figure of this combination of rafters is that of a funicular polygon formed by the weights G_1, G_2, G_3, &c. (*see* Vol. I. § 144), and this coincidence is quite explained, if we conceive the two halves of the weight G of each beam collected at its ends D, C, B, A, &c., and pulling downwards, that is, if we assume the weight G acting at each of these points.

If we take the beams very short, and very numerous, the axis of such a framing becomes a catenary.

§ 24. *Supported Rafters.*—If the head of a rafter rests on a pillar BC, Fig. 35, the thrust of the rafter is less than when it merely leans on a vertical wall. In this case, according to § 21*, the pressure on the head of this pillar is:

Fig. 35.



$$P = G \frac{s}{l} cos. \; \delta = \tfrac{1}{2} \, G \; cos. \; \delta,$$

and the horizontal thrust:

$H = P \; sin. \; \delta = \frac{1}{2} \, G \, cos. \; \delta \; sin. \; \delta = \frac{1}{4} \, G \; sin. \; 2 \, \delta.$

As the pillar supports a part of the weight $G = V = P \; cos. \; \delta = \frac{1}{2} \, G \; (cos. \; \delta)^2$, the beam does not, of course, press with its whole weight G on the foot A; but with a force:

$$V_1 = G - \tfrac{1}{2} \, G \, (cos. \; \delta)^2 = G \left[1 - \tfrac{1}{2} \, (cos. \; \delta)^2\right] = \tfrac{1}{2} \, G \left[1 + (sin. \; \delta)^2\right].$$

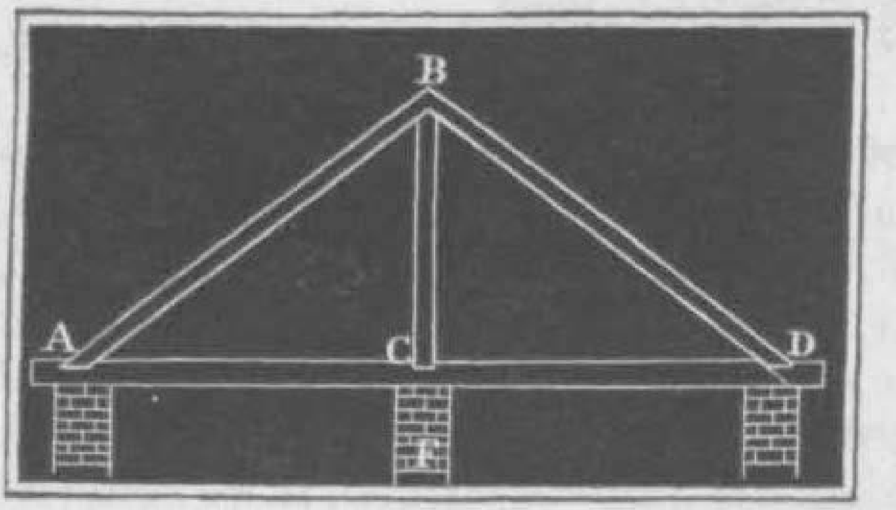
From this vertical pressure, and the horizontal thrust H, we get the angle ϕ, which the resultant R makes with the horizon, viz:

$$tang. \; \phi = \frac{H}{V_1} = \tfrac{1}{2} \cdot \frac{sin. \; 2 \, \delta}{1 + (sin. \; \delta)^2}.$$

If we introduce the depth $AC = b$ and height $BC = h$, we get $H = \dfrac{b \, h}{b^2 + h^2} \cdot \dfrac{G}{2}$, while in the case of the beam simply leaning, we had $H = \frac{b}{h} \cdot \frac{G}{2}$. If each unit of length of the rafter bears a load whose weight is γ, we have $G = \sqrt{b^2 + h^2} \cdot \gamma$, and therefore in the one case $H = \frac{b\,h\,\gamma}{2\sqrt{b^2 + h^2}}$, and for the other $H = \frac{b\sqrt{b^2 + h^2}}{2\,h} \cdot \gamma$, so that if the pillar *support* the rafter, the horizontal thrust is so much the less the lower the roof; while for roofs without such support, the thrust is greater as the roof is lower.

That the post BC may not be overturned by the horizontal force H, it is necessary to support it by a wall.

Fig. 36.



The relations of the forces now discussed, occur in the coupled roof, shown in Fig. 36, applicable in some cases, where the rafters are supported at the ridge by a central wall or column. The pillar takes up the weights $\frac{1}{2}\,G\,(cos.\ \delta)^2$, $\frac{1}{2}\,G\,(cos.\ \delta)^2$, and transfers, therefore, the vertical pressure $G\,(cos.\ \delta)^2$ to its support, and the horizontal thrust $H = \frac{1}{4}\,G\,sin.\ 2\,\delta$. There is no side support required for the pillar, as the horizontal thrust is equal on each side.

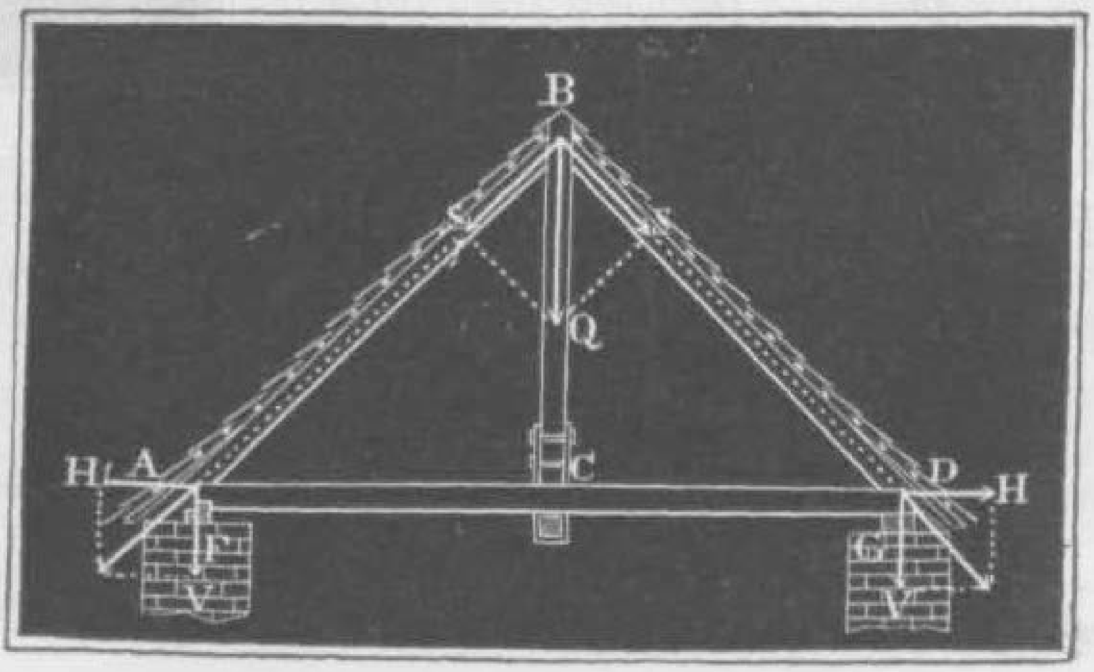
Example. For the roof in the example to § 22, the loading of one rafter $G = 2692$ lbs., $b = 20$ feet, $h = 30$ feet, therefore, $tang.\ \delta = \frac{3}{2}$, or $\delta = 56^\circ\ 18'\ 36''$; and, therefore, when a pillar is put in, the horizontal thrust is:

$$H = \frac{2692}{4}\,sin.\ 112^\circ\ 37'\ 12'' = 673\ sin.\ 67^\circ\ 22'\ 48'' = 621 \text{ lbs.}$$

The vertical pressure taken up by the pillar is $V = \frac{2692}{2}\,(cos.\ 56^\circ\ 18'\ 36'')^2 = 746{,}3$ lbs.; and, therefore, the beam supports a strain of only $2692 - 746{,}3 = 1945$ lbs.

§ 25. *King-posts.*—Whilst in the cases just considered the posts relieve the tie-beam (or walls in the absence of a tie) of a part of the thrust of the rafters, the *king-post*, BC, Fig. 37, acts in a very

Fig. 37.



different way; it carries a part of the weight of the tie-beam AD, and transfers it through the rafters AB and DB to act as thrust on the side walls, or rather as tensile strain on the tie. The force Q acting through the king-post, may be deduced from the scantlings of, and kind of load acting on the beam AD. If the load be uniformly distributed, it may be assumed, that the one half is supported by the side walls, the other half hangs on the king-post; but if the load be applied at the centre of the tie, it must be considered as acting entirely on the king-post. The force Q on the king-post is decomposed into two others in the direction of the rafters, the value of each of which is $S = \frac{Q}{2\,sin.\,\delta}$; and if we combine these forces with those arising from the weights G, G of the rafters, we get the horizontal thrust in A and D:

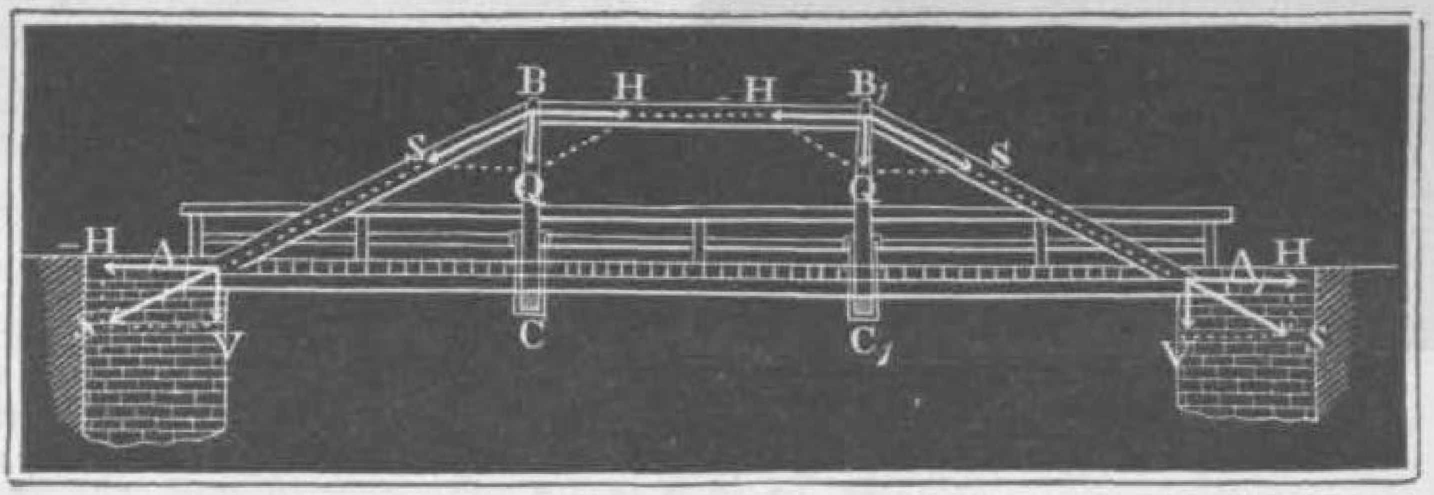
$$H = \tfrac{1}{2}\,G\,cotg.\,\delta + S cos.\,\delta = \frac{G + Q}{2}\,.\,cotg.\,\delta,$$

and the vertical pressure at that point:

$$V = G + S\,sin.\,\delta = G + \frac{Q}{2}.$$

For bridges and roofs of great span, more complicated framings, with two or more posts, and termed *trusses*, are applied. Fig. 38

Fig. 38.



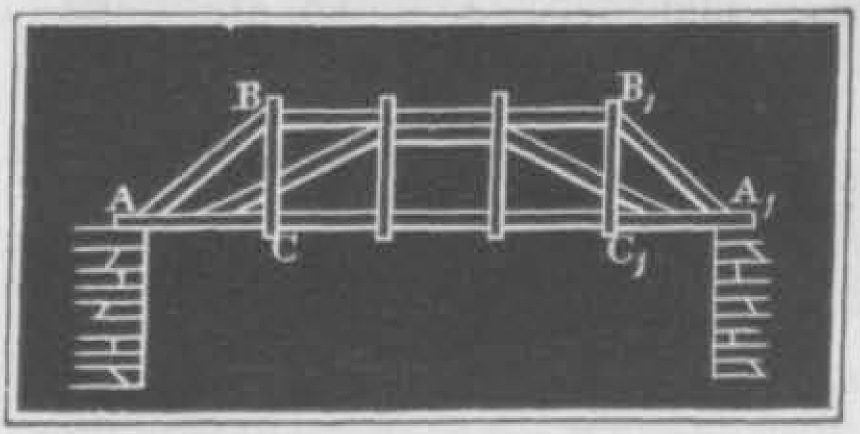
represents a *truss* with two posts, termed *queen-posts*, BC and B_1C_1, with a collar beam between them BB_1. The manner of calculating the strains in this framing is exactly similar to that for the simple couple with king-post. From the load on a queen-post Q, the horizontal thrust on the collar-beam tending to compress it, and acting on the side walls, if there be no tie, is $H = \frac{1}{2}\,Q\,cotg.\,\delta$, when δ is the inclination of the *rafters* or *braces* AB and A_1B_1 to the horizon. As this angle is frequently a small one, the thrust is considerable, and, therefore, care must be taken with the foot fastenings (*see* Vol. II. § 17). The scantlings of the braces and collar beams must be fixed by the rules in Vol. I. § 206, &c., so that they shall resist flexure and fracture, when exposed to forces

$$S = \frac{Q}{2\,sin.\,\delta}, \text{ and } H = \tfrac{1}{2}\,Q\,cotg.\,\delta.$$

The force Q depends on the loading of the bridge or roof. If the load be uniformly diffused, we shall do best to assume that each post carries $\frac{1}{3}$, and each side wall $\frac{1}{6}$ of the load.

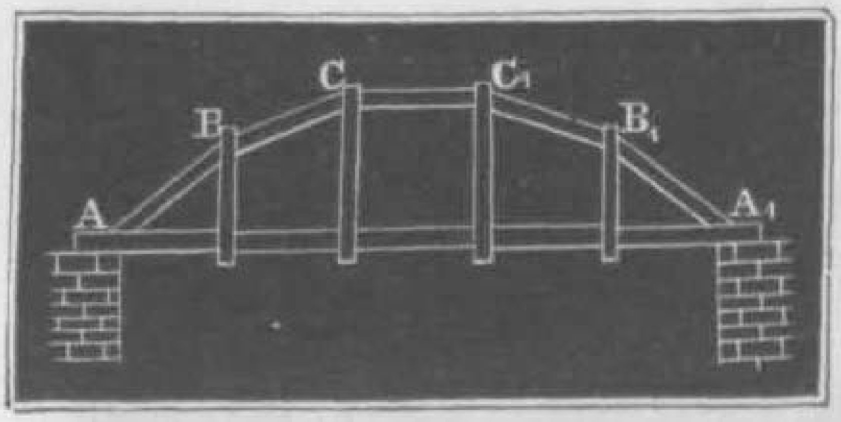
Example. Suppose the trussed bridge in Fig. 38, designed as one of two for a 60 feet span and 12 feet wide bridge: suppose each square foot of the bridge together with its load weighs 50 lbs., the weight of the bridge is $12 \times 60 \times 50 = 36000$ lbs., and the load on the queen-posts $= \frac{36000}{3} = 12000$. Therefore, for an inclination of the rafters of $22\frac{1}{2}^\circ$, the horizontal pressure $= \frac{1}{2}\, 12000$ *cotg.* $22\frac{1}{2}^\circ = 6000 \times 2{,}4142 = 14485$ lbs, and the thrust through each rafter $= \frac{6000}{sin.\ 22\frac{1}{2}^\circ} = 15679$ lbs. The half of these strains come on the pieces of each of the two trusses, so that on each collar there would be 7242,5 lbs., and on each rafter 7839,5 lbs. If we take the resistance of wood (Vol. I. § 206) at 7400 lbs., and if we strain only to $\frac{1}{20}$ of the absolute strength, we get for the section of each collar beam $F = \frac{7242{,}5 t.\ 20}{7400} = \frac{1448{,}5}{74} = 19{,}6$ square inches, and for each brace or rafter $\frac{7839.5\ .\ 20}{7400} = \frac{15679}{74} = 21{,}2$ square inches.

Fig. 39.



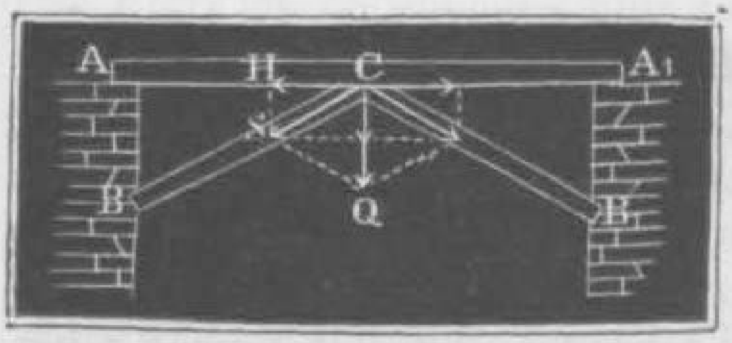
Remark. More composite trusses, as indicated in Figs. 39 and 40, are calculated in the same manner as the above. In each of these it may be assumed that each of the four posts or uprights carries one-fifth of the entire load, and that the remaining fifth rests immediately on the side walls. In the construction shown in Fig. 40, the directions of the different rafters are not optional, but dependent one upon the other. If Q be the weight on each post, and δ the inclination of the brace BC, and δ_1 that of AB, the horizontal thrust $H = Q$ *cotg.* $\delta = (Q + Q)$ *cotg.* δ_1, hence *cotg.* $\delta = 2$ *cotg.* δ_1 or *tang.* $\delta_1 = 2$ *tang.* δ

Fig. 40.



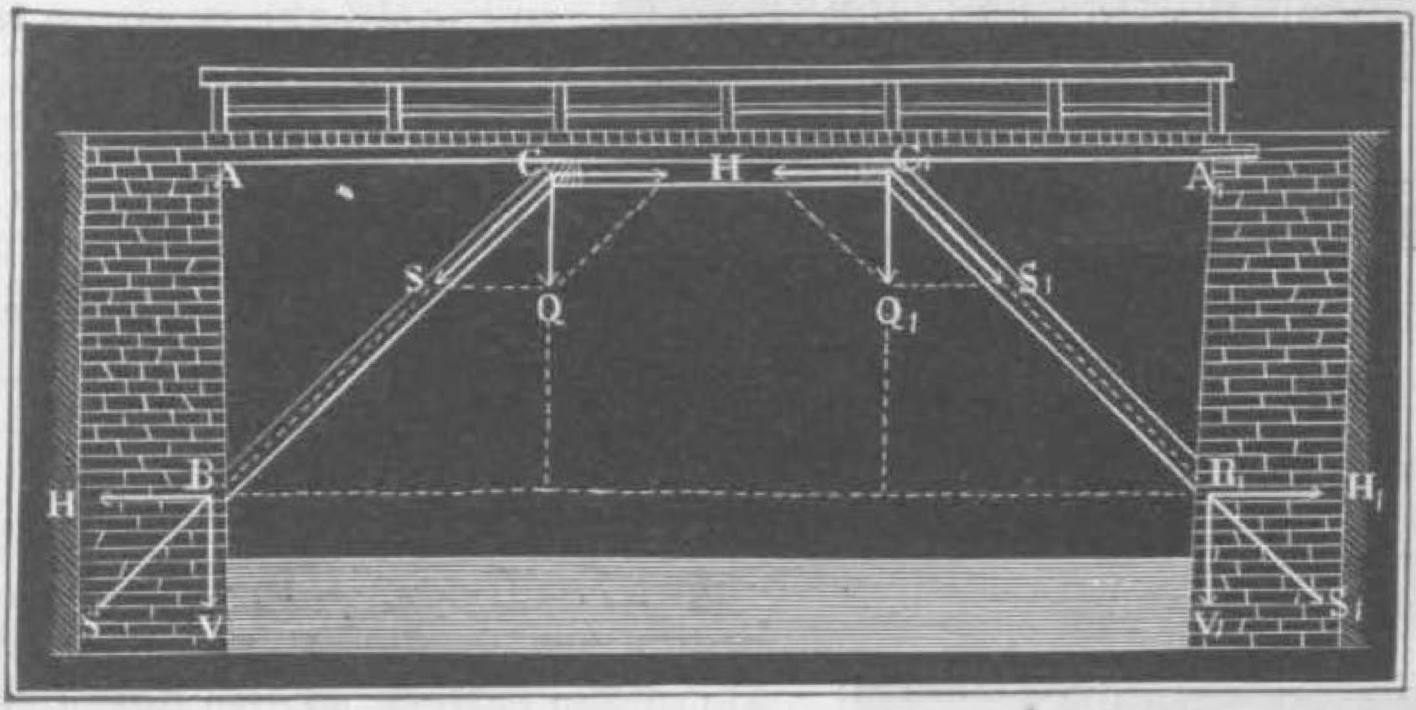
§ 26. *Timber Bridges.*—The framings in the foregoing section support the road-way or ceiling by suspension, but there are trusses applied for bridges, which support the road-way on the opposite principle of *sustaining* them. In these latter, the distribution of the pressure takes place exactly as in the former. In the simple case shown in Fig. 41, we have from the vertical force Q acting at the centre of the bridge AA_1, the horizontal thrust $H = \frac{1}{2}\, Q$ *cotg.* δ, and the strain on the *spear* or *strut* $BC = S = \frac{1}{2}\, \frac{Q}{sin.\ \delta}$, when δ is the inclination of the strut. In the example, Fig. 42, the forces are the same, but in this case Q may

Fig. 41.



be taken at $\frac{1}{3}$ of the whole load, whilst in the case Fig. 41, $Q = \frac{1}{2}$ the load. The piece CC_1 in Fig. 42 is termed a *straining cill*. If

Fig. 42.



there be a double set of struts or spears, as indicated in Fig. 43, there are four struts, and it may be assumed that each carries one-fifth of the whole load, or $Q = \frac{1}{5} G$. To prevent deflection of long spears, *braces* or *counter-braces* AD, A_1D_1 are added, particularly when there are several sets of spears. The distribution of the pressure in the case of spears of unequal length being used as in Fig. 44, is to be taken as exactly the same as in Fig. 43, only that in these the braces or suspending posts CD, C_1D_1 become the more requisite as the struts come to have considerable length. It is proper to take the weight of all the parts into calculation, and to reckon that half the weight of each part acts at its end.

Fig. 43.

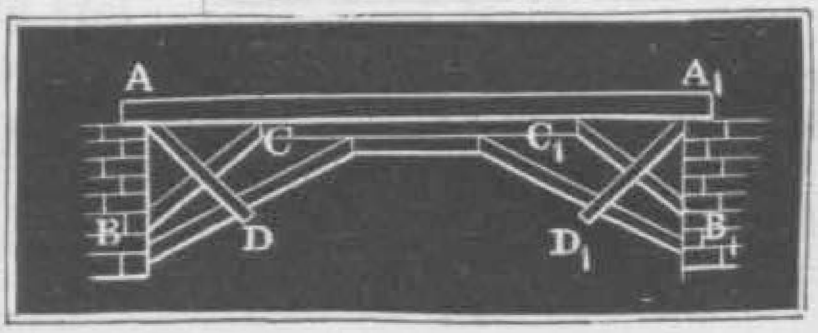


Fig. 44.

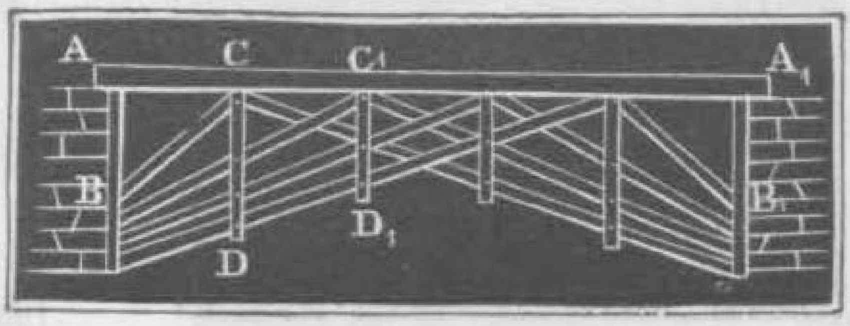
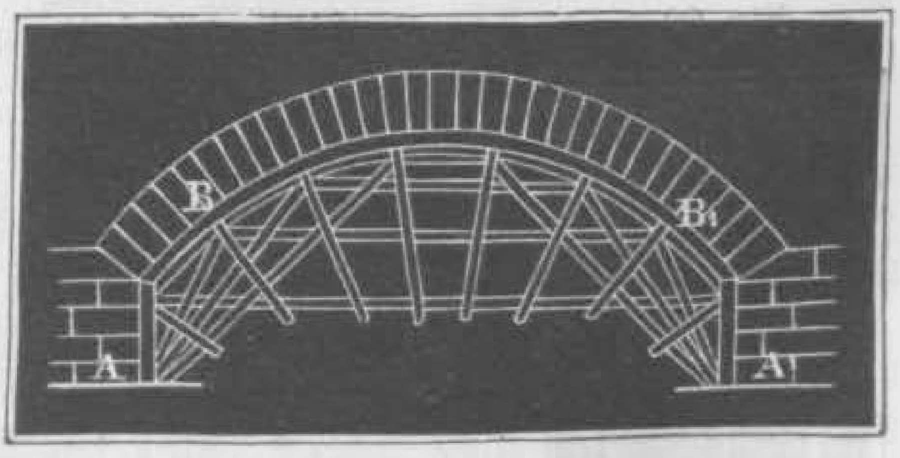
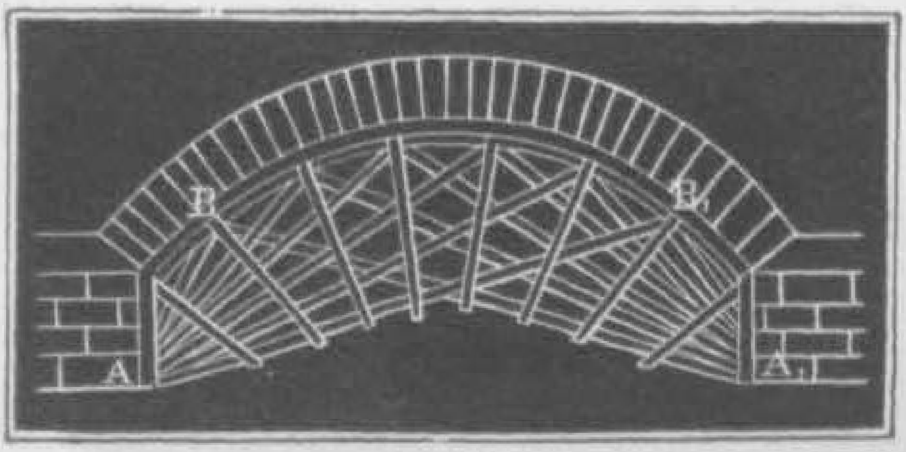


Fig. 45.



The centerings for bridges afford the most frequent application of the kind of framing we are now considering. Figs. 45 and 46 represent two such centres. The pressure which each simple frame ABB_1A_1 or ABA_1 undergoes and has to resist, may easily be determined by calculating the weight of the part of the arch bearing upon it.

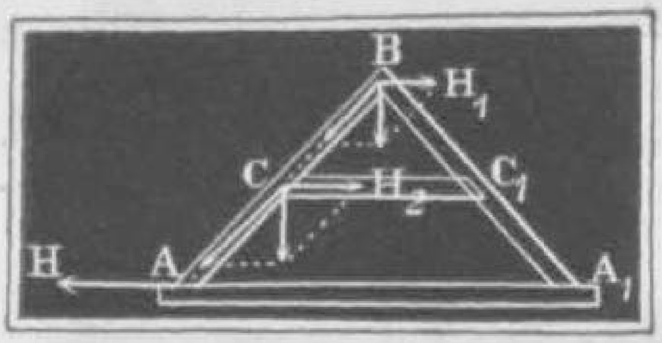
Fig. 46.



If the two spears abutting on each other have different inclinations to the horizon, as in the construction shown in Fig. 46, the strain on them is of course unequal. If the angles of inclination of two such spears $= \delta$ and δ_1, and if the vertical pressure at the abutting joint $= Q$, the strain along the spears $S = \frac{Q \cos. \delta_1}{\sin. (\delta + \delta_1)}$, $S_1 = \frac{Q \cos. \delta}{\sin. (\delta + \delta_1)}$ and the horizontal thrust of both $= H = \frac{Q \cos. \delta \cos. \delta_1}{\sin. (\delta + \delta_1)}$.

§ 27. *Roofs.* — In roofs, collar beams are applied to prevent deflexion of the rafters, as also *queen-posts*, *braces*, &c., and the nature of the forces may be traced, as in Figs. 47, 48, 49.

Fig. 47.



§ 28. *Posts.*—The strength of pillars and posts subjected to tensile or compressive strains, when these act in the direction of the axis, have been investigated, Vol. I. § 183 to § 206. It, however, not unfrequently happens, that the forces act

Fig. 48.

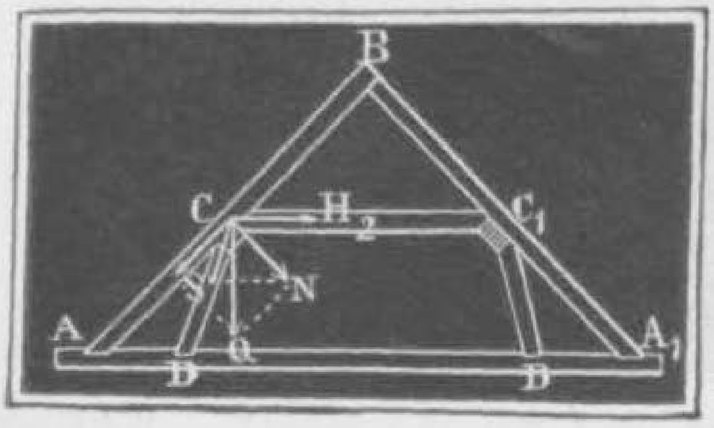


Fig. 49.

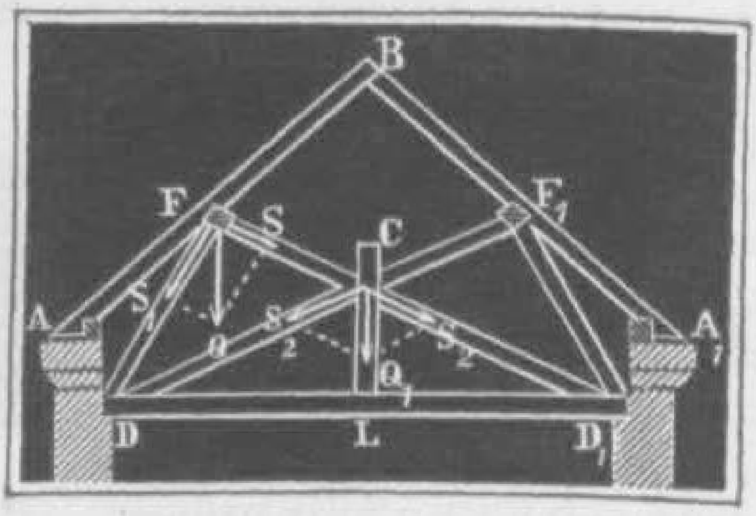
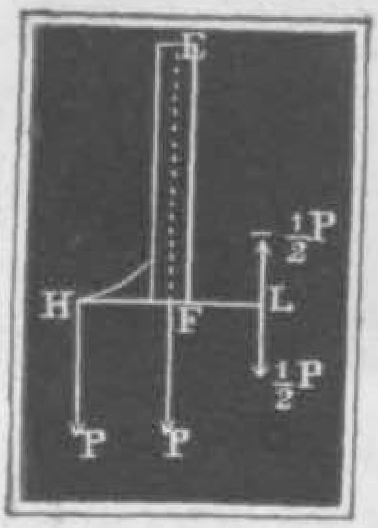


Fig. 50.



out of the axial direction, and we shall, therefore, examine this case. *EF* in Fig. 50, represents a suspending post to which a tensile strain P is applied excentrically. Let $F =$ the area l, the length EF of the post, $a =$ the leverage FH, or the distance of the direction of the force from that of the axis. Prolong FH in the opposite direction, and make $FL = FH = a$, and conceive that in L two equal and opposite forces $\frac{1}{2} P, -\frac{1}{2} P$ act: there results an axial force $FP = P$, and a couple $\frac{1}{2} P, -\frac{1}{2} P$. The former extends all the fibres uniformly by a quantity $\lambda_1 = \frac{P}{F \cdot E} \cdot l$, but the latter extends the fibres unequally on one side, and compresses them unequally on the other. If the post be rectangular, with the sectional dimensions b and h, where h is in the same plane as a, the moment of the force:

$$Pa = \frac{\lambda_2}{l} WE = \frac{\lambda_2 b h^3}{12 l} E \text{ (Vol. I. § 191)},$$

but the extension or compression of the fibres at the distance 1 from the axis: $\lambda_2 = \frac{12 P a l}{b h^3 E}$, and that of the extreme fibres

$= \frac{h}{2} \cdot \lambda_2 = \frac{6 P a l}{b h^2 E}$, therefore the greatest extension:

$$\lambda = \lambda_1 + \frac{h \lambda_2}{2} = \frac{Pl}{E}\left(\frac{1}{F} + \frac{6a}{b h^2}\right) = \frac{Pl}{Ebh}\left(1 + \frac{6a}{h}\right).$$

But for the force K producing rupture: $\frac{K}{E} = \frac{\lambda}{l}$, hence the modulus of strength:

$$K = \frac{P}{bh}\left(1 + \frac{6a}{h}\right), \text{ and inversely:}$$

$$P = \frac{bh}{1 + \frac{6a}{h}} \cdot K.$$

If the post be cylindrical, and its radius $= r$, we have (Vol. I. § 195).

$$Pa = \frac{\lambda_2}{l} \cdot \frac{\pi}{4} r^4 E, \text{ hence } \lambda_2 = \frac{4 P a l}{\pi r^4 E},$$

and the longitudinal extension:

$$\lambda = \lambda_1 + r\lambda_2 = \frac{P}{\pi r^2 E} l + \frac{4 P a l}{\pi r^3 E} = \frac{Pl}{\pi r^2 E}\left(1 + \frac{4a}{r}\right), \text{ hence}$$

$$P = \frac{\pi r^2 K}{1 + \frac{4a}{r}}.$$

If the force act at the periphery of the post, we have in the first case $a = \frac{1}{2} h$, and in the second $a = r$, and, therefore, for the rectangular section $P = \frac{bh}{4} K$, and for the cylindrical $P = \frac{\pi r^2 K}{5}$. Thus, theoretically, a rectangular post will carry only $\frac{1}{4}$, and a cylindrical one only $\frac{1}{5}$ when loaded in the direction of the side of what it will carry when fairly loaded. *Experiments* on cast iron give results of $\frac{1}{3}$ instead of $\frac{1}{4}$ for rectangular columns.

The same laws apply to the uprights *AC*, Fig. 51, but then λ must be taken as the *greatest compression.*

If the column be inclined, as in Fig. 52, and if its foot make an angle α, with the horizon, we may decompose P into two others $P_1 = P \sin. \alpha$, and $P_2 = P \cos. \alpha$, and in the equations for λ_1 and λ_2, we must substitute $P\ sin.\ \alpha$, for P, and besides this the extension λ_3 produced by the normal force $P\ cos.\ \alpha$, has to be introduced. If we substitute $P\ cos.\ \alpha$ for P, and l for a, we obtain $\frac{h\lambda_3}{2}$ for the greatest extension or compression produced by the force $P\ cos.\ \alpha$, and hence for rectangular sectioned beams this extension or compression.

Fig. 51.

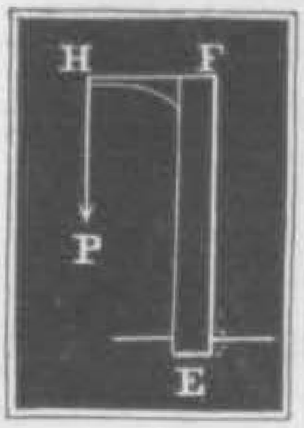
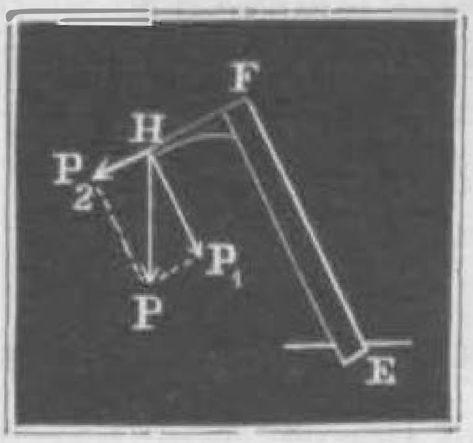


Fig. 52.



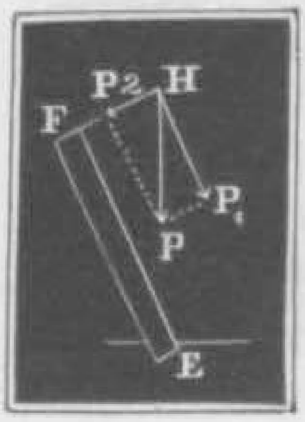
$$\frac{h}{2}\lambda_3 = \frac{6\,P\,l^2\,cos.\ \alpha}{b\,h^2\,E}, \text{ hence}$$

$$\lambda = \lambda_1 + \frac{h\,\lambda_2}{2} + \frac{h\,\lambda_3}{2} = \frac{Pl}{E}\left[\left(\frac{1}{b\,h} + \frac{6\,a}{b\,h^2}\right) sin.\ \alpha + \frac{6\,l\,cos.\ \alpha}{b\,h^2}\right]$$

$$= \frac{Pl}{E\,b\,h}\left[\left(1 + \frac{6a}{h}\right) sin.\ \alpha + \frac{6\,l}{h}\,cos.\ \alpha\right];$$

and therefore the tension:

Fig. 53.



$$P = \frac{b\,h\,K}{\left(1 + \frac{6a}{h}\right) sin.\ \alpha + \frac{6\,l}{h}\,cos.\ \alpha}.$$

If the arm FH be on the up-side of the beam, as shown in Fig. 53, we then have:

$$P = \frac{b\,h\,K}{\left(1 + \frac{6a}{h}\right) sin.\ \alpha + \frac{6\,l}{h}\,cos.\ \alpha}$$

and for round columns the expression becomes:

$$P = \frac{\pi\,r^2\,K}{\left(1 \pm \frac{4a}{r}\right) sin.\ \alpha + \frac{4\,l}{r}\,cos.\ \alpha}.$$

§ 29. If a loaded beam AB, Fig. 54, rests upon two uprights, the load P bears upon each in the proportion $\frac{l_2}{l}P$ on AD, and $\frac{l_1}{l}P$ on BE, when l_1, l_2, and l, represent the lengths AB, CA, and CB respectively.

Fig. 54.

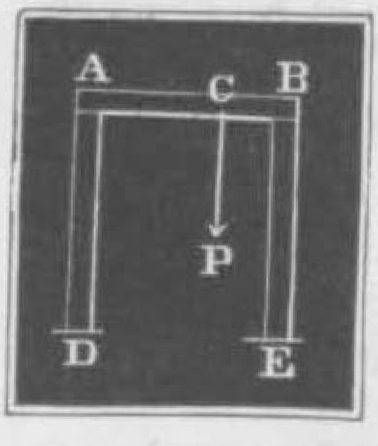


Fig. 55.

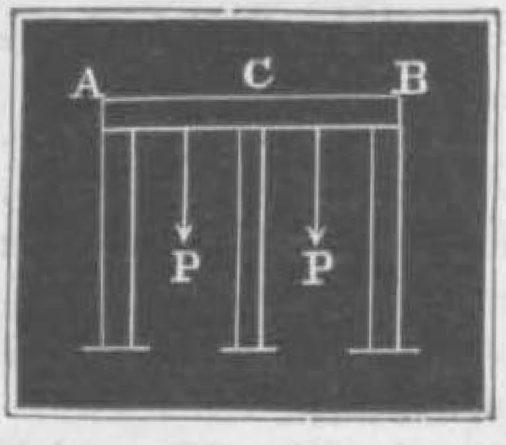
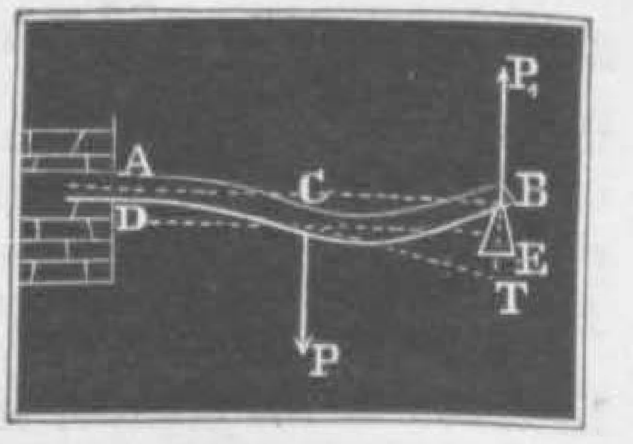


Fig. 56.



If a similar beam rests upon three or more uprights, the pressure on each can only be determined by aid of the theory of the elastic resistance of materials. If weights P and P act at the centre of the lengths AC and BC, and if we assume that the one part AC is independent of the other part BC, the pressure on the centre uprighth $= P$, and that on each of the othersh $= \frac{1}{2} P$. Buteif we consider the beam as an entire piece, the circumstances are different.

When a beam fastened by one end into a wall AB, Fig. 56, supports a weight P at C, and is supported at the other end B, the beam forms an elastic curve, horizontal at A, but inclining upwards at B. For simplicity's sake, let us assume P as acting at the middle C, and put the length $AB = 2\,l$. The deflexion BT of the outer half CB is equal to the deflexion of the inner half $AD = BE$ plus the tangent distance TE. But according to Vol. I. § 189, the height $BT = \frac{P_1 l^3}{3\,WE}$, if P_1 be the force on the end B required. Again the deflexion:

$$AD = \frac{P\,l^3}{3\,WE} - \frac{P_1}{2\,WE}\left(\tfrac{16}{3}\,l^3 - \tfrac{11}{3}\,l^3\right) = \frac{P\,l^3}{3\,WE} - \frac{5\,P_1\,l^3}{6\,WE},$$

and the tangent distance:

$$TE = CE\,.\,tang.\,\alpha = l\left(\frac{P\,l^2}{2\,WE} - \frac{3\,P_1\,l^2}{2\,WE}\right) = \frac{P\,l^3}{2\,WE} - \frac{3\,P_1\,l^3}{2\,WE};$$

and hence it follows:

$$\frac{P_1}{3} = \frac{P}{3} - \frac{5\,P_1}{6} + \frac{P}{2} - \frac{3\,P_1}{2},\text{ or } 16\,P_1 = 5\,P,\text{ therefore } P_1 = \tfrac{5}{16}\,P.$$

According to this view of the matter, the support B bears $\frac{5}{16}$ P, and the point of fixture A $\frac{11}{16}$ P. The same relations obtain in the case of a beam supported by three uprights, when the ends A and B are free to move up, but the middle part C kept horizontal. The uprights under A and B carry, therefore, each $\frac{5}{16}$ of the weight P, whilst the centre post carries $\frac{22}{16}$ P.

If the supports be inclined as shown in Fig. 57, there arises a horizontal thrust $H = \frac{1}{2}$ P $cotg.\ \delta$, with which the feet of the posts tend to spread. If, again, a beam resting upon two uprights be strengthened by two braces as shown in Fig. 58, we may, though only as an approximation, assume that at each end A, A_1 a pressure — $\frac{5}{22}$ P acts; whilst on each point C, C_1, there is a pressure of $\frac{16}{22}$ P. If δ be the angle of inclination BCA of the braces BC, the horizontal thrust in C and B, $= \frac{16}{22}$ P $cotg.\ \delta$, and the thrust along the brace $\frac{16}{22}\,\frac{P}{sin.\ \delta}$. If, again, l be the whole length AD, and l_1 the part BD of the support measured up to the brace, the horizontal strain on the upright $= \frac{l_1}{l}\cdot\frac{8}{11}\,P\ cotg.\ \delta$,

Fig. 57.

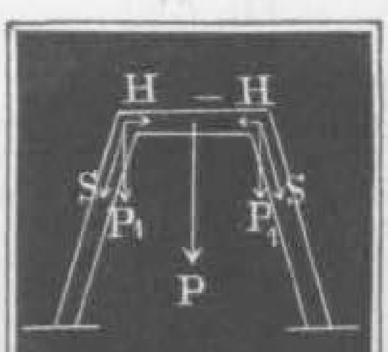
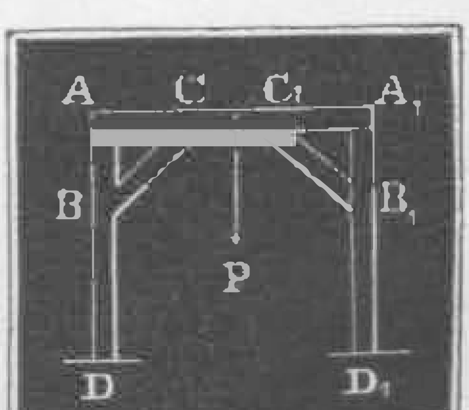


Fig. 58.



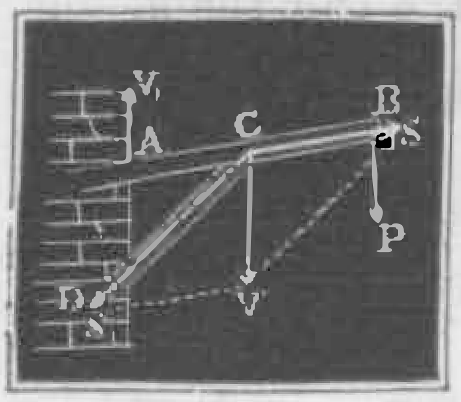
and, therefore, the column has not only to bear the vertical pressure $\frac{P}{2}$, but, likewise, a horizontal force $= \frac{l_1}{l}\ 1^{8}{}_{1}\ .\ P\ cotg.\ \delta$, creating flexure round B. In order, therefore, to insure the sufficiency of such a frame, the formula:

$$\tfrac{1}{2}\ P = b\,h\,K : \left(1 + 6\ .\ {}_1{}^{8}{}_{1}\ \frac{(l - l_1)\,l_1}{l h}\ cotg.\ \delta\right),$$

must he satisfied.

§ 30. *Braces or Struts.*—Fig. 59 shows a case of frequent occurrence. Where a beam AB, fixed in a wall or otherwise at one end, loaded at the other, is strengthened by a brace or strut CD. Let AB the length of the beam $= l$, and the part $AC = l_1$, the inclination of the beam $= \alpha$, and that of the strut $= \delta$. From the load P there arises a vertical pressure in C *downwards*

Fig. 59.



$$V = \frac{l}{l_1}\ P,$$ and a vertical pressure at A *upwards:*

$$V_1 = \left(\frac{l - l_1}{l_1}\right) P.$$ The first vertical pressure downwards resolves itself into two forces along the axes of the pieces

$$S = \frac{V\ cos.\ \delta}{sin.\ (\delta - \alpha)} = \frac{l\ P\ cos.\ \delta}{l_1\ sin.\ (\delta - \alpha)}, \text{ and}$$

$$S_1 = \frac{V\ cos.\ \alpha}{sin.\ (\delta - \alpha)} = \frac{l\ P\ cos.\ \alpha}{l_1\ sin.\ (\delta - \alpha)}.$$

The case shown in Fig. 60, where the beam is supported by a *tie-brace,* is to be treated in a manner exactly similar to the above. In most cases, the beam AB is horizontal, or $\alpha = 0°$, then we have

$$S = V\ cotg.\ \delta = \frac{l\ P}{l_1}\ cotg.\ \delta \text{ and } S_1 = \frac{l\ P}{l_1\ sin.\ \delta}.$$

Fig. 60.

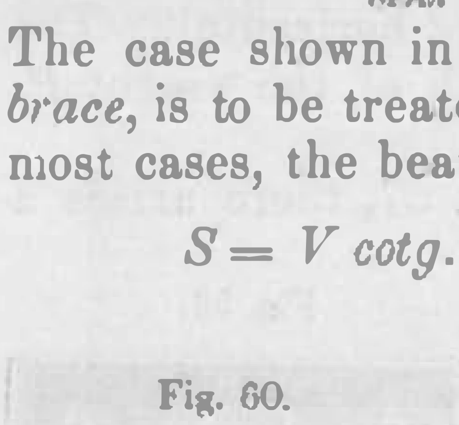
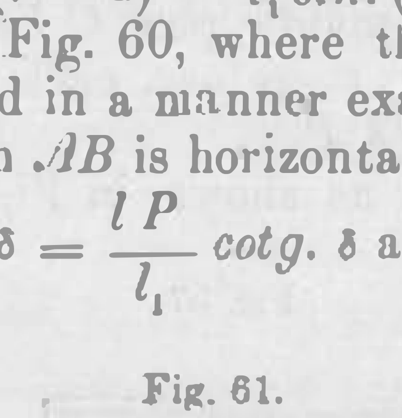


Fig. 61.



The dimensions of the brace have to be determined in proportion to the strain S_1 acting on it, and that of the beam with reference to the strain S compressing it, and likewise the cross strain arising from P, acting with the moment $P\,(l - l_1)$. Hence (§ 28):

$$P = b\,h\,K : \left(\frac{l}{l_1}\ cotg.\ \delta = 6\,\frac{(l - l_1)}{h}\right),$$

and by this equation, the section $b\,h$ of the beam must be determined. For the case shown in Fig. 61, we have to find the strain on the upright. The part DE of the upright is compressed by the force P, and strained across by the moment Pl, therefore we must put $P = \frac{b\,h\,K}{1 + \frac{6\,l}{h}}$, in order to get the required section $b\,h$. The piece AD, on the other hand, is under a tensile straine $= \left(\frac{l - l_1}{l_1}\right) P$, whilst the cross strain is the same, as for lower part; we have, therefore, in this case:

$$P = \frac{b\,h\,K}{\frac{l - l_1}{l_1} + \frac{6\,l}{h}}.$$

If at the foot of the upright there be placed a strut FG, this would take up the strain $S = \frac{l\,P}{a \cos. \alpha}$, if α be its inclination, and $a = EF$, and the force $S_1 = \frac{Pl}{h} \operatorname{tang.} \alpha$ passes through the uprights. Hence the part EF of the upright is strained by a force $= P - S_1$ or $S_1 - P$, the former when $a \operatorname{cotang.} \alpha < l$, and the latter when $a \operatorname{cotang.} \alpha > l$, or according as the strut falls within, or beyond, the point of suspension.

Example. In the framing, Fig. 61, suppose $P = 1500$ lbs., $AB = 12$ feet, the upright $EA = 24$ feet, the inclination of the braces $= 45°$, and the horizontal projection of each $= 6$ feet; required the necessary strength for the frame.

The braces have strains:

$$S_1 = \frac{l\,P}{l_1 \sin. \delta} = \frac{12\,.\,1500}{6 \sin. 45°} = \frac{3000}{0.7071} = 4243 \text{ lbs. to withstand.}$$

Taking 7400 as modulus of strength, we get, allowing 20 times absolute strength, the section of each brace $= \frac{4243}{7400}\,.\,20 = 11{,}5$ square inches. For the beam we may take according to Vol. I. § 198, $K = 12000$ lbs., for breaking across is here most likely to occur. Allowing 20 times the absolute strength, we have to put:

$$20\,.\,1500 = \frac{12000\,b\,h}{2\,.\,1 + \frac{6\,.\,l}{h}}, \text{ or } \frac{b\,h}{1 + \frac{18}{h}} = 5.$$

If now we make the depth of the beam double its breadth, we get:

$2\,b^2 = 5\left(1 + \frac{9}{b}\right)$, or $b^3 - \frac{5}{2}\,b = \frac{45}{2}$. From this we get the breadth of the beam 3,1 inches, and the depth 6,2 inches. For the upright, that is, for the centre part, by similar reasoning we get:

$$20\,.\,1500 = \frac{12000\,b\,h}{1 + \frac{6\,.\,12}{h}}, \text{ that is } \frac{b\,h}{1 + \frac{72}{h}} = \frac{5}{2}, \text{ or } b\,h = \frac{5}{2} + \frac{180}{h},$$

and if in this case we make $h = 2\,b$, we get $b^3 - \frac{5}{4}\,b = 45$, from which $b = 3{,}7$, and $h = 7{,}4$ inches.

§ 31. *Compound Beams.*—Beams laid upon one another, and united only by bolts, Fig. 62, have a resistance *equal only to the sum of the resistances of the individual beams.* If the beams only abut on each other, as in Fig. 63, and the butting joints be made to

Fig. 62.

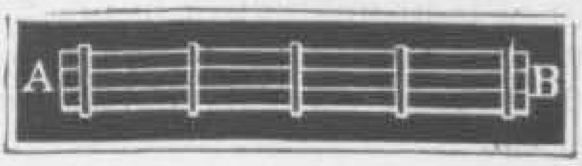
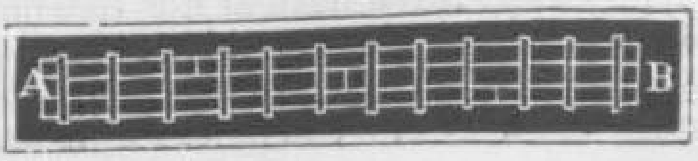


Fig. 63.



break joint, the strength of *one beam is lost to the whole.* If the beams be morticed, and tenoned as in Figs. 64 and 65, well strapped together, the strength of the combination is almost equal to that of a solid beam of the same dimensions.

Fig. 64.

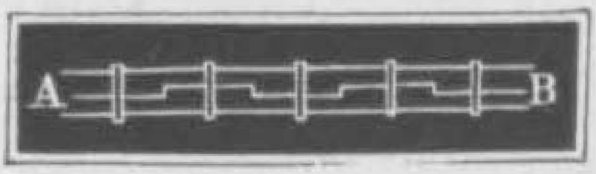
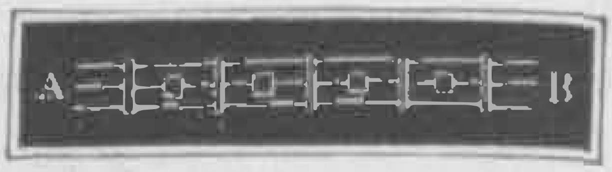
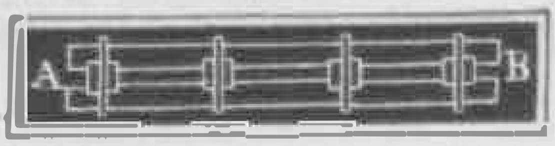


Fig. 65.



Beams are frequently *built* in this manner, to get great strength. The resistance of the elements of a beam increase, as their distance from the neutral axis. If, therefore, we separate two beams by thick tenons or wedges, and then strap or bolt them together, as in Fig. 66, their strength is considerably increased. If b be the breadth and h the depth, l the length and a the distance between the two beams, the strength of the combinations (Vol. I. § 200) is:

Fig. 66



$$P = \left(\frac{(a + 2\,h)^3 - a^3}{l\,(a + 2\,h)}\right) \frac{b\,K}{6}$$

If, for example, $a = 2\,h$, then $P = 14\,\frac{b\,h^2}{l} \cdot \frac{K}{6}$, whereas $P = 4\,\frac{b\,h^2}{l} \cdot \frac{K}{6}$, if the two beams had only been morticed together.*

The same relations obtain in the beam, shown in Fig. 67, united by St. Andrew's crosses or *lattice-framing.* In like manner, we determine the strength of wooden beams, composed of curved pieces, as in the bridge, Fig. 68, but it must be strictly borne in mind, that wooden framings lose much of their strength by deflexion. A principal advantage of such constructions is, that they are more stiff, and less liable to vibrate than

Fig. 67.

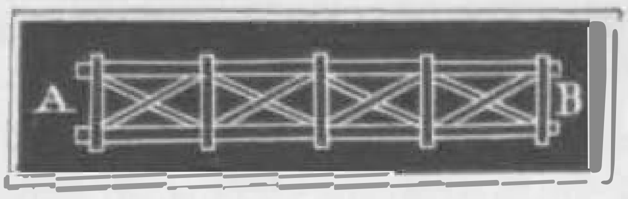
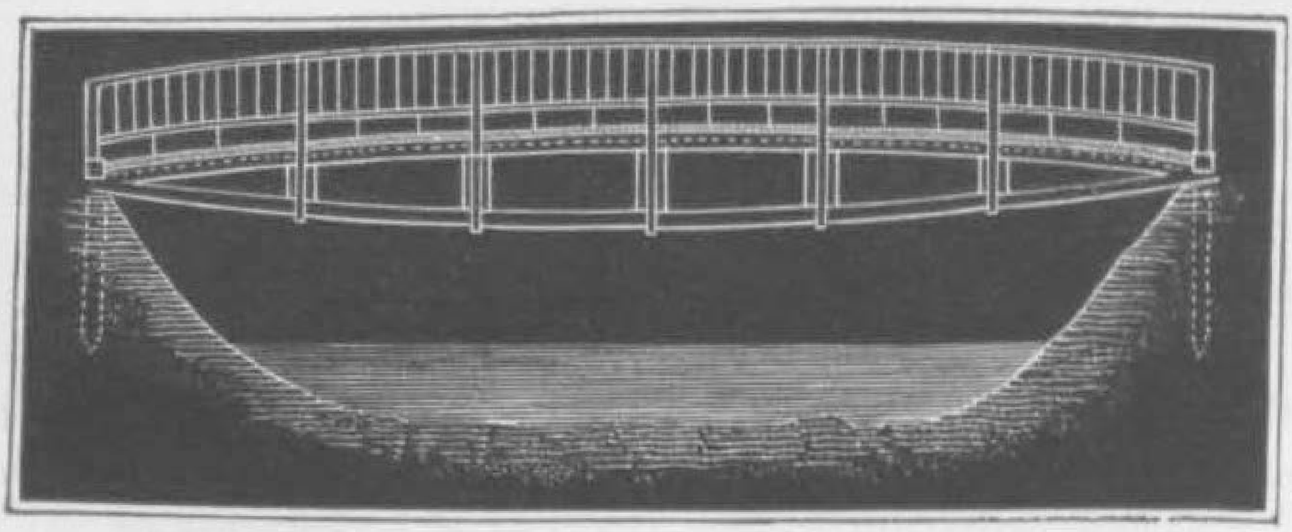


Fig. 68.

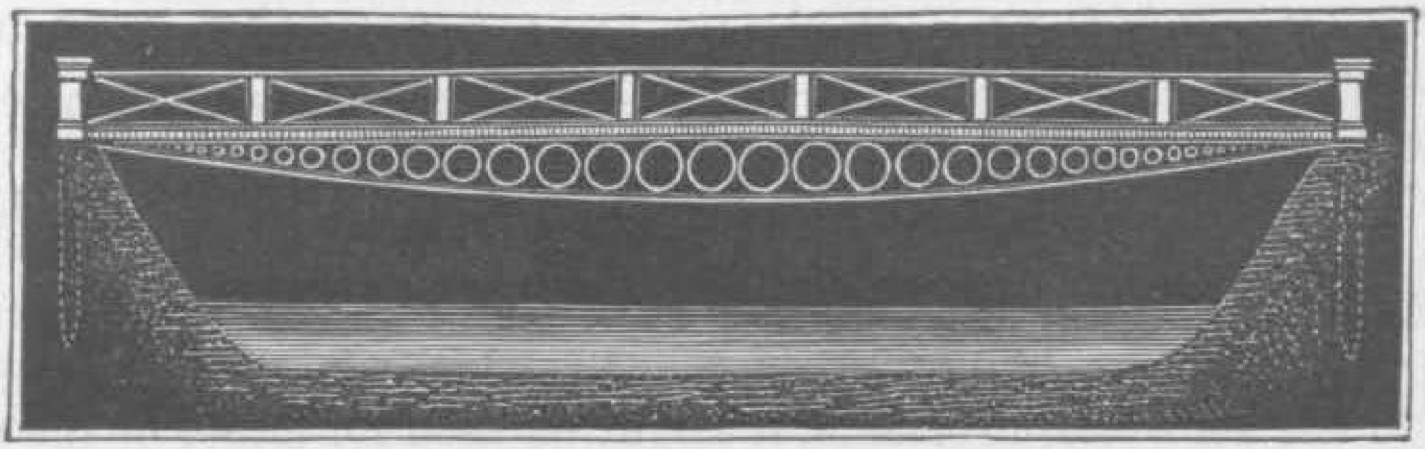


* Obvious as is the truth of this statement, and easy as is its application in practice, it is singular that so little use is made of it in the construction of timber bridges and other buildings in this country. It is evidently applicable to the double beam arches, often inserted in the so-called arch and truss bridges.—Am. Ed.

simple beams; and that, as they act only vertically on their points of support, they require no *abutments*, properly so called.

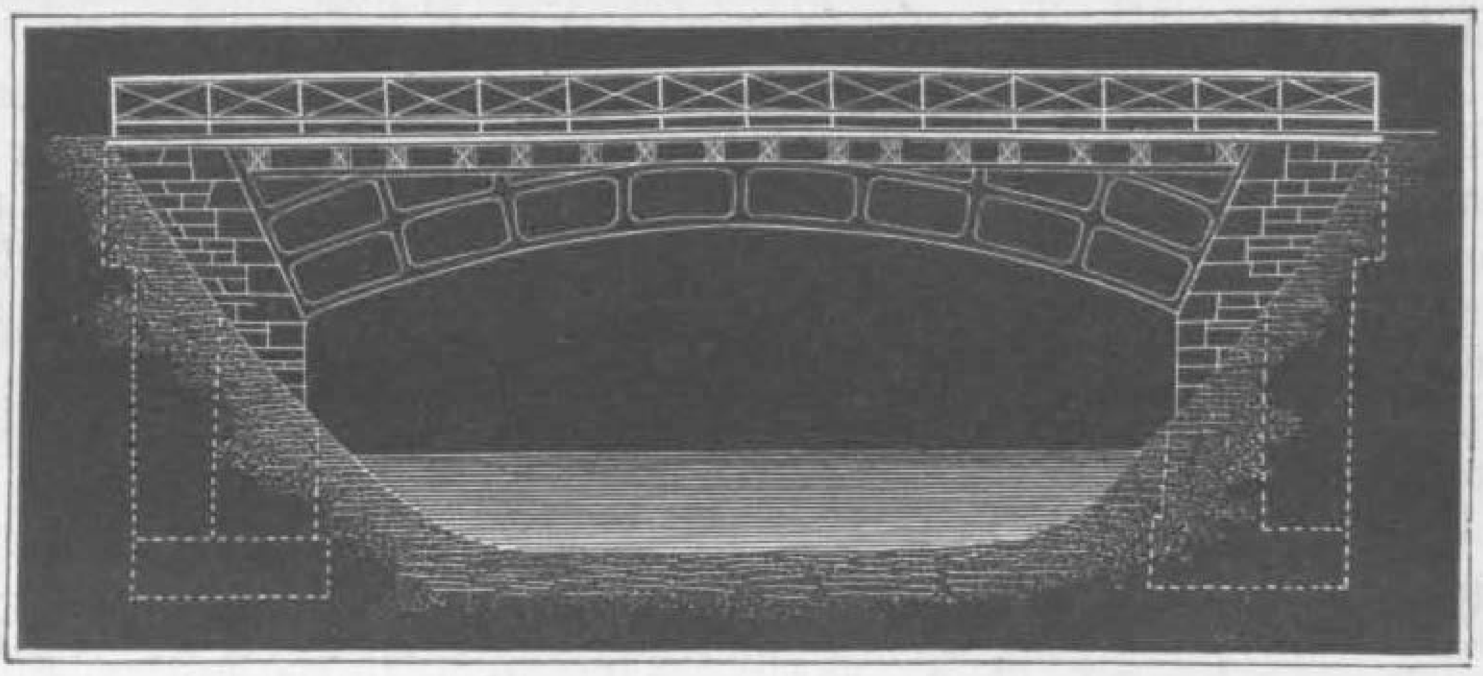
Curved beams, as shown in Fig. 69, have been frequently applied

Fig. 69.



in cast iron structures, and cast iron arches, as in Fig. 70, are a very usually employed bridge material. To judge of the strength of such

Fig. 70.



a structure, its line of resistance must be determined. If this fall everywhere *within* the arch, it shows that there is no cross-strain on the material, but only compression; but if the line of resistance fall *without* the arch, the weak point is where it runs furthest from the arch, and the resistance of the material to cross-strains, is that upon which the stability of the structure depends.

[*Proof of the strength of Complex Structures by means of Models.*— The plan of solving questions in practical mechanics and engineering by faithfully constructed models, presents the very obvious advantage of substituting the moderate cost of *experiment* for the often burdensome, sometimes ruinous, expense of *experience.* The conditions to be fulfilled in constructing models, so as to give reliable information in regard to the action or the stability of structures, may be stated as follows:—

1st. An entire correspondence must exist in the *model*, (at least, of all essential parts,) to the scale of dimensions and weights on which it is proposed to represent the *structure.*

2d. Identity not only in the nature, but also in the condition of materials employed in the model and structure respectively.

3d. Proportional accuracy in forming junctures; and proportional tension given by tightening screws, keys, wedges, and other mechanical means, by which the parts are compacted together.

In testing the model, modes of introducing, distributing, and withdrawing loads, conformable to those which practice will involve in regard to the structure, must be observed, so as to subject the model to shocks, jars, inequality of pressure and irregularities of application, at least proportional to those which the structure will be required to sustain.

Supposing the model of a bridge to have been constructed according to the above requirements, it might be used for either of the two following purposes:—

1. *To determine what weight the structure will bear when undergoing a given deflexion, or when on the point of breaking.*

2. *To ascertain whether the principle of construction be adequate to furnish a bridge of the proposed dimensions, and materials that can fulfil the specified duty.*

As a beam or bridge of uniform dimensions throughout will bear half as much weight accumulated at the centre as it could sustain if distributed throughout its length, the simplest mode of arriving at the result desired is to determine and apply to the centre of the model a weight which shall represent one-half the load supposed to come upon the structure.

The following formula applies to the loading of the model at its centre.

Let $l =$ the length, in feet, of the model between the points of support; p the weight in pounds which the model is to sustain at the centre, representing a load uniformly distributed over its length; $w =$ the weight of so much of the model as lies over the clear opening between its piers; $r =$ the ratio of dimensions between the *structure* and the *model; P* $=$ the load which the *structure* must be able to bear, when accumulated at the centre. Then it is evident that $r\,l =$ the length of structure between the piers. Since the relative resisting powers of similar beams or bridges are as the second powers of their corresponding dimensions, $\therefore r^2 : 1 ::$ resisting power of the *structure* : resisting power of the *model*. Hence, $r^2 (p + \frac{1}{2} w) =$ the absolute resisting power of the structure. Also, since the weights of similar structures are as the third powers of their corresponding dimensions—or, what is the same thing, as the third powers of their ratios of dimensions—therefore $r^3 w =$ the absolute weight of the structure; so that the weight P, which, by supposition, the structure can bear, accumulated at its centre, will be its absolute resisting power, diminished by half its own weight.

Hence, $P = r^2 \left(p + \frac{w}{2}\right) - \frac{r^3 w}{2} = r^2 p - \frac{r^3 w + r^2 w}{2} = r^2 p -$
$r^2 (r - 1) \frac{w}{2} = r^2 \left(p - \frac{w}{2} \times r - 1\right)$ [1].

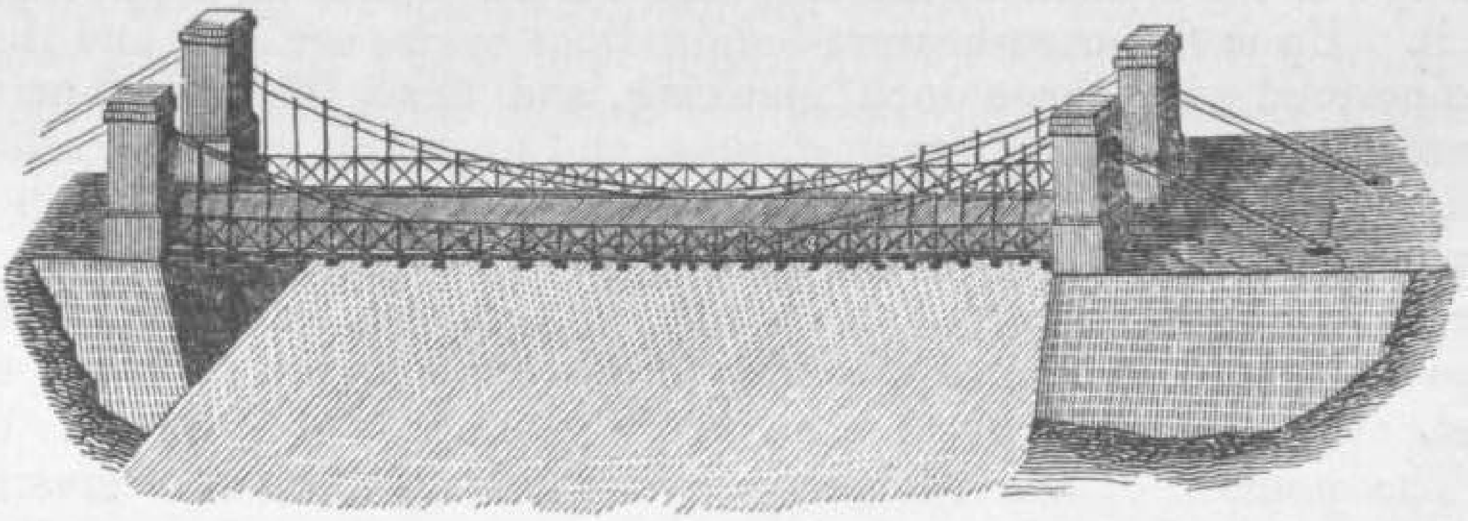
But as, by supposition, P is known, and it is desired to find p, the conversion of the last formula gives $p = \frac{P}{r^2} + \frac{w}{2}(r-1)$ [2].

Example. It is required to construct, on a given plan, a bridge having a clear opening between the piers of 150 feet, and capable of sustaining two tons per foot of its length, or 300 tons in all, equally distributed over its surface. A model is made on the scale of one inch to the foot, and weighing 136.3 pounds, exclusive of the part which rests directly upon the abutments. It is required to find what number of pounds must be suspended from the centre of the model, in order to prove whether any bridge constructed on the plan, with the relative dimensions and of the materials used in the model, will bear the load above specified.

Substituting the values of the several symbols in the second of the above equations, viz: $p = \frac{P}{r^2} + \frac{w}{2}(r-1)$, we obtain $p = \frac{300 \times 2240}{12 \times 12} + \frac{136.3}{2} \times (12-1) = 3082$ pounds; and twice this number, or 6164 pounds, is the weight which the model ought to bear, when distributed uniformly over its surface.]

§ 32. *Chain or Suspension Bridges.*—Suspension bridges involve considerations distinct from the principle of the stability of either stone, wood, or cast iron bridges, inasmuch as the road-way is suspended from chains or ropes, or is supported upon these. The former is the more frequent construction. Chains or cables drawn up with considerable force, between two or more piers or supports, pass over these to fastenings in rock or masonry, as shown in Fig. 71. The chains are formed of malleable iron bars, united by pins

Fig. 71.



or bolts: and cables of iron or steel wire, laid parallel or twisted together, are frequently employed instead of bar-chains. The

Fig. 72.

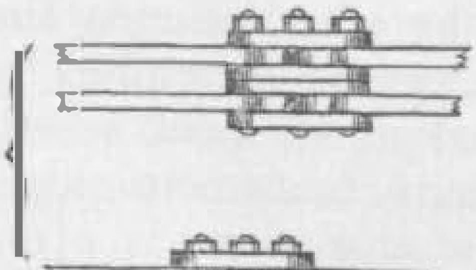
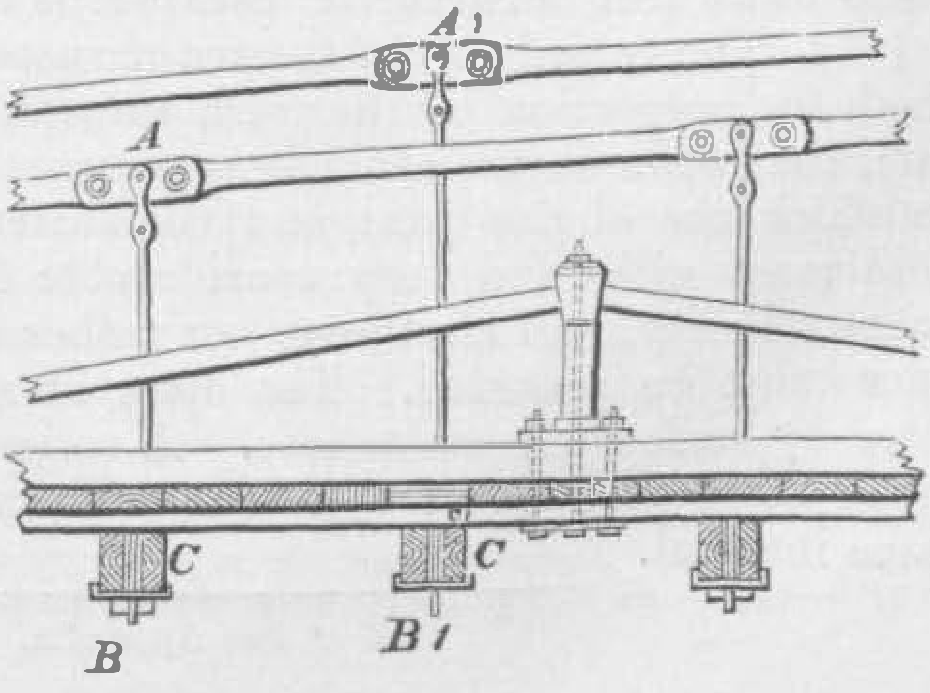


Fig. 73.



dimensions of the *links* or bars, depend upon those of the bridge. In large bridges they are made about 1 inch thick, from 3 to 9 inches deep, and from 10 to 16 feet long. Usually, several sets of bars are hung together, forming a compound chain united by coupling plates and bolts, as shown in Fig. 72 (or without coupling plates, according to Mr. Howard's patent plan). Wire cables are composed of wires of from $\frac{1}{20}$ to $\frac{1}{4}$ of an inch in diameter, and are made of any requisite diameter, varying from $\frac{1}{2}$ an inch to 8 inches. The suspending rods consist of wrought iron rods, or of wire ropes. The rods AB, A_1B_1, are hung by pins passing through the coupling plates as shown in Fig. 73, and suspending ropes are attached as shown in Figs. 74 and 75, by means of shackles with eyes, or by a simple loop. The cross-beams of the road-way C, C_1 are sometimes fastened to the suspending rods as shown in Fig. 74, sometimes as shown in Fig. 73. The rod goes either through the beam, and is then fastened by a *nut* resting on a metal plate, or *washer*, or a *stirrup*, or strap is put over the beam, a hook on the upper side of which goes into the eye of the shackle of the suspension rope, or into the loop formed on it. Upon the cross-bearers longitudinal beams are laid, and these are covered with three inch planking, and again three inch cross planking, according to circumstances, and upon this road-metalling, &c., is laid. In general there are two systems of chains, one above the other, on each side of the bridge, and hence the number of suspension rods is twice the number of joints in any one chain system. The distance from centre to centre of suspending rods is about five feet.

Fig. 74.

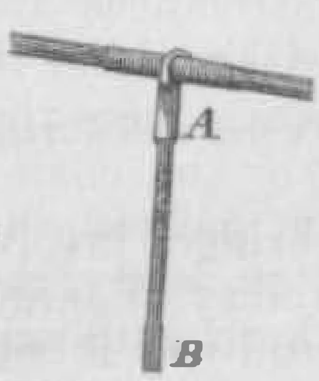
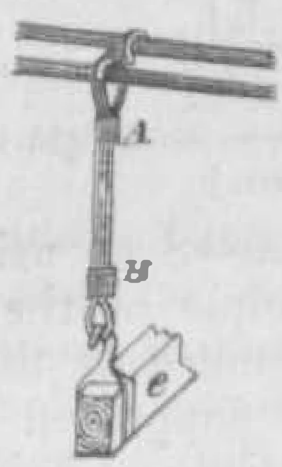


Fig. 75.



The *parapet* of the bridge ought to be framed so as to give the *greatest stiffness to the road-way.**

The width of road-way depends on the purposes which the bridge is to subserve. There should be 3 feet at least for a foot-path, and 7 to $7\frac{1}{2}$ for a carriage way. For a bridge for ordinary traffic, a total width of 25 feet between the parapets is sufficient.

§ 33. The versed sine of the arc of suspension bridges, is generally small in proportion to the cord, varying from $\frac{1}{4}$ to $\frac{1}{25}$, and, therefore, the strain on the chain is very great (Vol. I. § 144). The piers on which the chains pass, and the fastenings by which chains are held must withstand very considerable forces, and hence piers of great stability, and abutments, or rather *anchorage*, of great resistance must be provided. The span of suspension bridges is regulated by various circumstances. A series of smaller spans is often muchnmore economical than one or more large spans to cover the same interval.

* See Appendix.

The Menai bridge in England, the two bridges at Fribourg in Switzerland, the bridge at Roche Bernard in France, the bridge over the Danube at Ofen, are examples of large spans of from 600 to 720 feet; whilst there are innumerable instances of less span in every country. If the chain be not equally strained on the two sides of the pier, which always occurs when one side only is loaded, the chain slides forward towards the side on which there is the greater load. As, however, there would arise considerable friction between the rope and the head of the pier, under the pressure of the resultant force being on it, the pier must have stability to counteract a force equal to this friction. To prevent this action, special contrivances are adopted for diminishing the friction. These means consist, either in passing the chains over rollers or pullies, Fig. 76, which reduces the sliding friction to a rolling friction on a small axle, or the chains pass over a sector which rocks on the head of the pier, inclining to one side or the other as external forces act upon it; or, lastly, the pier is made as a column rocking on its foot, or on a horizontal axis at its foot. That the resultant of the forces acting on the chain may press vertically on the pier head, and thus be least strained by it, it is necessary that the parts of the chain on each side of the pillar should have equal inclinations to the horizon. If this equality cannot be obtained, as is not unfrequently the case for the land piers of bridges, the piers must be considerably strengthened.

Fig. 76.

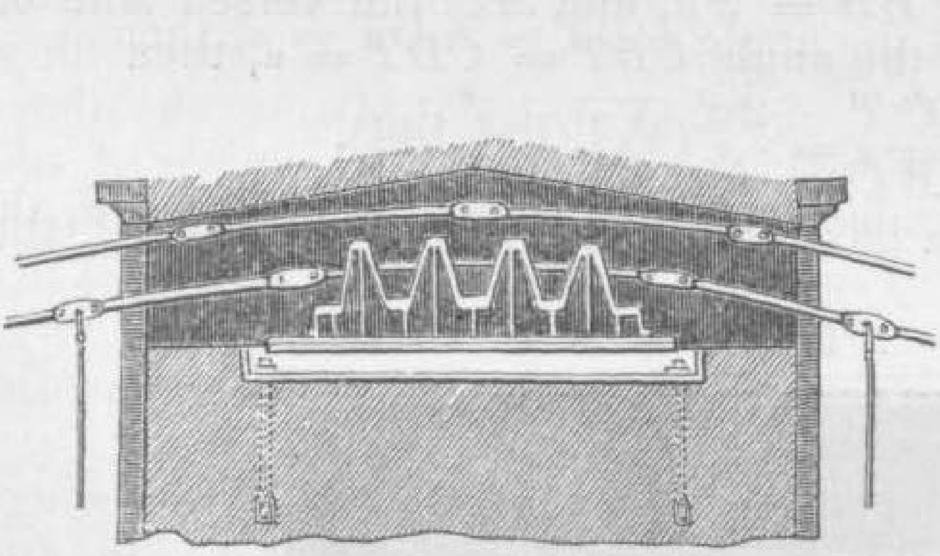
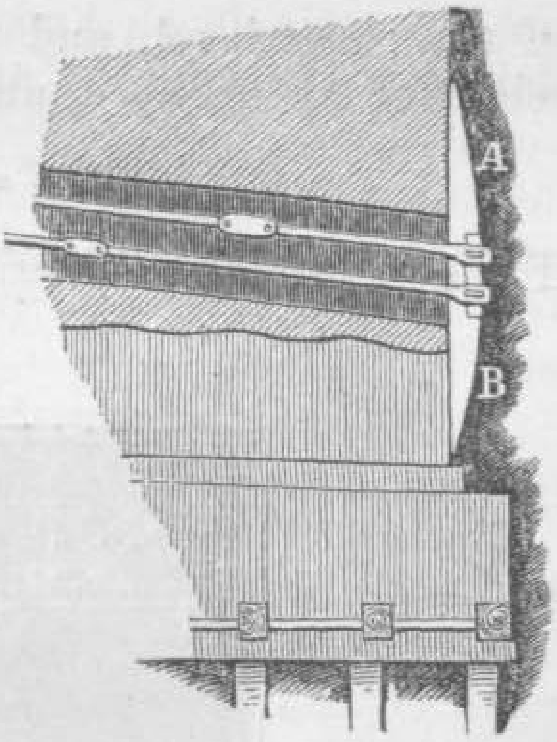


Fig. 77.



To fasten the ends of the chains to the land, various devices have been practised, the general plan of which is to carry the chains by wells or drifts into the rock or soil, and there to fasten them to broad iron or wooden piles, or planking as at *AB*, Fig. 77, which abut upon substantial retaining walls of masonry, or against an arch, or against the rock itself. The fastenings can thus be examined at any time, and adjusting wedges for compensating the influences of expansion and contraction be conveniently manipulated.

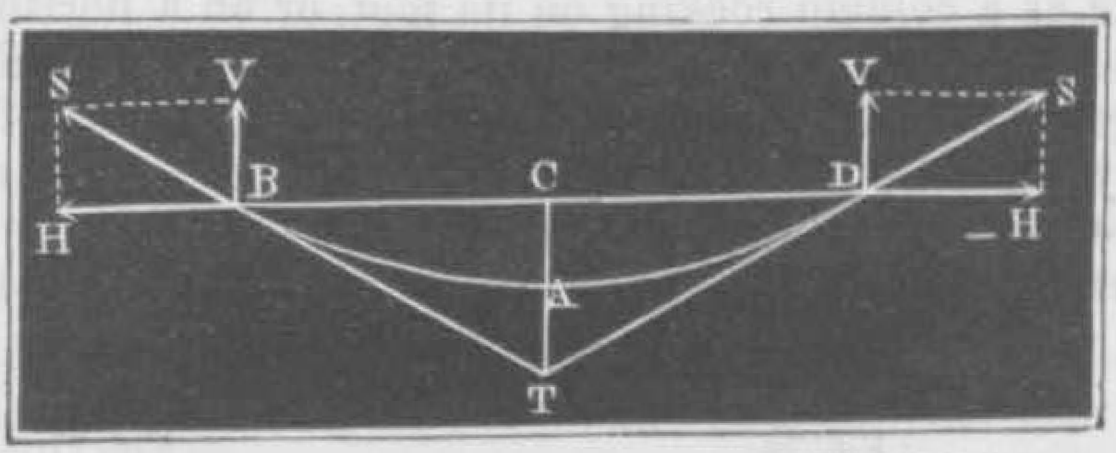
Remark. On the subject of suspension bridges, the most complete treatise is that of Navier, "Rapport et Mémoire sur les ponts suspendus, Paris, 1823." The papers of Mr.

Davies Gilbert, in the "Transactions of the Royal Society of London, 1826," are important in the history of these bridges. In Moseley's "Engineering and Architecture" there is a very elegant investigation of the properties of these structures. The treatise of Drewry on "Suspension Bridges, 1832," is a very excellent resumé of the general practice in respect to suspension bridges. The account of the suspension bridge over the Vilaine, at La Roche Bernard, by Leblanc, Paris, 1841, is very instructive. There is a treatise of Seguin, "Mémoire sur les ponts en fil de fer," worthy of attention. There are many memoirs in the "Annales des ponts et chaussées" on this subject; and, in the volume for 1842, there is an account of a bridge made of *ribbons* of hoop iron.

§ 34. The curve formed by the chain or cable of a suspension bridge, lies between the parabola and the catenary, and is very nearly an ellipse. The parabola approximates the curve in the loaded bridge, the catenary in the unloaded (compare Vol. I. § 144 and § 145, &c.). We shall consider the curve asa parabola, or the bridge in its loaded state.

If the two points of suspension B and D, Fig. 78, of a chain, be

Fig. 78.

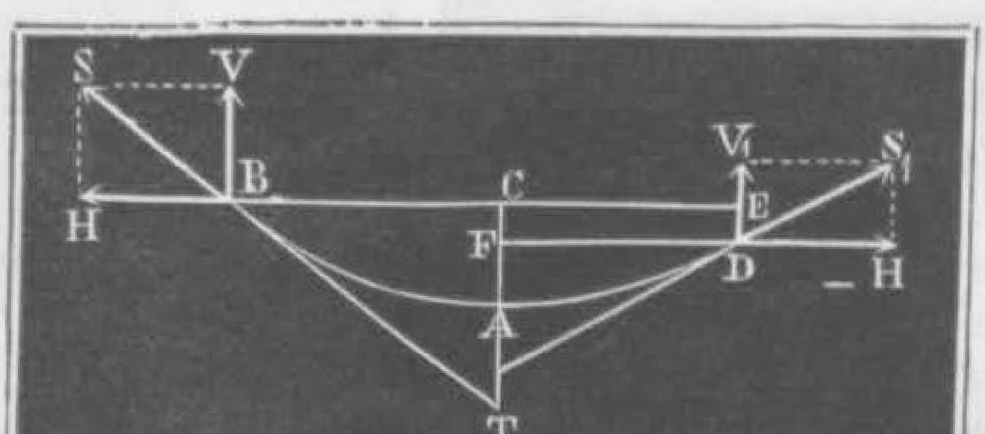


on the same level, and if $BD = 2\,b$, and AC the versed sine or height of the arc $= a$, and the angle $CBT = CDT = \alpha$, then

$$tang.\ \alpha = \frac{C\,T}{B\,C} = \frac{2a}{b} \text{ (Vol. I. § 144).}$$

If the points of suspension be at different levels, as in Fig. 79, the

Fig. 79.



apex of the curve is not in the centre, and the ends of the chain have different inclinations. If we put the co-ordinates AC and BC $= a$ and b, and the co-ordinates AF and $FD = a_1$ and b_1, we put the whole span $BE = s$, and the difference of $DE = h$, we have: $h = a - a_1$, $s = b + b_1$, and $\frac{a}{a_1} = \frac{b^2}{b_1^{\,2}}$, we have, therefore, from h, s, and a:

$1,\ a_1 = a - h,\ 2,\ b = \frac{s}{1 + \sqrt{\frac{a_1}{a}}},\ 3,\ b_1 = s - b = \frac{s}{1 + \sqrt{\frac{a}{a_1}}},$

and for the angles of inclination α and α_1:

$$tang.\ \alpha = \frac{2\,a}{b},\ \text{and}\ tang.\ \alpha_1 = \frac{2\,a_1}{b_1}.$$

The length of the parts of the chain $AB = l$ and $AD = l_1$, is expressed by:

$l = b\left[1 + \frac{2}{3}\left(\frac{a}{b}\right)^2\right]$, and $l_1 = b_1\left[1 + \frac{2}{3}\left(\frac{a_1}{b_1}\right)^2\right]$, (Vol. I.h§ 147).

If we have the distance e between the suspension rods, their number for a length $BC = b$, is $n = \frac{b}{e}$; and if in the equation $x = \frac{y^2}{b^2}\,a$, we substitute for y the values o, e, $2e$, $3e$, $4e$, &c., we get for the lengths of the suspension rods:

$$0, \frac{e^2}{b^2}\,a, \frac{4e^2}{b^2}\,a, \frac{9e^2}{b^2}\,a, \text{ \&c., or } 0, \frac{a}{n^2}, \frac{4a}{n^2}, \frac{9a}{n^2}, \text{ \&c.,}$$

to each of which a few inches are to be added.

From the weight G of the loaded half of the chain AB, the horizontal tension of the whole chain:

$H = G\ cotg.\ \alpha = \frac{b}{2\,a}\,G$, and the entire tension on the end:

$$S = \frac{G}{sin.\ \alpha} = \frac{2\,a\,G}{\sqrt{b^2 + 4a^2}}.$$

If we know the modulus of strength of the chains and suspension rods, we can determine the sectional dimensions they should have. According to French experience, the greatest load that should be brought on chains, is 12 kilogrammes per square millimetre (or about 8 tons on the square inch), and for cables of iron wire 18 kilog. per square millim., or about 12 tons per square inch. The *suspension rods* are made much stronger in proportion, as they have to resist the *shocks* of loaded wagons, &c., passing along the bridge. The load on them is reduced to from $1\frac{1}{2}$ to 3 tons per square inch of section.

§ 35. *Sectional Dimensions of the Chains and Ropes.*—In order to determine the dimensions of the parts of a suspension bridge, we have to take into consideration, not only the weight of the road-way, but also the greatest weight of men, as troops, or of cattle, or of wagons, that can be brought to bear upon it. This has been taken as 42 lbs. per square foot of surface by Navier, but in the case of a dense crowd of persons, it might amount to 72 lbs. per square foot. Having assumed a certain maximum load, the dimensions of the cross and longitudinal beams have to be determined, and hence we find the entire weight of the road-way. If we put the sum of this constant weight, and the maximum load that may come on to the bridge $= G_1$, and the modulus of strength of the suspension rods $= K$, we get for the section of these $F_1 = \frac{G_1}{K}$. From this we have the weight of these rods, which has to be added to that of the road and load, in order to put the total load on the chain G. If we put the section of the chains $= F$, and the specific gravity of the iron $= \gamma$, we have, retaining the notation as above, the weight of the chains:

$$G_2 = Fl\gamma = Fb\left[1 + \tfrac{2}{3}\left(\frac{a}{b}\right)^2\right]\gamma,$$

and hence the total load on one-half the bridge:

$$G = G_1 + G_2 = G_1 + Fb\left[1 + \tfrac{2}{3}\left(\frac{a}{b}\right)^2\right]\gamma,$$

and the strain at the point of suspension

$$S = \frac{G}{sin.\ \alpha} = \frac{G_1 + Fb\left[1 + \tfrac{2}{3}\left(\frac{a}{b}\right)^2\right]\gamma}{sin.\ \alpha}.$$

But for the necessary security $S = FK$ (where K is the modulus of strength), therefore:

$$FK\ sin.\ \alpha - b\left[1 + \tfrac{2}{3}\left(\frac{a}{b}\right)^2\right]\gamma = G_1,$$

i. e., the section of the chains:

$$F = \frac{G_1}{K\ sin.\ \alpha - b\left[1 + \tfrac{2}{3}\left(\frac{a}{b}\right)^2\right]\gamma}.$$

Example. The dimensions of the parts of a suspension bridge of 150 feet span, 15 feet deflexion, and 25 feet in width are required. Suppose 45 suspension rods on each side, we have then 44 equal parts of 3,409 feet each. The length of these rods, commencing at the centre would be 0, $\frac{15}{22^2} = 0{,}031$, $4 \cdot \frac{15}{22^2} = 0{,}124$, $9 \cdot \frac{15}{22^2} = 0{,}279$, $16 \cdot \frac{15}{22^2} = 0{,}496$, $25 \cdot \frac{15}{22^2} = 0{,}775$ feet, &c., or if we add to each 2 inches, the length becomes: 2, 2,37, 3,49, 5,35, 7,95, 11,30 inches, &c.

The maximum load on the half bridge, we shall take according to Navier $75 \times 25 \times 42$ lbs. $= 78750$ lbs., and if the road-way weighs a little less than a ton per foot of length $G_1 = 157500$, and the section of all the rods of one-half of the bridge: $F_1 = \frac{157500}{2190} = 72$ square inches. The whole bridge is suspended on 90 rods, and, hence the section of each rod is $\frac{72 \cdot 2}{45 \cdot 2} = 1{,}6$ square inches, or the diameter of the rods must be 1,427 inches. According to the rules for the quadrature of the parabola, the mean length of a suspension rod $= \frac{1}{3}$ that of the largest, therefore, $= \frac{1}{3} \cdot 15 = 5$ feet, and if as above, we add 2 inches to it, then it $= 5\frac{1}{6}$ feet, or 62 inches. Thus the volume of all the rods is $90 \times 62 \times 1{,}6 = 8928$ cubic inches, and the weight taken at 0,29 lbs. per cubic inch $= 2598$ lbs. The half of this added to the above-found weight of half the road-way gives $G = 158794 . 5$ lbs., and, hence, according to the formula:

$$F = \frac{G_1}{K\ sin.\ \alpha - b\left[1 + \tfrac{2}{3}\left(\frac{a}{b}\right)^2\right]\gamma},$$

if, $G_1 = 158794{,}5$, $K = 17500$, $b = 75 \times 12 = 900$, $\frac{a}{b} = \frac{15}{72} = 0{,}2083 \ldots$, $\gamma = 0{,}29$,

and $sin.\ \alpha = \frac{2a}{\sqrt{b^2 + 4a^2}} = \frac{30}{\sqrt{75^2 + 30^2}} = \frac{1}{\sqrt{6{,}25 + 1}} = \frac{1}{\sqrt{7{,}25}}$.

$$F = \frac{158794{,}5}{17500 \cdot 0{,}3714 - 900 \cdot 0{,}29\,(1 + \tfrac{2}{3} \cdot 0{,}2083^2)} = \frac{158794{,}5}{6499{,}5 - 268{,}5} = \frac{158794{,}5}{6231{,}0}$$

$=25{,}48$ square inches, and, therefore, for 4 chains the section of each would be 6,37 square inches.

§ 36. *Elongation of Chains.*—The chains are elongated by the load, and, therefore, the deflexion is increased. Changes of temperature also, produce variations in the length of the chains. We must know the effects of both these. If the deflexion changes from a to a_1, the length.

$$l = b\left[1 + \tfrac{2}{3}\left(\frac{a}{b}\right)^2\right], \text{ and } l_1 = b\left[1 + \tfrac{2}{3}\left(\frac{a_1}{b}\right)^2\right],$$

and hence the elongation of the chain:

$$\lambda_1 = l_1 - l = \tfrac{2}{3}\left(\frac{a_1^2 - a^2}{b}\right) = \tfrac{2}{3}\frac{(a_1 - a)(a_1 + a)}{b},$$

or if Δ be the increase in the deflexion, and if we put as an approximation $a + a_1 = 2\,a$, $\lambda_1 = \tfrac{4}{3}\frac{a}{b}\Delta$, and, therefore, for the whole chain $\lambda = \tfrac{8}{3}\frac{a}{b}\Delta$, inversely $\Delta = \tfrac{3}{8}\frac{b}{a}\lambda$. From the weight G of the half bridge, the horizontal tension or tension at the apex, $H = G\ cotg.\ \alpha$, and the tension at the ends: $S = \frac{G}{sin.\ \alpha}$, therefore, the mean tension $= \frac{H + S}{2} = \frac{G(1 + cos.\ \alpha)}{2\ sin.\ \alpha}$, and the extension of the chains caused by this force $\lambda = \frac{(1 + cos.\ \alpha)}{2\ sin.\ \alpha} \cdot \frac{G}{FE} \cdot 2\,l$, (Vol. I. § 183), for which we may put as an approximation: $\lambda = \frac{2\,G\,b}{F\,E\,sin.\ \alpha}$. If we introduce this value into that for Δ, we get the increase in the deflexion for the loaded chains:

$$\Delta = \tfrac{3}{8} \cdot \frac{b}{a} = \frac{2\,G\,b}{FE\ sin.\ \alpha} = \tfrac{3}{4}\frac{G}{FE\ sin.\ \alpha} \cdot \frac{b^2}{a},$$

or $sin.\ \alpha = \frac{2\,a}{\sqrt{b^2 + 4\,a^2}}$, or approximately $= \frac{2\,a}{b}$, we get

$$\Delta = \tfrac{3}{8} \cdot \frac{G}{FE} \cdot \frac{b^3}{a^2}.$$

Malleable iron expands 0,0000122 of its length for a rise of temperature of one degree of centigrade (=,0000068 for 1° Fahr.). This increase is, therefore, $0{,}0000122 \cdot 2\,l\,t$ for the length of chain l, and a rise of t degrees of temperature, or $0{,}0000244\ l\,t$. Putting this in the expression for Δ, we get the increase of deflexion for a rise of temperature t:

$$\Delta = \tfrac{3}{8} \cdot \frac{b}{a} \cdot 0{,}0000244 \cdot lt, \text{ or approximately } = 0{,}00000915 \cdot t\,\frac{b^2}{a}.$$

In like manner the contraction is determined for decrease of temperature.

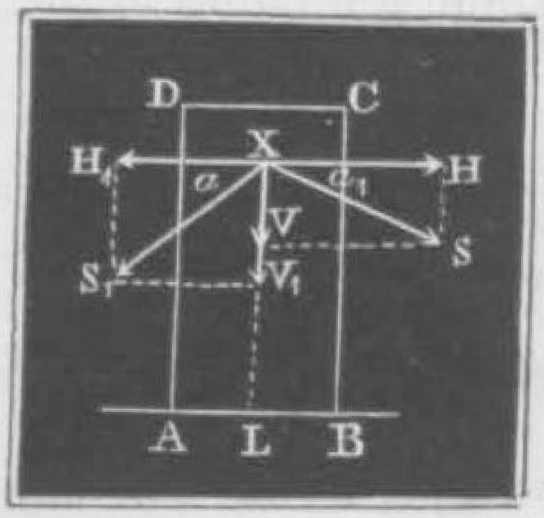
Example. Retaining the values of the example in the last paragraph, we get the increase of the height of the arc corresponding to the load, taking the modulus of elasticity of malleable iron = 29000000 (Vol. I. § 186), and adding 6841,8 lbs. the half weight of the chains to the load 158794,5 lbs.

$$G = 158794,5 + 6841,8 = 165636,3 \text{ lbs.}$$

$\Delta = \frac{3}{8} \cdot \frac{165636,3}{25,48 \cdot 29000000} \cdot \frac{900^3}{180^2} = \frac{1397}{739} = 1,9$ inch. For a change of temperature of 20° C. this change of deflexion is:

$$0,00000915 \cdot 20 \cdot \frac{900^2}{180} = 0,4 \text{ inches.}$$

Fig. 80.



§ 37. *Piers and Abutments.*—The proportions of the piers and abutments form an important consideration.

If S and S_1 be the tension on the ends of the chain, Fig. 80, and α and α_1 the angles of inclination, the vertical pressure on the pier:

$$V_2 = V + V_1 = S \sin. \alpha + S_1 \sin. \alpha_1,$$

and the horizontal pressure, as the horizontal tensions counteract each other,

$$H_2 = H - H_1 = S \cos. \alpha - S_1 \cos. \alpha_1.$$

If, now, h be the height, b the breadth, and d the depth or thickness of a pier, the density of the masonry of which $= \gamma$, its weight is $b\,d\,h\,\gamma = G$, and the total vertical pressure $= V_2 + G = S \sin. \alpha + S_1 \sin. \alpha_1 + b\,d\,h\,\gamma$. In order, however, that the horizontal force $H_2 = H - H_1$ may not turn the pier on the edge B, it is requisite that the statical moment:

$$H_2 \cdot LX = H_2\, h = (S \cos. \alpha - S_1 \cos. \alpha_1)\, h$$

should be less than the statical moment:

$$(V_2 + G)\, B\,L = (S \sin. \alpha + S_1 \sin. \alpha_1 + b\,d\,h\,\gamma)\, \frac{b}{2},$$

i.e. it is requisite that:

$$b^2 + \frac{S \sin. \alpha + S_1 \sin. \alpha_1}{d\,h\,\gamma}\, b > \frac{2\,(S \cos. \alpha - S_1 \cos. \alpha_1)}{d\,\gamma}, \text{ or}$$

$$b^2 + \frac{V + V_1}{d\,h\,\gamma}\, b > \frac{2\,(H - H_1)}{d\,\gamma}.$$

For the sake of security, the greatest value of $S \cos. \alpha$ and the least value of $S_1 \cos. \alpha_1$ are to be taken, that is to say S is to be taken as completely loaded, and S_1 as unloaded. This formula assumes that the forces S and S_1 are entirely transferred to the pier head, which, of course, only takes place when the friction on the pier head exceeds the difference $S - S_1$ of the tensions. According to Vol. I. § 175, this friction is:

$$F = \left[\left(1 + 2 f \sin. \frac{\beta}{2}\right)^n - 1\right] S_1,$$

where f is the co-efficient of friction, n the number of links on the pier head, and β the central angle corresponding to *one link*, it is hence requisite that:

$$S - S_1 < \left[\left(1 + 2 f \sin. \frac{\beta}{2}\right)^n - 1\right] S_1, \text{ or}$$

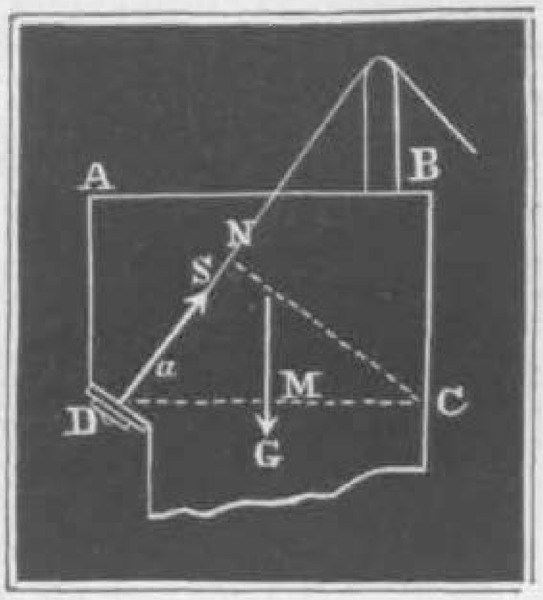
$S < \left(1 + 2f\, sin.\, \frac{\beta}{2}\right)^n S$. Unless this condition be fulfilled, the chain will slide on the pier head, and therefore we have only to put $S = \left(1 + 2f\, sin.\, \frac{\beta}{2}\right)^n S_1$, or for ropes $S = e^{f\alpha} S_1$ (Vol. I. § 176), in the above formula. If the chain or cable be laid upon pulleys, this difference is much less, and, therefore, the requisite thickness of pier is less. If the radius of the pulleys,$=a$, and the radius of the axes on which they turn $= r$, then:

$$S = S_1 + f\frac{a}{r}(S\, sin.\, \alpha + S_1\, sin.\, \alpha_1),$$

for the friction reduced to that of the axis may be put,$= f\frac{a}{r}(S sin.\, \alpha + S_1\, sin.\, \alpha_1) = f\, \frac{a}{r}(V + V_1)$. If the rope passes over rollers, then the friction is so much reduced, that we may put $S = S_1$.

From the tension S on the land or back chains, we can determine the dimensions of the retaining wall AC, Fig. 81.

Fig. 81.



The strain S tends to turn the masonry AC round C, and acts with a leverage $CN = CD\, sin.\, \alpha = l\, sin.\, \alpha$, if α be the angle of inclination SDC of the rope to the horizon, and l the length CD of the wall. The height of the wall resists with the moment:

$$G\,.\,CM, = h\,d\,l\,\gamma\,.\,\frac{l}{2} = \tfrac{1}{2}\,h\,d\,l^2\gamma,$$

where h is the height BC, d the depth, and γ the weight of the masonry. For equilibrium $S\,l\,sin.\,\alpha = \frac{1}{2}\,h\,d\,l^2\,\gamma$, and, therefore, the requisite width of wall $l = \frac{2\,S\,sin.\,\alpha}{h\,d\,\gamma}$. To insure stability this must be doubled. That such a wall may not be pushed forward, the friction $f\,(G - S\,sin.\,\alpha)$ must be greater than the horizontal force $S\,cos.\,\alpha$, or, $f\,G > S\,(cos.\,\alpha + f\,sin.\,\alpha)$, i.,e. $l > \frac{S}{h\,d\,\gamma}\left(\frac{cos.\,\alpha}{f} + sin.\,\alpha\right)$, in which f may be taken $= 0{,}67$.

Example. For the suspension bridge mentioned in previous paragraphs, the vertical force of the loaded chaint $V = 165636{,}3$ lbs., and that of the unloaded: $V_1 = V - 78750 = 86886{,}3$ lbs., if now we suppose friction pulleys to be applied, the radius of each pulley being to that of its axis as $\frac{a}{r} = \frac{1}{4}$ and $f = \frac{1}{4}$, the friction at the pulleys would be $\frac{1}{4}\,.\,\frac{1}{4}\,.\,(165636{,}3 + 86886{,}3) = 15782{,}6$ lbs., or much less than the difference of the tensions, and therefore the chains would move, and the pulleys turn till the tension on the one had so far increased, and that on the other so far decreased that the difference would be only 15782,6 lbs. If now the height of the pier be 16 feet, the thickness 4 feet, and the weight of the masonry 130 lbs. per cubic foot, we have for the necessary width of piers:

$$b^2 + \frac{252522,6}{16 \,.\, 4 \,.\, 130} \,.\, b = \frac{2 \,.\, 15782,6 \cos. \alpha}{4 \,.\, 130}, \text{ i. e. } b^2 + 30,4\, b = \frac{15782,6 \,.\, 0,9285}{260} = 56,36.$$

Therefore, $b = \dfrac{56,36 - b^2}{30,4} = 1,75$ feet. This would, in practice, be made 4 to 5 feet.

The requisite length of retaining wall, when $h = 16$ and $d = 16$ feet, is:

$l = \dfrac{2\ S \sin. \alpha}{h\, d\, y} = \dfrac{2 \,.\, 165636,3}{16 \,.\, 10 \,.\, 130} = 15,9$ feet, which would be made 20 to 25 in practice.

STRENGTH OF MATERIALS.*

The strength of an engineer's work depends upon its *proportions*, the *materials* of which it is composed, and the *manner of putting them together*.

As to stability, a structure may yield, under the pressures to which it is subjected, either by the *slipping of certain of its surfaces of contact upon one another*, or by their *turning over upon the edges of one another*. The former case very rarely occurs.

The *strength of materials* depends upon their physical constitution, viz: *form, texture, hardness, elasticity, and ductility.*† The resistance of materials in buildings is tested in reference to various strains—compression—extension—detrusion—deflexion under a cross strain, and fracture under a cross strain.

A. *Compression.*—In prismatic pieces of *stone*, *wood*, or *cast iron*, which absolutely *crush* under a strain, *the strength is directly proportional to the transverse area of the piece.*

Pieces exposed to compression are not fairly *crushed*, but in some measure broken across, where their height is to their diameter or least lateral dimensions in the case of,

Stone, more than as		6	to	1?
Wood,	"	" 4	to	1
Cast iron,	"	" 3½	to	1
Wrought iron,		" 2½	to	1

The manner in which materials yield under a crushing strain is very remarkable, as is exhibited by the experiments of Rondelet, Vicat, and E. Hodgkinson, the latter of whom has found, that the plane of rupture is always inclined at the same angle to the base of the column, when its height is within the limits above mentioned. The angle of rupture depends upon the nature of the material. In cast iron, for instance, it varies from 48° to 58° in different makes of iron, though confined to narrow limits for different prisms of the same make.—*See* "Report British Association," 1836, and Moseley's "Engineering," p. 550.

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* Professor Weisbach has treated this subject as it is usually given in elementary works on mechanics. Excepting as exhibiting approximately the laws of the phenomena, the "theory of the strength of the materials" has many practical defects. These we shall not here enumerate; but have put together, in as concise a form as possible, what we consider to be the most valuable part of our present knowledge on this subject to engineers or architects engaged in the execution of works.

† *See*, on this subject, Poncelet's "Mécanique Industrielle."

TABLE OF THE RESISTANCE OF MATERIALS TO CRUSHING.

		lbs. per sq. inch.
Granite, Scotch		10804 to 8184
" Cornwall		6292
Sandstone, Dundee		6490
" Derby		3110
Marble (white)		9583
Limestone (Portland)		6550
Stourbridge brick		1695
Deal	unseasoned	6780
	seasoned	7290
Beech	unseasoned	7730
	seasoned	9360

		lbs. per sq. inch.
Oak	unseasoned	6480
	seasoned	10000
Mahogany		8198
Larch	unseasoned	3200
	seasoned	5568
Poplar	unseasoned	3100
	seasoned	5100
Cast iron, *good common*		109800
" " *Stirling's toughened*		145500
Wrought iron		56000?

The effect of *seasoning* or drying timber, in increasing its strength, is never to be lost sight of. In wrought iron, a strain of 28000 lbs. reduces the length, and causes a slight lateral bulging, corresponding to the slight reduction in length; that is to say, for a compressive strain of about $\frac{3}{8}$ths of the absolute crushing-strain, wrought iron is quite "*crippled.*"

Stirling's process of *toughening* cast iron, consists in adding to it proportions of *malleable scrap*, varying according to the nature of the cast iron in its normal state.

Scotch hot blast, No. 1, will take 28 to 30 lbs. of scrap per cent.
" " " No. 2, " 20 " " "

Welsh and Staffordshire hot or cold blast iron require a less addition of scrap.

This process increases the strength of all cast irons, from 50 to 80 per cent.

The strength of pieces, such as pillars, that *break across*, but are not *crushed* under compression, may be calculated by the following formulas, as found by Mr. Hodgkinson's "Experimental Researches on the Strength of Pillars," published in the Phil. Trans., 1840, and in his edition of "Tredgold on Cast Iron," published 1846.

For stone: $b = \frac{ad^4}{l^2}$? For timber: $b = \frac{ad^4}{l^2}$.

For cast iron. Solid pillar, round ends $b = \frac{ad^{3,76}}{l^{1,7}}$.

" " flat ends $b = \frac{ad^{3,55}}{l^{1,7}}$.

The length 1 *being not less than* 30 d.

Hollow pillars, round ends $b = a\frac{D^{3,76} - d^{3,76}}{l^{1,7}}$.

The length 1 *being not less than* 15 d.

Hollow pillars, flat ends $b = a\frac{D^{3,55} - d^{3,55}}{l^{1,7}}$.

The length 1 *being not less than* 30 d.

Wrought iron, round ends $b = \frac{ad^{3,76}}{l^2}$.

" " flat ends $b = \frac{ad^{3,55}}{l^2}$.

When the length is from 30 *to* 90 *times the diameter.*

The laws indicated by the formulas do not hold good for shorter columns.

TABLE OF THE VALUES OF a, (D and d being in inches, l in feet, and the result b being the *crushing-weight in* lbs.

Granite	flat ends	25000 ?
Sandstone		15000 ?
Marble		24000 ?
Dantzic oak		24542
Red deal		17511
Cast iron solid pillar,	flat ends	98922
" " " "	round ends	33379
Hollow pillars,	flat ends	99318
" "	round ends	29074
Wrought iron	flat ends	299617
" "	round ends	95844

The numbers here given are co-efficients, and have no meaning, apart from the special position they occupy in the formulas.

In all pillars of cast iron, whose length is thirty times the diameter or upwards, the strength of those with flat ends seems to be *three times* as great as the strength of those of the same dimensions with rounded ends: when l is less than 30 d, the ratio of the strength of pillars of the same dimensions with flat and with rounded ends, is very variable.

When pillars are reduced in length below the proportion above indicated, there is a *falling off* of their strength, *nearly in proportion to the reduction in the length of the pillar;* and this obviously must be the case, as the strength to resist *flexure*, under a compressive strain, increases as the fourth power of the diameter, whilst the resistance to *crushing* increases only as the square of the diameter.

For pillars of less length than 15 times their diameter, there is a falling off in the resistance, on account of the change produced in the position of the molecules of the material by the great weight necessary to break them: Mr. Hodgkinson has, however, given a formula which includes this case, and by which the strength of the pillars, however short, may be deduced from the results of the formulas for long columns, when the *crushing strength* of the material is known. The formula is $y = \frac{b\,c}{b + \frac{3}{4}c}$ in which b is the strength of the pillar, as calculated by the rules for long pillars, and c the crushing weight of the material, and $y =$ the strength of the short pillar.

In *similar pillars*, the strength is nearly as the square (1,865 power) of the diameter, or of any other lineal dimension; and as the area of the section is as the square of the diameter, the strength is nearly as the area of the transverse section.

The strength of pillars not less than 30 times their diametera that of cast iron with rounded ends being set = 1000

Wrought iron	is = 1745
Cast steel	= 2518
Dantzic oak, square ends . .	= 108,8
Red deal	= 78,5

In all long pillars, whose ends are firmly fixed, the power to resist breaking is equal to that of pillars of the same diameter and half the length, with the ends rounded or turned, so that the strain runs through the axis.

B. *Extension.*—When a tensile strain *passes up the centre* of a piece of stone, wood, or metal, the *resistance is proportional to the transverse area of the piece.*

TABLE OF THE RESISTANCE OF MATERIALS TO RUPTURE BY TENSILE STRAIN.

Stone.	Portland	857 lbs. per sq. inch.
	Fine sandstone . . .	215
	Brick	275 to 300
Glass		3565
Hydraulic lime, best . . .		168
Good		142
Mean quality		100
Common lime		43
Timber.	Deal	12857 to 11549
	Beech	17850
	Oak	9198 to 12780
	Mahogany	16500
	Larch	9700 to 10220
	Poplar	7200
Cast iron	(Hodgkinson)	13505 to 17136
"	(Rennie)	19200
"	(Cubitt)	27773?
"	Stirling's toughened . .	28000
Wrought iron	bars	65520 to 56000
	Wire (hard) . . .	128000 to 65360
	Wire (annealed), half the strength of hard.	
	Plates	52100
Brass wire	(hard)	98960 to 63000
	Annealed	49000
Gun metal (hard)		36368
Copper rolled		35000
" cast		19200
Ropes.	Hemp	1 ton per lb. weight per fathom.
	Wire, (Newall and Co.) .	2 tons per lb. weight per fathom.

In reference to the above table, it may be stated that it contains numbers which are the *mean values of the tensile strain,* as deduced after a careful weeding of the experimental results that have hitherto been published.

[*Thermotension, or the Effect of Heat on the Tenacity of Iron.*—The following table exhibits the effect of heat on the tenacity of iron, both while actually hot and also subsequent to the application of a strain at high temperature. The comparisons are made on thirty-two different specimens of iron, the origin of which is designated in the first column of the table. The temperature at which either the "hot fracture" or the hot strain was made on each bar, and which produced the strengthening effect of "thermotension," is contained in the second column. The third contains the number of trials made on each specimen of iron to ascertain its strength in its ordinary state and temperature, as it came from the hammer or the rolls, and before being put under strain at a high temperature. Column fourth shows the number of times the specimen was broken, or at least strained, at the temperature marked in column third. Column fifth gives the number of fractures made on the specimen to obtain the average strength after being heated, strained, and then cooled again to ordinary temperature. Columns six, seven, and eight, contain the absolute strength given in the three different states respectively. Column nine exhibits the per centage increase of strength by treatment with thermotension, and ten, the difference in strength between the iron at ordinary temperature in its original state, and that which it possessed while heated as in column third. In three cases only does it appear that the strength had been diminished by heating up to the point at which the trials were made. One of those trials was at 766°, one at 662°, and the third at 552°. The average temperature at which the effect was produced was 573.78, at which point the tenth column shows that the strength of thirty varieties of iron, was 5.9 per cent. greater than at ordinary temperatures, say at 60 or 80 degrees.

It also appears that the average gain of tenacity in thirty-two samples of iron, by the process above mentioned, was 17.85 per cent., ranging from 8.2 to 28.2 per cent. In a report by the Editor to the Bureau of construction, equipment, and repairs of the Navy Department of the United States, it is proved that the average gain of *length* of bolts of iron treated at the Washington Navy Yard, by this same process, was 5.75 per cent., and the gain of *strength* 16.64, making together the gain of *value* 22.4 per cent. The addition of 5.75 to 17.85, gives 23.6 per cent. for the total gain of value. In many instances the experiments proved the gain of *length* to exceed 7 per cent. The total elongation of a bar of iron, broken in its original cold state, is from two to three times as great as the *same force* would produce upon it if applied at a temperature of 573°, which force will, moreover, not break the bar at that temperature.

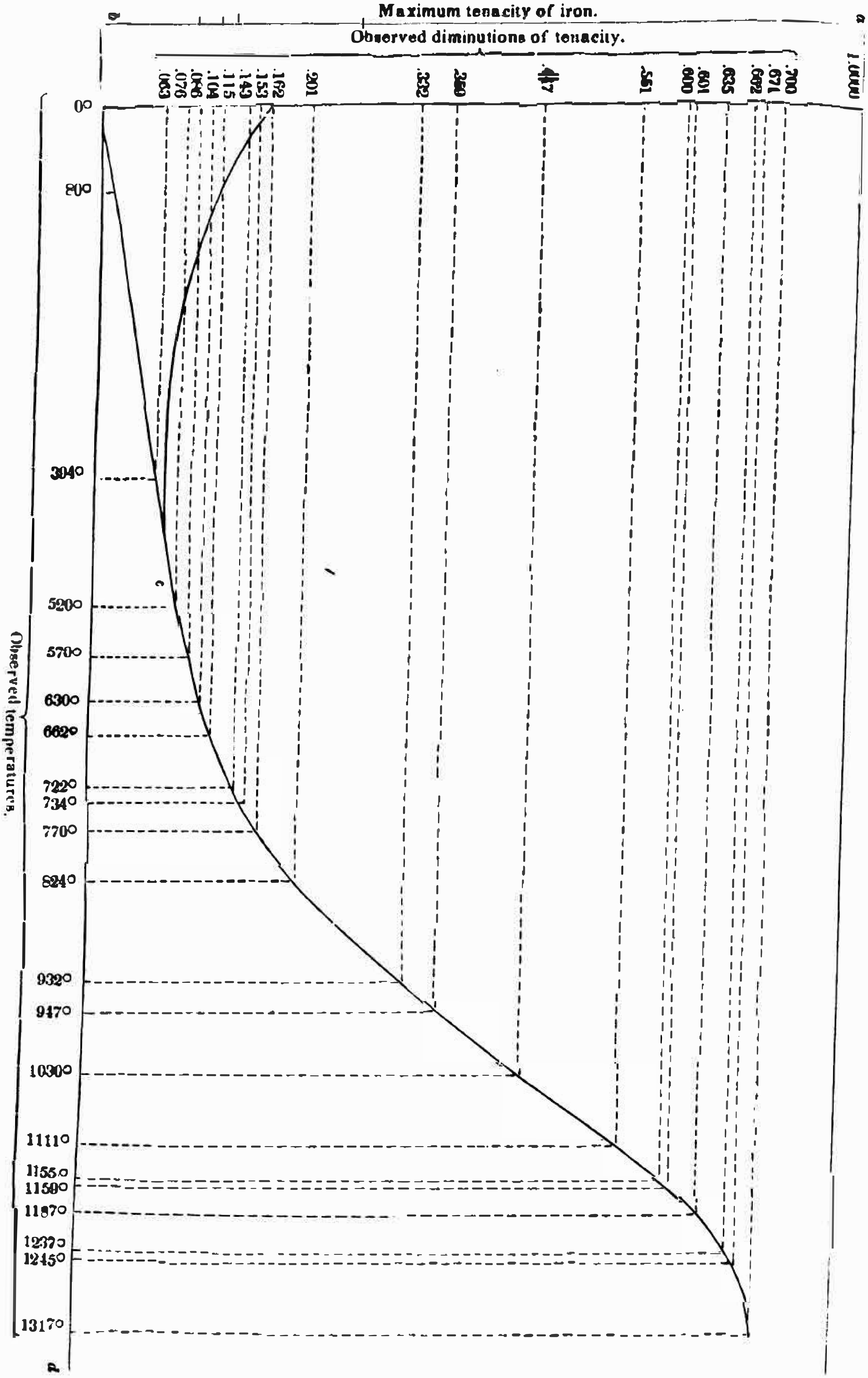
TABLE EXHIBITING THE EFFECT OF HEAT ON THIRTY-TWO VARIETIES OF MALLEABLE IRON.

NAME AND ORIGIN OF THE SPECIMENS OF IRON TRIED.	Temperature of iron when proved hot.	No. trials for average ordinary strength.	No. of trials at high temperatures	No. of trials after heating and straining.	Average strength in ordinary state and at ordinary temperature.	Strength exhibited while hot.	Average strength after applying thermo tension.	Gain of tenacity per cent. by the treatment with thermotension.	Diff. per ct between original strength and that shown when the bar was hot.
Salisbury (Conn.), gun bar	554°	4	1	5a	59.271	60 450	65 090	8.2	+ 2.0
Maramec (Mo.), bar iron	528	7	1	3	53.775	54.273	59.044	9.8	+ 0.9
Phillipsburgh (Pa.). wire	500	4	2	4	79.720	80.438	84.187	10 4	+ 0.9
Ellicott's Baltimore boiler plate	770	3	1	5	56 644	56.614	63.132	10.7	+ 0.0
" " " "	662	4	1	7	58.899	58.181	64,820	10.9	1 2
Salisbury (Conn.), gun bar	550	3	1	7	59.654	60.323	66.638	11 7	+ 1.1
" " "	590	4	1	9	59 032	62.952	67.384	13.5	+ 6.6
Swedish bar iron	530	2	1	3	58 012	59.775	66.334	14.3	+ 3 0
Nashville (Tenn.). bar iron	520	7	1	5	54.934	58451	62600	14 5	+ 6.4
Salisbury (Conn.), gun bar	572	5	1	10	58 305	58.895	63,558	15.0	+ 1.0
Ellicott's Baltimore forged bar	394	1	1	1	57.182	63,322	65.960	15.3	+10.7
Spang & Son,* hammered plate	766	3	1	1	57664	54819	66 500	15.8	— 4.9
Blake & Co.,* hammered plate	572	6	1	4	60 532	62,278	66,941	16.3	+ 2.8
Salisbury (Conn.), gun bar	580	4	1	5	55.977		65,883	17.0	no hot frac.
" " "	564	4	2	8	54,644	60.215	64 363	17.6	+10.2
" " "	576	5	2	5	58 299	64 278	68.938	18.4	+10.0
" " "	630	4	1	6	57.433	60,010	67 509	18 7	+ 4.5
English "best-best" cable bolt	560	3	1	10	62466		71,000	19.3	no hot frac.
Spang's Pittsburgh hamm'd plate	552	2	1	3	56 762	55 932	62,736	19 4	— 1.4
Nashville (Tenn.), bar iron	560	7	1	6	52,729	58.534	62,127	19 5	+11.0
Mason & Miltenberger,* piled	574	4	1	2	55 426	60.083	68,339	19 5	+ 8.4
Nashville (Tenn.), bar iron	562	4	1	2	52,194	59,623	62,433	19.8	+14 2
Ellicott's Baltimore boiler iron	553	2	1	3	61.519	66.450	73.898	20.1	+ 8 0
Schaenberger's Pittsburgh boiler	630	1	1	3a	53 803	56.159	64.926	20.6	+ 4.4
Maramec (Mo), bar iron	564	5	1	9	49.974	52.158	58.126	20 6	+ 4.3
Nashville (Tenn.), bar	578	5	1	5	52.406	59.192	62,951	21.1	+12 9
Russia sable bar	584	5	1	3	70.071	77.161	92470	21 5	+ 1 4
Maramec (Mo.), bar	576	6	1	4	43 386	50.067	53.368	23 0	+15.4
Ellicott's hammered bar	394	1	1	1	53.176	56.570	69 767	25 8	+ 6.4
Salisbury (Conn.) gun bar	575	3	1	6a	52873	60.988	66,685	26 6	+15.3
Maramec (Mo.), bar	574a	5	2	4	45.586	51,437	58,252	28 1	+12.8
Blake's Pittsburgh hamm'd plate	564	6	1	4	52937	58,284	65.425	28 2	+10 1
Mean,	573 7	129	36	153			Mean,	17.85	+ 5.9

Fig. 82 represents the tenacity of wrought iron at various temperatures from 0° up to 1317° as measured in parts of the total maximum tenacity, the line *a b* representing that maximum, and the line 0°*d* (indefinite towards *d*) being the scale of observed temperatures, in degrees Fahrenheit marked below it. The vertical dotted lines, or ordinates of the curve, therefore, exhibit temperatures, and the corresponding horizontal ones, or abscissas, show diminutions from the maximum strength, at the temperature observed. Thus, at a temperature of 1030°, the diminution from maximum tenacity is .4478, and, consequently, the remaining strength is 55.22 per cent. At 1187° the diminution is .6352, and the remaining strength 36.48 per cent., and at 1245° (a dull red heat in daylight) the diminutiou is .6715, and the remaining cohesion only 32.85 per cent., &c.

* Of Pittsburgh.

Fig. 82.


Maximum tenacity of iron.
Observed diminutions of tenacity.
.063
.076
.096
.104
.115
.143
.153
.162
.201
.332
.380
.417
.561
.600
.601
.635
.662
.671
.700
1.0000
0°
80°
394°
520°
570°
630°
662°
722°
734°
770°
824°
932°
947°
1030°
1111°
1155°
1159°
1187°
1237°
1245°
1317°
Observed temperatures.

For a more full exposition of the effect of heat on the tenacity of iron under direct tension, and for investigations of the relation between temperature and tenacity, reference may be had to the "Report on the Strength of Materials for Steam Boilers," page 212—218.

At page 75 of the same report, will be found the law of tenacity as affected by temperature for rolled copper. In that metal no increase of strength takes place from increase of temperature in any part of the scale; and the law eliminated from about 180 comparisons of different experiments on several specimens of copper, is, that the diminutions of strength by augmentations of temperature follow the principle of a parabola, of which the ordinates representing the elevation of the temperature above 32°, have to the abscissas representing the diminutions of tenacity, a relation expressed by saying, that *the third powers of the temperature are proportional to the second powers, of the diminution of strength which they produce.* This law was ascertained in the following manner: Putting $t=$ any observed temperature above 32°; $t'=$ any other observed temperature above the same point; $d=$ the diminution of tenacity by the former temperature and $d'=$ that by the latter: also making $x=$ that power of the temperature according to which the diminution of tenacity takes place; we have, by the supposition $t^x : t'^x :: d : d'$, or $\frac{t^x}{t'^x}=\frac{d'}{d}$.

From this we derive the expression $x=\frac{log.\ d'-log.\ d}{log.\ t'-log.\ t}$.

Example. At a temperature of 1016° the tenacity of a bar of copper was found to have been diminished 66.91 per cent. below its strength at 32°; at the temperature of 492° it was 21.33 per cent. below what it was at 32°; according to what power of the temperature did the tenacity vary?

Here $x=\frac{log.\ .6691-log.\ .2133}{log.\ (1016-32)-log.\ (492-32)}=1.50$; hence $t^{1.5} : t'^{1.5} :: d : d'$, or $t^3 : t'^3 :: d^2 : d'^2$.

Transforming this into an equation, we get $\left(\frac{t'}{t}\right)^3=\left(\frac{d'}{d}\right)^2$, and $\frac{d'}{d}=\left(\frac{t'}{t}\right)^{\frac{3}{2}}$, or $d'=d\left(\frac{t'}{t}\right)^{\frac{3}{2}}$. From this $\frac{3}{2}$ $(log.\ t'-log.\ t)+log.\ d=log.\ d'$; by which, knowing the diminution d at any one temperature t, we are enabled to calculate what it will be at the temperature t'.]

In reference to cast iron, the first or lower numbers (p. 70) are the results of Mr. Hodgkinson's experiments; the higher number is the result of numerous experiments made for Mr. Thomas Cubitt by Mr. Dines.* This difference is chiefly of importance in respect of there being a discrepancy so wide, between results stated by two careful experimenters. In reference to the experiments on Mr. Morries Stirling's toughened iron, they were made by the same direct means as were all Mr. Hodgkinson's experiments. The tensile strain of cast iron is seldom brought directly into action; and the part it plays in the resistance to cross strains is evidently not that for which the direct strength shown by Mr. Cubitt's experiments can be attributed to it.

* See Mr. Henry Law's edition of Gregory's "Mathematics for Practical Men," p. 375.

The elongation of wrought iron, under a given tensile strain, may be judged of from the following experiment.*

Load per square inch in producing an elongation of				Load per square inch, producing fracture.	Total elongation divided by original length.
$\frac{1}{400}$	$\frac{1}{100}$	$\frac{1}{50}$	$\frac{1}{20}$		
lbs.	lbs.	lbs.	lbs.	lbs.	lbs.
34700	40980	46124	52122	56834	.086

According to Vicat, the elongation of iron wire for a load of 1428 lbs. per square inch, or $\frac{1}{40}$ the breaking strain, amounts to 0,000057.

Mr. E. Hodgkinson's experiments have proved, in like manner, that no material is so elastic as to recover itself perfectly from even very small loads allowed to act for a considerable time, and the defect of elasticity is nearly as the square of the weight applied.

The *modulus* or *co-efficient* of elasticity, is a term first suggested by Dr. Thomas Young, to denote the measure of the elastic reaction, or the energy of the resistance of any substance, and is represented thus: $E = \frac{P}{A\,i}$.

Where E is the co-efficient of elasticity, P the weight in pounds, producing the proportional elongation $i \left(= \frac{l}{L}\right.$ where $l =$ the elongation, and L the original length) in a bar with a base of sectional area A.

Rigidity is expressed by the ratio $\frac{E.A}{L}$.

Thus, the *elastic resistance* of a prism of any material, is really only the *rigidity* referred to the unit of length of the prism.

[* In the report of the Committee of the Franklin Institute, on the materials for steam boilers, p. 219–20, will be found very numerous observations on the elasticity of iron, of which the following may be cited as the results of direct measurement.

	Recoil, when relieved from strain, in parts of original length.		Breaking weight in lbs. per square inch.	Total elongation after fracture.
Bar 49 Boiler plate from Juniata blooms.	$\frac{1}{168}$ 51.030 lbs. per square inch.	$\frac{1}{348}$ 54.860 lbs. per square inch.	57.565	6.9 per cent.
Bar 226.	$\frac{1}{800}$ 43.800 lbs. per square inch.		49.053	6.25 per cent.
Bar 228.	$\frac{1}{313}$ 34.804 lbs. per square inch.		40.643	
Bar 230.	$\frac{1}{180}$ 47.155 lbs. per square inch.		49.368	

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TABLE OF DATA CONNECTED WITH THE ELASTIC RESISTANCE OF MATERIALS.

Name of material.	Tensile strain per square inch for limit of elasticity.	Proportional elongation for strain of limit of elasticity.	Ratio of strain in column 1 to that causing rupture.	Modulus of elasticity.
				lbs.
Oak	2856	0.00167	0.23	1,713600
Yellow pine	3332	0.00117	0.33	1,856400
Red pine	4498	0.00240	0.44	2,142000
Larch	2470	0.00192	0.30	1,285200
Beech	3355	0.00242	0.30	1,385160
Bar iron, ordinary quality	17,600	0.00062	0.30	28,400000
" " Swedish hammered } selected	24,400	0.00093	0.44	29,365000
" " English rolled } selected	18,850	0.00072	0.37	29,465000
Wire. No. 9, unannealed	47,532	0.00165	0.49	28,825000
" " annealed	36,300	0.00129	0.58	28,081000
Steel plates, tempered blue	93,720	0.00222	0.67	42,600000
Steel wire of commerce	35,700	0.00120	0.50	29,500000
Cast iron	. . .	. . .	. . .	17,000000 to 13,000000

NOTE.—By "Limit of Elasticity," is meant the limits within which displacement of the parts of materials under strain may be called into play without *permanent palpable derangement, or crippling.*

C. Detrusion is the resistance that the coherence of the particles of materials opposes to their sliding on each other, under a detrusive strain.

The resistance to detrusion, or the "force necessary to *shear* across" any material, is called into play at the joints, and in the bolts of framings of timber and iron, and the rivets of steam boilers, &c.

The resistance of deal to detrusion in the direction of fibre is 592 lbs. per square inch.

The resistance of cast iron to detrusion is about 73000 lbs. per square inch, as deduced from experiments on crushing.

The resistance of wrought iron to detrusion, or to a force "shearing it across" is 45000 to 50000 lbs. per square inch, or from 70 to 80 per cent. of the resistance to a direct tensile strain.

D. Deflexion.—When a *beam* is deflected by a cross strain, the side of the beam which is bounded by the concave surface is *compressed*, and that bounded by the convex surface is *extended*. The surface at which extension terminates and compression begins, is termed the neutral surface.

The property of *elasticity*, inherent in all substances in a greater or less degree, causes them to resume their original form, very nearly, when, under forces of compression, extension, or deflexion, they have undergone a limited change of form. Up to this limit, the amounts of extension and compression for a given cross strain are nearly equal, and, therefore, the neutral surface lies very nearly, if not accurately, in the centre of gravity of the cross section of the beam.

Beyond this limit the position of the neutral surface changes, as the flexure increases; because, in stone and cast iron at least, the resistance to compression is greater than the resistance to extension, whilst the amount of deformation, under the compressive strain, is less than under an equal tensile strain. In wrought iron, as it possesses great *ductility*, this limit occurs much later than in cast iron. In timber, the resistance to extension is greater than that to compression, and its want of homogeneity renders the limit alluded to very variable.

The general law of deflexion is, that it increases, *cæteris paribus*, directly as the cube of the length of the piece, and inversely as the breadth and cube of the depth.*

E. Fracture.—The theory of deflexion, which gives the displacements of the parts of beams before the conditions even of *crippling*, has few practical applications; while equations for the resistance to fracture, which is what is essential in practice to be known, are more simply established.

The hypothesis for the theory of resistance of materials to fracture or rupture, propounded by Galileo, consists in placing the horizontal axis of equilibrium at the lowest point of the section of rupture, or in supposing the material incompressible; and he considered the *internal force developed at each point of the section as constant for every point.*

The hypothesis commonly attributed to Mariotte and Leibnitz, consists in like manner in placing the horizontal axis of equilibrium at the lowest point of the section, and in supposing the internal force developed at each point *proportional to the distance of that point from the axis of equilibrium.*

The hypothesis now generally adopted, consists in admitting that the resistance of each point, at the instant rupture is going to take place, continues *proportional to the extension and compression*, and, therefore, *that the axis of equilibrium*, or *neutral surface, has the same position as in the case of a very small deflexion.*

Experiments have proved that none of these hypotheses is true, and, that, according to the physical constitution of the material, the formula deduced from the one or the other may be taken as representing experiments. Experiments on cast iron are best represented by the deduction from Galileo's hypothesis; those on stone, by Mariotte's, and those on timber and wrought iron, by the modern hypothesis, announced by Hooke, and first developed by Dr. T. Young.

The formula commonly employed for reducing experiments, or for calculating dimensions by aid of experiments, on beams of uni-

* For the most complete development of this subject, the student is referred to Mr. Moseley's work, "Engineering and Architecture," Part V.

form rectangular section, fixed at one end and loaded at the other, is $W = \frac{f\,b\,d^2 t}{n\,l}$.

On Galileo's hypothesis	$n = 2$
On Leibnitz and Mariotte's . . .	$n = 3$
On Young's hypothesis	$n = 6$
The mean of experiments gives for cast iron	$n = 2.63$
" " stone . . .	$n = 3$
" " wrought iron and wood	$n = 6$?

To answer the imperfection of the theory, however, f^1 is substituted for f; or for the resistance to a direct tensile or compressive strain there is substituted a co-efficient of the composite resistance to fracture, under a cross strain.

The most convenient general formula in use for calculating the resistance to fracture under a cross strain is $W = \frac{f^1 I}{l\,c_1}$.

Where $W =$ the breaking weight, $I =$ the moment of inertia of the cross section of the beam, round an axis passing through its centre of gravity, c_1 the distance of the neutral surface, from the side at which the material gives way; and l the length. The beam is supposed fixed in the circumstances above mentioned.

For a beam supported at each end, and loaded in the middle, this becomes $W = \frac{4 f^1 I}{l\,c_1}$, and for beams of triangular section: $W = \frac{2}{3}\frac{f^1 b d^2}{l}$.

For a beam supported at each end, if the load be uniformly distributed over it, we have $W = \frac{8 f^1 I}{l\,c_1}$, and for beams of rectangular section, $W = \frac{4}{3} f_1 \frac{b\,d^2}{l}$.

If the weight of the beam G be taken into account, the above formulæ become respectively $W + \frac{1}{2} G = \frac{2 f^1 I}{l\,c_1}$, and $W + G = \frac{8 f^1 I}{l\,c_1}$.

For the forms of transverse section commonly met with in practice, the values of I in terms of the breadth b, and depth d, of the beam, are as follows:

1. Rectangular section.	$I = \frac{1}{12} b\,d^3$.	$c_1 = \frac{1}{2} d$.
2. Circular section.	$I = \frac{1}{4} \pi r^4$.	$c_1 = r$.
3. I shaped and hollow rectangular, b_1 and d_1 being the breadth and depth of hollow.	$I = \frac{1}{12}(b\,d^3 - b_1\,d_1^3)$.	$c_1 = \frac{1}{2} d$.
4. Hollow cylinder, or annular section, $r_1 =$ radius of hollow.	$I = \frac{1}{4} \pi (r^4 - r_1^4)$.	$c_1 = r$.
5. Inverted ⊥ (Mr. Hodgkinson's for cast iron). When A_1 . A_2 A_3 are the *areas*, and d, d, d, the *depths* of the top flange, the bottom flange and the uniting rib respectively.	$I = \frac{1}{12}(A_1 d_1^2 + A_2 d_2^2 + A_3 d_3^2)$. $+ \frac{1}{4}(d_1 - d_3)^2 A_1 - (d_2 + d_1)^2 A_2$. $- \frac{1}{4}\left(\frac{(d_2 + d_3) A_2 - (d_1 + d_3) A_1}{A}\right)$.	c_1 depending on the form of the beam.

The following table contains values of f^1, or *modulus of rupture*, being deductions from experiment by the formula $f^1 = \frac{3\,w\,l}{2\,b\,d^2}$, all dimensions, that is, l, b, and d, being in inches.

Name of material.	Modulus of rupture. lbs.	Working load. lbs.
Stone (Rochdale)	2358	235
" Yorkshire flag . . .	1116	112
" Caithness slate . . .	5142	514
Beech	9336	1550
Birch	9624	1600
Deal (Christiania)	9864	1640
" Memel	10386	1700
Fir	6700	1100
Larch	6894	1150
Oak, English	10000	1700
" Dantzic	8742	1500
Cast iron	30000 to 46900	5000 to 8000
" Hot blast mean	36900	6000
" Cold blast mean	39987	6500
" Stirling's toughened	46750	7800
Wrought iron	54000	9000

The following table, drawn up by Mr. Hodgkinson, gives the relation between the resistances to *crushing*, rupture by *tension*, and by *cross strain*.

Material.	Assumed resistance to crushing per square inch.	Mean resistance to rupture by extension per square inch.	Mean transverse strength of a bar, 1 inch square and 1 foot long.
Timber . . .	1000 or 1	1900 or 1.9	85.1 or 0.045
Cast iron . .	1000 or 1	158 or 0.16	19.8 or 0.02
Stone . . .	1000 or 1	100 or 0.1	9.8 or 0.01
Glass . . .	1000 or 1	123 or 0.125	10. or 0.01

From this table we get an idea of the extent to which the mutual dependency of the fibres or particles of the material comes into play when the pieces are bent.

This table indicates, too, that the resistance of the same area of cross section must vary according to the disposition of the material compressed and extended in the section. Mr. Hodgkinson has proved, in reference to this, that for cast iron, one mode of disposing the ironhin the section gives a greater strength per square inch of the section than another, in the ratio of 40 to 23, and the principle holds in other materials.

For the inverted ⊥-shaped girder, the strongest form is that in which the bottom flange is six times the area of the top flange. When, in these girders, the *length*, *depth*, and *top flange* are constant, and the thickness of the vertical rib between the flanges small and constant, *the strength is nearly in proportion to the area of the bottom flange.* Again, in beams of this form which vary only in depth, the strength is nearly as the depth.

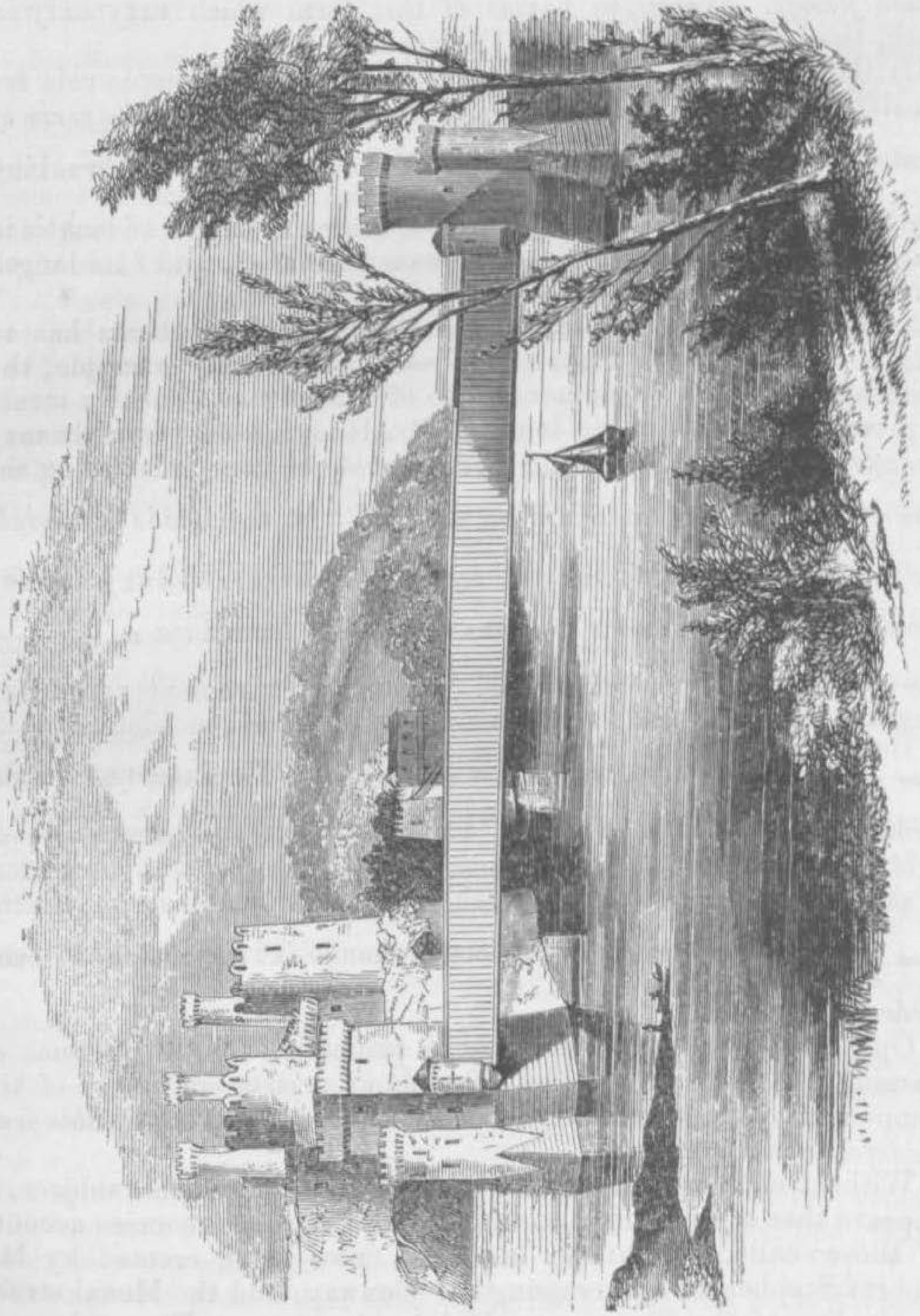
Mr. Hodgkinson has hence deduced the following simple rule for calculating the strength of cast iron beams *approaching the form of greatest strength*, viz: $W = \frac{2.166\, a\, d}{l}$ in which $W =$ the breaking weight in tons; $a =$ the area of bottom flange at centre of length in square inches, $d =$ the depth of the beam in inches, and l its length in feet.

As it is very usual to express the load a girder or beam has to bear, in terms of its length, or $W =$ to $w\,l$, (as, for example, the girders of railway bridges have to be of dimensions to bear a strain of 2 tons per foot of their length,) Mr. Hodgkinson's formula may be converted into the following very simple one for calculating the area of the bottom flange, vize $a = \frac{w\,l^2}{2{,}116\, d}$ in which w is *the weight per foot* of the girder, of the load upon it. Further, as d is generally a simple fraction of $l = x\,l$, we may make the formula $a = \frac{w\,x\,l}{25{,}992}$. For example, it is a usual and generally convenient proportion to make $d = \frac{1}{18}\,l$, and hence, for railway girders, in which $w = 2$, $a = \frac{l + 16}{12.99}$, which we may put $a = \frac{l + 16}{13}$. Engineers now make girders of proportions such as to bear 6 times the greatest load likely to come upon them. Hence, as there are generally 4 girders to take the load in a railway bridge, our formula may be written $a = \frac{l + x}{8.662}$ for the area of the bottom flange (at its centre) of each girder.

Open cast iron girders are bad in principle. Of all systems of framing girders or beams, the principle of perfect continuity of the component parts, involved in Mr. Fairbairn's patent *malleable iron girders*, is the best.

Without entering further into an examination of this subject, it appears that the present is a fitting place to give a concise account of the so-called "TUBULAR BRIDGES," now being erected by Mr. Robert Stephenson for crossing the Conway, and the Menai straits on the line of the Chester and Holyhead railway. The problem of passing both these points with the "Holyhead road," was solved by Telford in 1825, by the erection of the well known Conway and Menai suspension-bridges. Suspension-bridges have been rejected as inapplicable to railways, and Mr. Stephenson has proposed, nay, has already completely settled the practicability of carrying out the *girder system* to meet the case. A girder to span 462 feet is an

Fig. 83.



original and bold conception; and now that it may be said *to have been executed*, an attempt, if only imperfect, to sketch the progress of engineering art in the direction that has led to this master-piece, cannot but be useful.

The circumstances demanding or necessitating the erection of a bridge of great span, occur but seldom, and the double condition of erecting the bridge without centering, still more rarely.

The deep and rapid rivers of Switzerland, seem first to have called forth constructive skill for this purpose. In the year 1757, Jean Ulrich Grubenmann, born at Taffen, in the canton Appenzell, erected the celebrated bridge at Schaffhausen, over the Rhine, in lieu of a stone bridge that had been swept away by the stream. In designing his bridge, Grubenmann took advantage of a rock about midway across, for the erection of a pier to support the ends of two frames or *compound girders of carpentry*, the one of 170 feet, the other 193 feet clear-bearing, or span.

In 1778, Grubenmann and his brother constructed the Wettingen bridge over the Limmat, on the same principle that had guided them so successfully to the erection of that at Schaffhausen. This bridge had a clear span of 390 feet.*

To Chretien von Michel, an engraver at Bale, we are indebted for the preservation of a record of the details of construction of these two bridges, viz.: "Plans, coupes et élévations des trois Ponts de Bois les plus remarquables de la Suisse, publiés d'après les dessins originaux, Basle, 1803."

Both these bridges were burnt by the French in 1799, the one having stood 42 years, the other 21 years. Over the one, stones weighing 25 tons each had passed; and over the other a division of the French army with its artillery, in extreme haste. ("Emy, Traité de la Charpente.") The points of construction in Wittingen bridge, to which we would direct especial attention, are:—

1. The *continuity* of the framing, especially in its vertical plane, as perfect as the nature of the materials allow.

2. The introduction of a *roof* as an integral part of *the constructive strength of the* bridge, and of the disposition of the greater *mass* of the timber towards the top and bottom, while the intermediate more slender part, or *rib*, is stiffened at every 15 feet by strongly framed uprights on the outside and inside. The timbers are laid nearly horizontally, accurately bedded on, and indented into each other, and bolted together by numerous wrought iron through-bolts.

3. The circumstance that the two side frames of each were raised ready framed into their positions. This latter is an inference from the fact, that powerful *screw-jacks* placed on a scaffolding, supported

* [The single arch wooden bridge, built by Lewis Wernwag, over the river Schuylkill, at Fairmount, Philadelphia, had a span of 340 feet 4 inches, and a rise of the arch in the centre of nearly 19 feet or above $\frac{1}{18}$th of the chord line. This bridge had a triple beam arch of timber surmounted by king-posts and truss braces, with longitudinal ties above, the whole being strengthened by screw-bolts. See a figure of it in Rees' Cyclo., Amer. Edition, vol. 34.—AM. ED.]

on piles ("des *verins* placés sur des échafaudages établis sur *pilotes*"), were used in raising the bridge at Schaffhausen, and that the Limmat, near the convent of Wettingen, is of great depth.

Fig. 84.

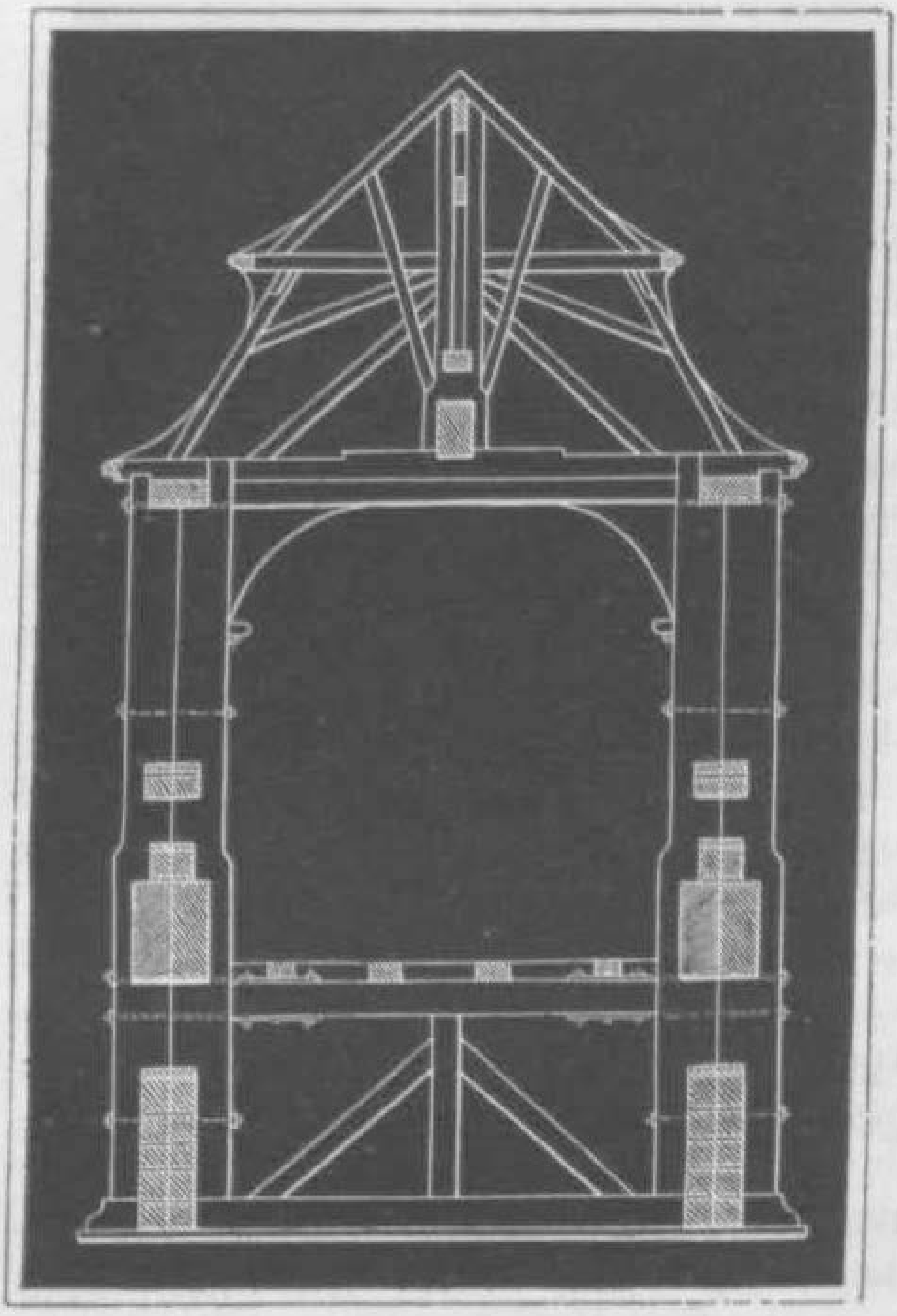


Fig. 85.

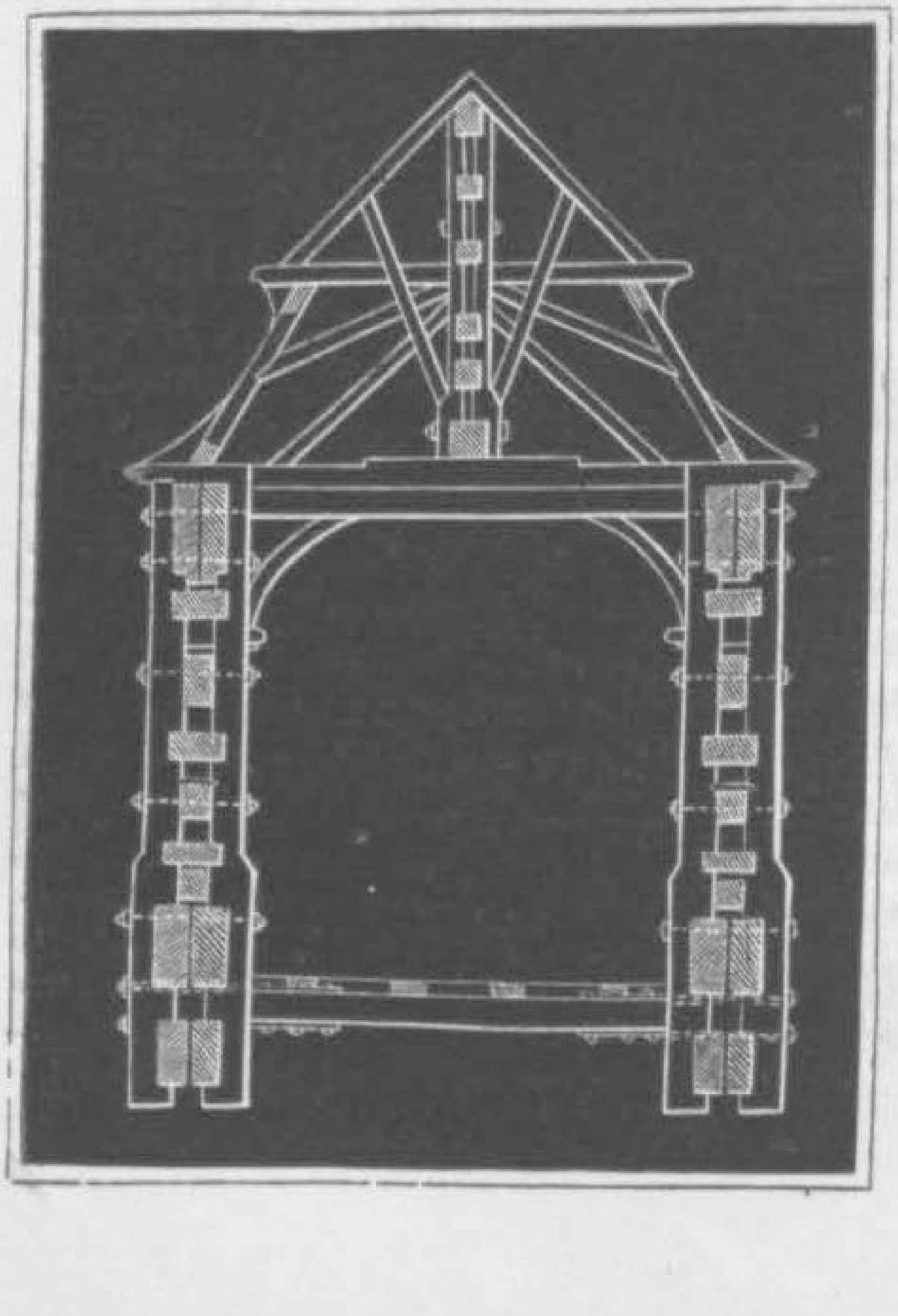


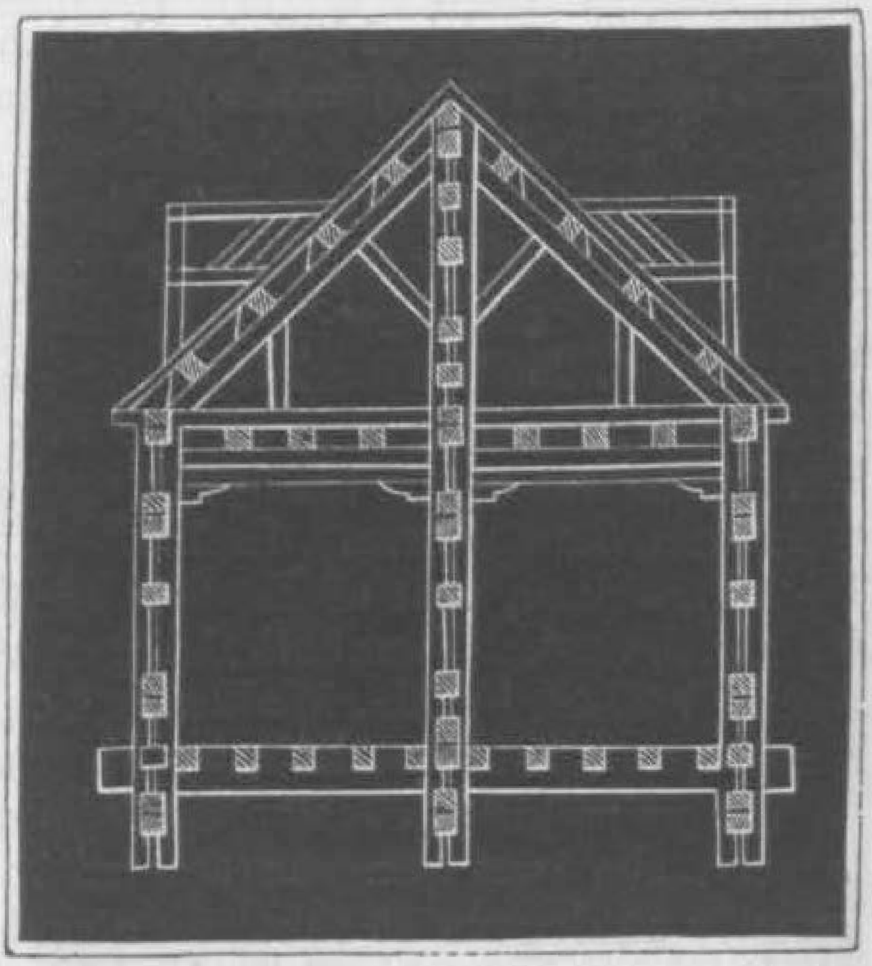
Fig. 84 is a section of the bridge of Wettingen at the ends, and Fig. 85 a section at the centre. They sufficiently illustrate what we have said above in reference to the principle of continuity, and the disposition of the roof and timber of the frames generally, in reference to the strength of the bridge.

At the period when the Wettingen bridge was erected by the Apenzell carpenter, the science of the strength of materials had scarcely begun to be formed. Galileo's theory, partially corrected by the hypothesis of Hooke and Leibnitz, and by the experiments of Mariotte and Buffon, began to attract notice; but our present knowledge of the mechanism of the transverse strain, resulting from the later experiments of Duhamel, Rondelet and Barlow, and the theories founded upon them were undeveloped. Yet we find the essential elements of these theories fully recognized in the construction of the bridges erected by the brothers Grubenmann. Art is the mother of Science.

This was the largest bridge ever *erected* on Grubenmann's principle; but, in 1772, there was exhibited, at the Hotel d'Espagne, rue Dauphine, a model of a bridge designed by one M.

Claus, for Lord Hervey. This was the model of a bridge 900 feet span, to be thrown across the Derry. The model was 20 feet long, or $\frac{1}{45}$ of the full size. The engravings were executed by Lerouge, and Fig. 86 is taken from the plate. It is a transverse section of the bridge at about $\frac{1}{3}$ of the span from the abutment or pier. The scale being about $\frac{1}{363}$.

Fig. 86.



Grubenmann's principle is adopted. The frames are here again nearly continuous. They consist of beams laid *nearly horizontal*, indented into each other, bolted together by innumerable long wrought iron bolts, forming the side ribs, and these were stiffened laterally by uprights. The floor and roof are so framed with the trusses or ribs, as to form one great double *box*, or hollow girder, nearly every pound in the weight of which is available towards the absolute strength of the whole.

This bridge was never executed; but we see in it a still more perfect adoption of the plan of making the floor and roof a part of the framing, and also a recognition of the fact that wood has double the resistance to extension, that it has to compression; and, hence, the timbers of the upper part are arranged conformably to this fact. This was clearly recognized by Grubenmann, but not so perfectly worked out in the construction of his bridges, as was done by Claus. The introduction of a roof, as an integral part of the structure, is, of course, limited to cases in which the span is such as necessitates a *depth* of girder of 16 to 18 feet at least. The proportion of the depth to the length of the bridge of Wettingen is nearly $\frac{1}{12}$. (For further details of the construction of the model, *see* "Emy, *Traité de la Charpente*, Vol. II. p. 398, and plate 134.)

In Great Britain, the problem of erecting bridges of wide span had scarcely ever been mooted till about the beginning of this century, when the joint influence of the inventions of her Dudleys, Brindleys, Hargreaves, Arkwrights, Smeatons, Watts, Corts, Wyatts, Mylnes, Rennies, Telfords, so rapidly developed the long latent industrial genius of the country, that in the short space of half a century, from being as low as any, she became the first in the scale of nations for perfection in internal communication, manufacturing skill, and in productiveness of the useful metals, especially *iron*.

In the year 1800, the subject of replacing Old London Bridge, occupied the attention of nearly every engineer of eminence, and of many men of acknowledged scientific attainments. At this period, the success of the Wearmouth Bridge, designed by Mr. Wilson, in 1793, and erected in 1796, by Rowland Burdon, and of that of Buildwash, erected by Telford, 1796, seems to have drawn the attention of the most distinguished engineers to this material, as that best facilitating the execution of bridges of great span. The wonderful progress of the iron trade at this period, also, had its influence. The question of rebuilding London Bridge was shelved at this period; but Messrs. Telford and Douglas gave in designs for spanning the Thames by a single arch of 600 feet span, and the practicability of the design was supported by the opinions of Playfair, Robison, Watt, Southern, and others. In 1808—12, Staines Bridge was erected by Mr. Wilson, that of Boston by Mr. Rennie, and that at Bristol by Mr. Jessop. Vauxhall Bridge was commenced, 1813, by Rennie, finished 1818, by Mr. Walker. The magnificent Southwark Bridge was erected 1814 to 1818, by Messrs. Rennie, father and son.

The principle of construction adopted in all these, was that of the *arch*. The cast iron was framed so as to render the structure as strictly analogous to that of an arch of voussoirs as possible. We shall here only notice that the adoption of this principle involves a prodigious expenditure of cast iron, to insure the lateral stability, essential in the voussoir principle, beyond what is necessary for the vertical strength required to bear the load.

The use of cast iron as the framing of machinery, floor-girders, lock-gates, swivel-bridges, &c. &c., became more and more usual in the construction of works executed after 1808, at which period Brunel, by demonstrating the practicability by using cast iron as the framing of his block-machinery, gave new confidence in adopting the recommendation of Smeaton, on this subject, made 50 years earlier.

In 1817, Barlow's Essay on the "Strength of Timber, Iron, and other materials," was published, and English engineers were thus put far on the way of making "principles of science rules of their art." A few years afterwards, Tredgold's Essay "On the strength of cast Iron and other Metals," was published; and this remarkable work of a most remarkable man, together with Barlow's work, had—all engineers will admit—a powerful influence in extending the rational use of iron in construction. Ten years later, Mr. Eaton Hodgkinson, of Manchester, began a course of inquiry on the strength of iron, which, while it has earned for him and his coadjutor, Mr. Fairbairn, a high reputation for scientific knowledge and skill, has, even more directly than the earlier works mentioned, contributed to the present important position of iron as a material in construction.

During this period, too, the dependence of England on Russia and Sweden, for malleable iron, was put an end to, by the improvements and vast extension of the Welsh and Staffordshire rolling-mills, which, towards 1810, began to stock the markets with iron, equal for all ordinary purposes to that which, up to this period, had been chiefly supplied by foreigners.

It is a distinguishing element in the engineer's art, to adopt the material best suited, economically speaking, to the work he has to accomplish.

In 1806, the price of bar iron, larger size, was £20 per ton; in 1816, it was £10 per ton; in 1828, it was £8 per ton; and in 1831, it was £5 to £6 per ton.

Thus this material has gradually come into the domain of applications in construction, from which its high price had long excluded the consideration of its qualifications. Roofs of great span began to be formed of combinations of cast and malleable iron. The eligibility of the one to resist strains of compression, and of the other to resist tensile strains, became familiar to those engaged in practical construction.

In 1825, a new engineering era had arisen. As the genius of Brindley, under the mighty influence of the policy of a Chatham, had created the inland navigation of England, the genius of a Stephenson, under the influence of the policy of a Huskisson, created the railway system. Steam navigation advanced from mere essays to a system of vast importance. The demands of the ship builder, the locomotive maker, the railway engineer, gave rise to new exertions of the iron masters. Blooms were puddled, of sizes hitherto deemed impracticable. It became usual to have bars rolled, and pieces forged of sizes exceeding those which, within a few years, had been deemedh wonderful or isolated examples. In this respect, the complacent dictum of a celebrated engineer, that "no difficulty can arise in engineering or mechanical art, that is not certain to be overcome," has been fully borne out.

In the construction of the London and Birmingham Railway, the Great Western Railway, the Midland Counties Railway, and others, the engineers made ample use of cast iron, and examples of girders of 50, 60, even 70 feet in length are to be found on these lines of railway. The scientific principles of construction of such girders were not at once recognized or learned, and we consequently find excess of iron in most instances, and mistaken construction in others. There was no time for gathering exact knowledge, though extant. A limited experience of successful cases led to endless repetitions of girders of not very happy proportions, and "trussed" in the wrong direction. The outcry made in England on the subject of hot blast iron being so inferior in quality, so treacherous, &c. &c., the consequent high price demanded for castings of what was termed good iron, had considerable influence in limiting the applications of iron in railway bridges. Stone and brick were preferred for the few bridges of great span erected. Suspension bridges were tried and failed. Of the wooden bridges erected, that over the Tyne at Scotswood, by Mr. Blackmore, deserves mention as involving the best principles of construction. The path so well opened up by Grubenmann had long been lost. The system of the Bavarian engineer, Wiebecking, and applied by him successfully to the bridge at Bambery, 215 feet span, and others, were extensively made known by his published writings, whilst the better principle of Grubenmann was overlooked. The essential part of Wiebecking's system consists in putting the main *strength* of the frame in arches of curved timbers trenailed together, on to which the rest of the timbers of each truss is framed, *suspending the horizontal ties*, from which the road-way is supported. Wiebecking's system, with certain modifications, was adopted in France by M. Emmery, about 1830, and by the Messrs. Green, of Newcastle, about 1840. In imitation of Wiebecking's plan, too, the *bow and string* fashion of open cast iron girders was adopted, small as is the analogy between *wood* and *iron*. Beginning with the bridge over the Regent's Canal at Camden Town, this fashion of girder has been many times repeated, on various scales; and is in execution even at the present moment, for spans of 120 feet, in the high level bridge at Newcastle-upon-Tyne.

In the mean time, in America, Town's lattice frame bridges, and Long's diagonal frame bridges, had been invented, and railway bridges of 150 to 180 feet clean span, had been executed according to each system. In the largest application of Long's system, the depth of the frame is about 20 feet, and the sides and floor, and roof are connected together, so as to form one *box-like girder*. The diagonal framing, even when carried out in the form of lattice work, makes but an imperfect continuity in the framing, or ribs connecting together the top and bottom rails or flanges; but this is *the principle* aimed at, and the bridges are to be considered as very successful engineering. They have been adopted in England, in a few cases, the largest being that of an occupation bridge on the Birmingham and Gloucester railway; but wooden structures are avoided in that country, on account of the extreme variations in the hygrometric state of the atmosphere.

Of the many lattice bridges erected in America, the most interesting in reference to our subject, is the iron tubular lattice bridge in the great hotel, Tremont House, at Boston. This is *an elliptical tube of lattice or trellis work*, the height being 7 to 8 feet, the minor axis of the ellipse being 4′—6″, the span about 120 feet. The top is stiffened by a longitudinal bar. The flooring of wood on the bottom, is about three feet 6 inches wide, and helps to stiffen the whole. This foot bridge had been several years in use in 1843, and its perfect rigidity, it may be here mentioned, at once suggested the applicability of the plan for carrying a railway across the Menai straits.

Among the circumstances concurring to the result consummated by Mr. Stephenson, the success of iron ships of enormous dimensions, in resisting the strain they have to undergo, is certainly a prominent one. The Great Britain steam-ship, for example, is 253 feet in length. It is mainly composed of sheet and angle iron, of less than half an inch in thickness; it is thus, like other iron ships, a mere shell; and yet from its perfect *continuity*, and the nature of the materials, has, unimpaired, withstood lateral strains under whichh a vessel, on almost any other construction, must have broken up.

Such was the state of preparation of engineers' minds for solving the problem of carrying a railway across the Menai straits by girders, when, early in 1845, Mr. Stephenson's "aërial tunnel" was spoken of. On the 5th of May, 1845, he announced his plan before a committee of the House of Commons.

Few inventors can explain the development in their minds of an original conception. Invention in art consists of two distinct intellectual efforts—first, in seizing the ideal conception of the object to be made for a given end; and second, in the contrivance of the suitable arrangement of materials (or of mechanism, in the case of a machine) for that object. The nature of the first conception seems always to depend on the existing state of analogous objects, and, hence, the two parts of the process are generally intimately connected, though not inseparable. In Mr. Stephenson's case, the two processes seem to have been separated. For as early as April, 1845, Mr. Eaton Hodgkinson and Mr. Fairbairn seem to have been consulted as to experiments on the strength of cylindrical tubes of riveted sheets of iron, and as to the necessity of a combination of the girder plan with suspension chains, for his great bridges. We learn from a communication of Mr. Hodgkinson's to the Mechanical Section of the meeting of the British Association, held at Southampton, in 1846, "that a number of experiments were made upon *cylindrical* and *elliptical* tubes, and a few upon rectangular ones;" but, inasmuch as a girder has to resist in its vertical direction much more than in its horizontal, the oblong rectangular form should have immediately suggested itself as the best; and, therefore, these first experiments were works of supererogation.

Mr. Hodgkinson's experiments were, therefore, at once directed to ascertaining what should be the distribution of the metal in hollow rectangular girders, to secure a maximum of strength with a minimum of weight. Mr. Hodgkinson, whose investigations, published in 1840, had proved experimentally that hollow columns have a greater resistance to compression than the same weight of material in a solid column (as the usual theory had indicated, and the practice of Wiebecking and Gauthey thirty years earlier, and of Polonceau, in 1839, had testified), now made further experiments to ascertain the relative resistance of circular and rectangular tubes, with the object of disposing of the malleable iron, of which the girders were to be made in this hollow form, on the upper side, *i. e.*, the part *compressed* by the strain.

As might have been anticipated, the "*buckling*" of the plates on the top had to be prevented by particular contrivances, or by greatly increasing their substance beyond that of the bottom or extended side.

The following are some of the leading results of Mr. Hodgkinson's experiments.

Expcriments on two similar tubes.

Length of tube.	Weight of tube.	Distance between supports.	Depth of tube.	Breadth of tube,	Thickness of metal in 16ths of an inch.	Breaking weight in tons.	Ultimate deflexion.
	cwt. qu.		feet		Top Bottom Side		Inches
31′—6″	20—3		2	1′—4″	6 4 2	26,1	2¾
47—0	61—1	45	3	2′—0	9 6 3	65,5	3¾

This breaking weight in tons is in *excess* of the results deduced from the usual formula, when the value of I (the moment of inertia), is calculated by our formula 5 (page 79), when f^1 is taken = 56000. To ascertain the power of such tubes to resist a lateral strain—as from the action of wind—the smaller of these two tubes, after being well repaired, was laid on its side and broken. The mean of two experiments gave 15,2 tons as breaking weight, which is about 25 per cent. above the result of calculation by our formulas, when the value of f^1 is taken as indicated. Experiments on the strength of sheet iron, however, give the tensile resistance as high as 62000 lbs. per square inch, and if we introduce this as the value of f^1, the experimental results would almost exactly correspond with the received theory.

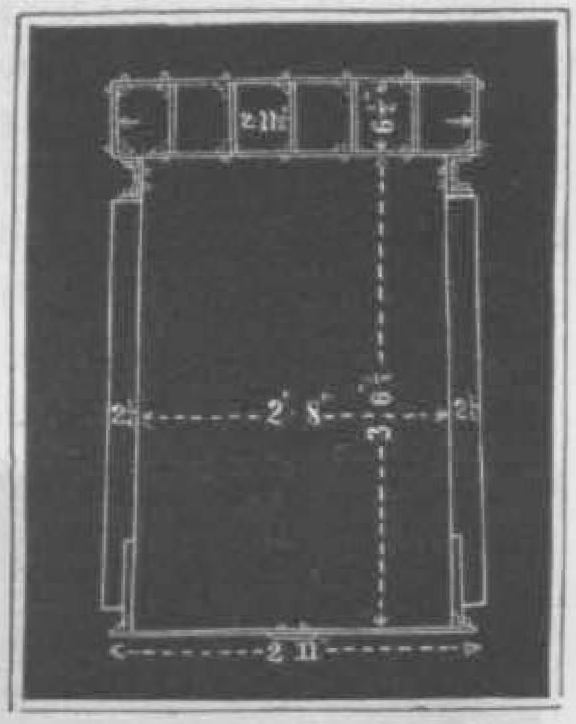
Mr. Hodgkinson's experiments on the resistance of sheet iron tubes to compression, show (as his experiments on cast iron columns made in 1839, had previously done, and as Euler's theory indicates), that rectangular tubes are weaker than square ones, and both of these much weaker than cylindrical tubes; so much so, indeed, *that the substitution of cylindrical for square or rectangular tubes, would,* according to Mr. Hodgkinson's experiments, *effect a saving of one-fourth of the metal in the top.*

Mr. Fairbairn, at the same meeting of the British Association, September, 1846, made the following communication of "Experiments on the Tubular Bridge, proposed by *Mr. R. Stephenson*, for crossing the Menai straits. These experiments, says Mr. Fairbairn, have put us in possession of facts, which greatly increase our knowledge of the properties of a material, whose powers, when it is properly put together, are but imperfectly understood; for exclusive of the rapidly increasing use of wrought iron in the construction of ship-boilers, &c., its application to bridges of the tubular form is perfectly novel, and originated with Mr. Robert Stephenson. Experiments of the most conclusive character were those made on a model tube on a large scale, containing nearly all the elements of the proposed bridge, and the various conditions with regard to form and construction, which had been developed by the previous inquiries (above alluded to). It occurred to Mr. Fairbairn that the strongest form would be that, wherein the top and bottom consisted of a series of pipes, with riveted plates on their upper and under sides. This form of top, says Mr. Fairbairn, would possess great rigidity, and is well adapted to resist the crushing forces to which it is subjected; and the bottom section appeared equally powerful to resist tension.

Mr. Fairbairn thought that this is the strongest form that could be devised; but practical difficulties present themselves in its construction, as an easy access to the different parts for the purposes of painting, repairs, &c., is absolutely necessary. The scale of the model tube was exactly one-sixth of the length, breadth, depth, and thickness of metal of the bridge intended to cross one span of the straits, 450 feet, (since increased to 462 feet.) In each of the experiments, the weights were laid on at the centre, about one ton at a time, and the deflection was carefully taken as well as the defects of elasticity after the load was removed.

"The rectangular model tube, Fig. 87, was 80 feet long, 4′—6″ deep, 2′—8″ wide, 75 feet between the supports. The thickness of the plate: bottom .156 inch, sides .099 inch, top .147 inch, sectional area of bottom 8,8 inches, weight of the tube 4,86 tons = 10,889 lbs. First experiment, breaking weight 79,578 lbs = 35½ tons. Ultimate deflexion 4,375 inches, permanent set under strain of 67,842 lbs. .792 inch. With the strain of 35½ tons, the *bottom* was torn asunder, directly across the solid plates, at a distance of 2 feet from the centre of the shackle, from which the load was suspended. One of the principal objects of this inquiry was to determine the ratio between the top and bottom of the tube. From the experiments immediately preceding this, it appeared that the ratio of the area of the top to that of the bottom, in a *rectangular tube* (of thin sheet iron), should be as 5 to 3.

Fig. 87.



"The plates forming the top of the model tube were somewhat thicker than intended, and consequently gave (as former experiments indicated) a preponderating resistance to that part. To obviate this disparity, two additional strips, 6½ by $\frac{5}{16}$, weighing about 4 cwt. were riveted along the bottom, extending 20 feet on each side the centre. This raised the area of the bottom to nearly 13 inches, being about the ratio of 5 to 3, or 23,5 to 13. With these proportions, and having repaired the fractured part by introducing new plates, the experiments proceeded as before.

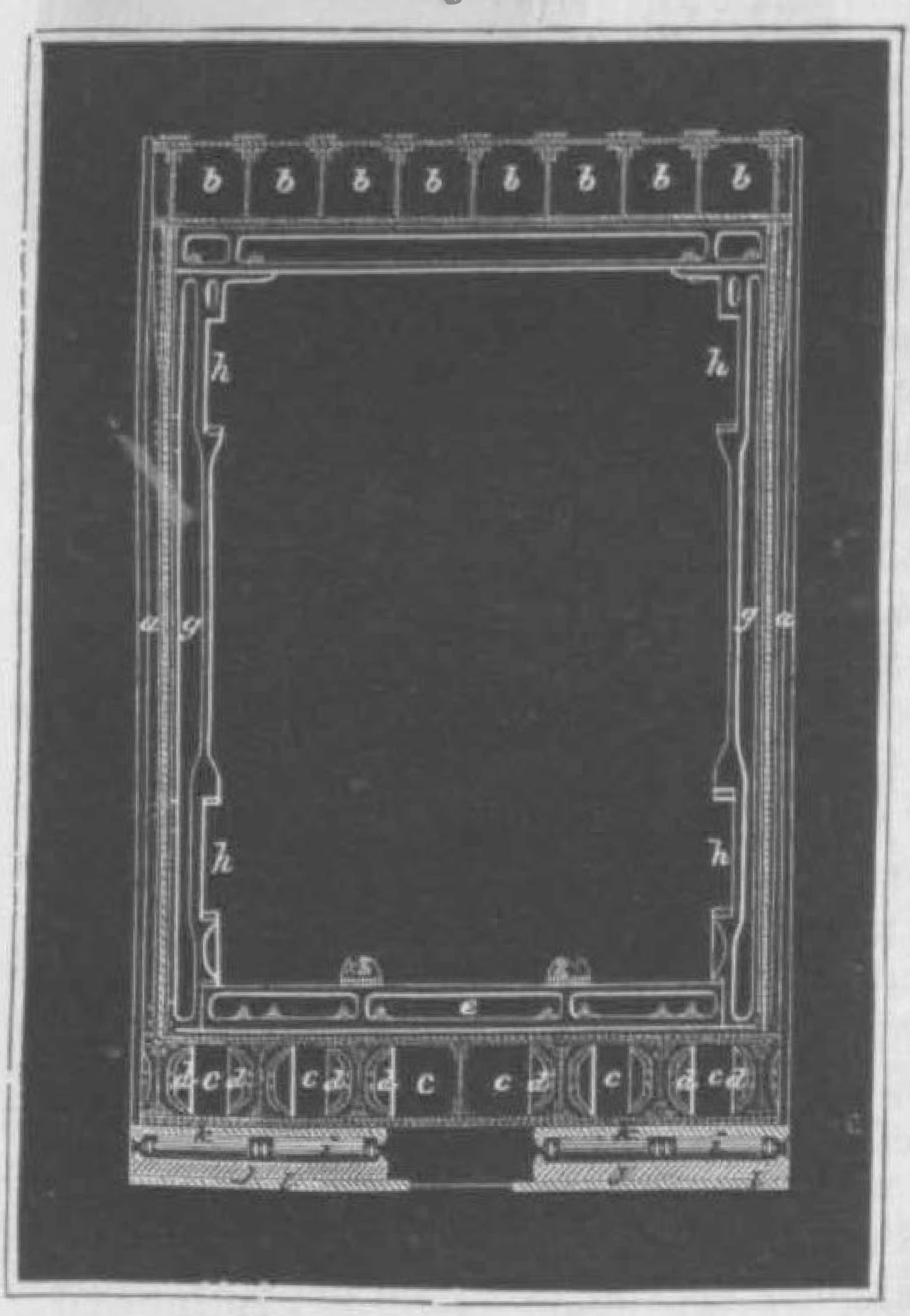
"Second experiment. Breaking weight 97,102 lbs. = 43,3 tons. Ultimate deflexion 4,11 inches. In this experiment the tube failed by one of the ends giving way, which caused the sides to collapse. The weak point in this girder was evidently a want of stiffness in the sides. To remedy this evil and keep them in form, vertical ribs, composed of light angle iron, were riveted along the interior of each side at distances of 2 feet; and, having again restored the injured parts, the tube was a third time subjected to the usual tests.

"Third experiment. Breaking weight 126,138 lbs. = 56,3 tons, ultimate deflexion 5,68 inches. The tube was torn asunder through the bottom plates. The cellular top gave evident symptoms of yielding to a crushing force by the puckerings of *each side*, which gradually enlarged as the deflection increased. These appearances became more apparent as the joints of the plates on the top side *had sheared off a number of the rivets*, and the holes had slid over each other to an extent of nearly $\frac{3}{16}$ of an inch."

On Mr. Fairbairn's most admirably stated facts, we shall only remark, that a *cellular bottom* would probably be found to be the *weakest* and not the strongest form in which the iron could be distributed there; for there is no tendency to *buckle* in the bottom; and to resist the transverse strain of passing loads (in the actual bridge), the separation of the plates composing the bottom, should only be such as to allow of the introduction of connecting plates or joists to stiffen it, so as to make the *bottom* a roadway. Again, the ratio of the areas of the *top* and *bottom* above deduced, is evidently not an *absolute* quantity, but refers only to the particular *form of cells* adopted in these experiments. Theory and experiment indicate this to be the true view of the case.

These experiments were used in determining the dimensions of the bridges already erected, and now in construction. In reference to the Conway Bridge, the first tube of which was erected in March, 1848, the following particulars are taken from the Civil Engineers' and Architects' Journal, for June, of this year.

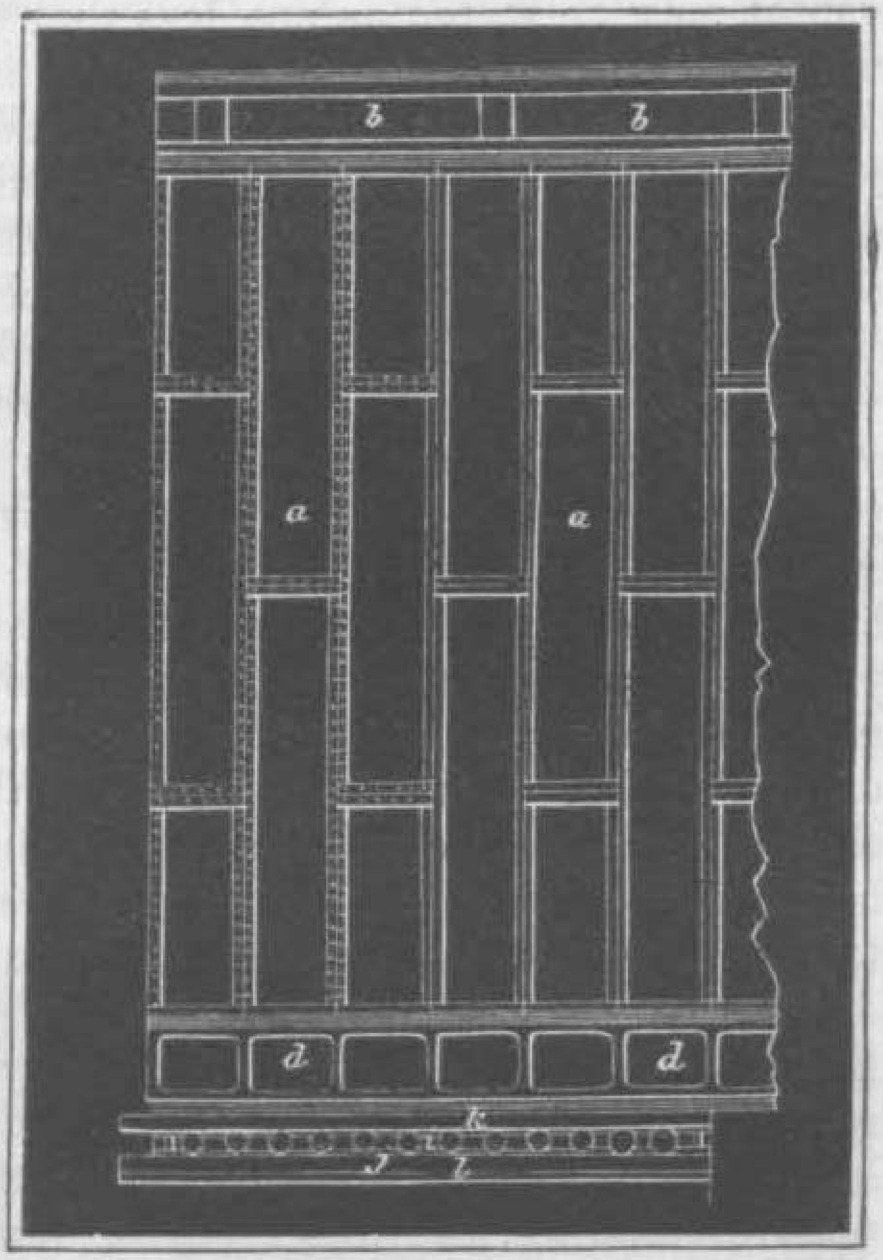
Fig. 88.



"Fig. 88 exhibits a transverse section of one of the tubes. Fig. 89 is a side elevation, of 12 feet in length, of the tube, resting on the masonry. The tube consists of sides *a*, *a*, of wrought iron plates, from 4 to 8 feet long, and 2 feet wide, by $\frac{1}{2}$ inch thick in the centre, and $\frac{5}{8}$ths of an inch thick towards the end of the tube, riveted together to T-angle-iron ribs, placed on both sides of the joints, and angle-gussets at the feet of the ribs to stiffen them; a ceiling (or top flange), composed of 8 cells or tubes *b*, each $20\frac{1}{4}$ inches wide, and

21 inches high; and a floor containing 6 cells or tubes *c*, 27½ inches wide, and 21 inches high. The whole length of the tube is 412 feet; it is 22 feet 3½ inches high at the ends, and 25 feet 6 inches high in the centre, (including the tubes at top and bottom, running the whole length,) and 14 feet wide to the outside of the side plates. The upper cells are formed of wrought iron plates, ¾ inch thick in the middle, and ½ inch thick towards the ends of the tube, put together with angle-iron in each angle of the cells; and over the upper joints is riveted a slip of ½ inch iron, 4½ inches wide. The lower cells consist of ⅜ inch iron plates for the divisions, and the top and bottom of two thicknesses of plate, each 12 feet long, 2 feet 4 inches broad, and ½ inch thick in the centre, and ¼ inch thick at the ends, and so arranged as to break the joint; and a covering plate of ½ inch iron, 3 feet long, is placed over every joint on the underside of the tube. The external casing is united to the top and bottom cells by angle-iron, on both the inside and outside of the tube. The ends of the tube, where it rests on the masonry, are strengthened by cast iron frames *d*, to the extent of 8 feet of the lower cells. The tube was constructed on a platform erected on the shore of the river, close to where it was to cross; and, when finished, six pontoons were placed under the tube at low water, and at high water they lifted the tube off the piles upon which the stage was erected. It was then floated to its destination, and placed between the two towers, part of the masonry being left undone until the tube was put into its proper position, and as it was raised by means of hydraulic lifting presses, the masonry was built up under the tube. In order to allow of the free expansion and contraction of the tube, the ends rest on 24 pairs of iron rollers *i*, connected together by a wrought iron frame, and placed between two cast iron plates *j*, *k*, 12 feet long by 6 feet wide, and 4 inches thick. The lower plate is laid on a flooring of 3 inch planks *l*, bedded on the stone-work.h Fig. 88, *h*, *h*, are uprights, into which are fitted the cross-lifting girders for attaching the chains of the hydraulic presses.h'

Fig. 89.



The weight of each tube of the Conway Bridge has been stated to be 1300 tons, but whether this is the weight including the fixtures for the rails, or of the tube *per se*, is not recorded in the papers to which we have had access. The total length being 420 feet, the weight may be stated as not less that 62 cwt. or 3,1 tons per "foot running." It is difficult to conceive anything more admirable than this final result, when we learn that, under the passage of the heaviest goods-trains, there is no sensible motion, by deflection, among the parts of the tube-girders. No method of construction, hitherto adopted for large spans, could have accomplished this absolute security with so small a weight of materials.

In what precedes, we have endeavored to trace the progress of a particular part of the engineer's art, with a view to encourage young engineers to look upon their art as capable of being formed into a science; and we yet venture to add, in reference to what we have said as to the separate intellectual efforts involved in Mr. Stephenson's invention—that the adoption of the results of the above-mentioned experiments, the execution of the designs ultimately determined upon, and the erection of the tubes, are details requiring the highest order of skill and practice in the execution of works; but, *the "keeping hold of the original idea,"* until brought to the form of ascertaining, experimentally, the best shape or arrangement of the materials, is certainly the essence of invention—marks out the Engineer-in-chief's work unmistakably—is the element in the grand result that commands the homage paid to engineering genius, by the less gifted of the profession, and secures to itself, envy or jealousy notwithstanding, the due meed of fame and public applause. The co-operation of Mr. Hodgkinson, Mr. Fairbairn, and Mr. Clark, has doubtless been of very great service to Mr. Stephenson, and the Holyhead Railway Company, in working out the details of a design, in proposing which, however, they had no share; but it seems impossible to associate the names of others with that of Stephenson in this work, further than we associate the names of Davies, Gilbert, Rhodes, and Provis, with that of Telford, in connection with the Menai Suspension Bridge. Just as well might "all but the original idea" of the block machinery be claimed by Maudslay, who made it for Brunel. The same of the safety-lamp, by the tinsmith, who made the first for George Stephenson. The same of the hot-blast, by Mr. Wilson or Mr. Condie, who first applied it for Mr. Neilson; and the same of many other cases, in which, from imperative circumstances, the inventor has found himself necessitated to delegate to others the actual execution of his "original idea."

Whilst the experiments on the Tubular Bridges were in progress, letters-patent were granted to Mr. Fairbairn, for "improvements in constructing iron beams;" in which he claims "the novel application and use of plates or sheets of iron, united by means of angle iron and rivets, or by other means, for forming or constructing, by such combination, hollow beams or girders for the erection of bridges or other buildings."

Although the *principle* of Mr. Fairbairn's patent is perfect, the limits of its economical application, is, as far as we can judge, to cases of great span, and where extreme strains are likely to be suddenly brought upon the structure.

The system successfully applied by Wiebecking, for wooden bridges, and in cast iron, by Reichenbach and Gauthey, and, latterly, to a certain extent, by Polonceau, in the beautiful Pont du Carrousel, over the Seine, has recently been revived in malleable iron. On the extension of the London and Blackwall Railway, and elsewhere, Mr. Locke has introduced a hollow sheet iron *arch*, with suspended tie and diagonal bracings, to carry the roadway. We object to this form of bridge, as being a retrogression in the *principle* of construction, and consequently offering no economical advantage, but the contrary.*

[STRENGTH OF CYLINDRICAL STEAM BOILERS, TUBES, AND FIRE-ARMS.

It has been generally supposed that the rolling of *boiler-plate* iron, gives to the sheets a greater tenacity in the direction of the length, than in that of the breadth. Supposing this to be correct, it has frequently been asked how the sheets ought to be disposed in a cylindrical boiler of the common form, in order to oppose the greatest strength to the greatest strain. It has also been asked whether the same arrangement will be required for all diameters, or whether a magnitude will not be eventually attained, which may require the direction of the sheets to be reversed?

To determine these questions in a general manner recourse must be had to mathematical formulas, assuming such symbols for each of the elements as may apply to any given case of which the separate data are determined either by experiment or by the conditions of the case. The *principles* of the calculation require our first notice.

1. To know the force which tends to burst a cylindrical vessel in the longitudinal direction, or, in other words, to separate the *head* from the curved *sides*, we have only to consider the actual area of the head, and to multiply the number of units of *surface* by the number of units of *force* applied to each superficial unit. This will give the total *divellent* force in that direction.

To counteract this, we have, or may be conceived to have, the tenacity of as many longitudinal bars as there are lineal units in the circumference of the cylinder. The united strength of these bars constitutes the total retaining or *quiescent* force; and, at the moment when rupture is about to take place, the *divellent* and the *quiescent* forces must obviously be equal.

2. To ascertain the amount of force which tends to rupture the

* Of all the designs for iron bridges hitherto planned and executed—when we consider the situation, the extent, the elevation above the water, and the scenery by which it is surrounded—the most imposing is that of the wire bridge over Niagara river, a short distance below the Falls. This bridge is still in progress. It was planned and executed by Mr. Charles Ellet, Jr.—Am. Ed.

cylinder along the curved side, or rather along two opposite sides, we may regard the pressure as applied through the whole breadth of the cylinder upon each lineal unit of the diameter. Hence the total amount of force which would tend to divide the cylinder in halves by separating it along two lines, on opposite sides, would be represented by multiplying the diameter by the force exerted on each unit of surface, and this product by the length of the cylinder. But even without regarding the length, we may consider the force requisite to rupture a *single band* in the direction now supposed, and of one lineal unit in breadth; since it obviously makes no difference whether the cylinder be long or short in respect to the ease or difficulty of separating the sides. The *divellent* force in this direction is, therefore, truly represented by the diameter multiplied by the pressure per *unit of surface.* The retaining, or *quiescent* force, in the same direction, is only the strength or tenacity of the two opposite sides of the supposed band. Here, also, at the moment when a rupture is about to occur, the *divellent* must exactly equal the *quiescent* force.

3. In order to estimate the augmentation of *divellent* force consequent upon an increase of diameter, we have only to consider, that, as the diameter is increased, the product *of the diameter, and the force per unit of surface*, is increased in the same ratio. But unless the thickness of the metal be increased, the *quiescent* force must remain unaltered. The *quiescent* forces, therefore, continue the same—the *divellent* increase with the diameter.

4. Again, as the diameter of the cylinder is increased, the area of its end is increased in the ratio of the *square* of the diameter. The *divellent* force is, therefore, augmented in this ratio. But the retaining force does not, as in the other direction, remain the same, since the *circumference* of a circle increases in the same ratio as the diameter. The *quiescent* force will, consequently, be augmented in the simple ratio of the diameter, without any additional thickness of metal. So that, on the whole, the total tendency to rupture in this direction will increase only in the *simple* ratio of the diameter.

5. Since we have seen that the tendency to rupture, in both directions, increases in the simple direct ratio of the increase of diameter, it is obvious that any position of the sheets which is right for one diameter, must be right for all. Hence there can never be a condition, in regard to mere magnitude, which will require the sheets to be reversed.

6. The foregoing considerations being once admitted, we may proceed to ascertain what is the true direction of the greatest tenacity in the sheet, if any difference exist, and what that difference might amount to, consistently with equal safety of the boiler in both directions.

7. Let $x =$ the diameter of the cylinder.

$f =$ the force or pressure per unit of surface (pounds per square inch, for example).

$T =$ the tenacity of metal which, with the diameter x, and the force f, will be required in the lineal unit of the circumference, in order to hold on the head.

Then will the whole *quiescent* force be $3.1416\,x\,T$, while the *divellent* will be $.7854\,x^2 f$; consequently, $.7854\,x^2 f = 3.1416x$ as above stated.

Dividing by $.7854\,x$, we have $xf = 4T$; and we derive immediately—

$$x = \frac{4T}{f},$$

$$f = \frac{4T}{x},$$

$$\text{and } T = \frac{xf}{4}.$$

That is, the tenacity of the *longitudinal bar of the assumed unit in width, will be one-fourth of the product of the diameter into the pressure*, measuring the tenacity by the same standard as the pressure, whether in pounds or kilograms.

8. Now assuming the tenacity required in the *circular band* of the same width to be t, we shall, agreeably to what has already been said, have the *divellant* force expressed by xf, and the *quiescent* by $2t$, so that $xf = 2t$ and $t = \frac{xf}{2}$. Also $f = \frac{2t}{x}$; and $x = \frac{2t}{f}$.

Having thus obtained two expressions for each of the quantities x and f, we may, by comparing them, readily discover the relative values of T and t,—thus:

$$\left.\begin{aligned} x &= \frac{4T}{f} \\ x &= \frac{2t}{f} \end{aligned}\right\} \text{hence } \frac{4T}{f} = \frac{2t}{f} \text{ whence } 4T = 2t, \text{ or } t = 2T. \text{ From which}$$

it follows, that *under a known diameter, and with a given force or pressure, the tenacity of metal in a cylindrical boiler of uniform thickness, ought to be twice as great in the direction of the curve as in that of the length of the cylinder, and that if this could be the case the boiler would still have equal safety in both directions.*

In whatever direction, therefore, the rolling of metal gives the greatest tenacity, in the same direction must the sheet always be bent in forming the convexity of the cylinder. It follows that if we suppose the tenacity precisely equal in both directions, the liability to rupture by a mere internal pressure *ought to be twice as great along the longitudinal direction as at the juncture of the head.* This supposes the strain regular, and the riveting not to weaken the sheet.

9. To know how large we may safely make a cylindrical boiler, having the absolute tenacity of the metal, in the *strongest direction*, and with a known thickness, we have only to revert to the formula $x = \frac{2t}{f}$. That is, *the diameter will be found by dividing twice the*

tenacity by the greatest force per unit of surface, which the boiler is ever to sustain.

10. When knowing the absolute tenacity of a metal, or other material reckoned in weight, to the bar of a given area in its cross section, we would determine the *thickness* of that metal which ought to be employed in a boiler of given diameter, and to sustain a certain force, we may use the formula $t = \frac{xf}{2}$, and dividing the latter number of this equation by the *strength* of the square bar, which we may call s, we obtain the thickness demanded in the direction of the curve, which we may denominate p; so that $p = \frac{xf}{2s}$; this will give the thickness of the boiler plate either in whole numbers or decimals. Thus, suppose the diameter of a cylindrical boiler is to be thirty-six inches—that it is to be formed of iron which will bear 55,000 lbs. to the square inch, and is to sustain 750 lbs. to the square inch—what ought to be the thickness of the metal?

$$\text{Here } x = 36$$
$$f = 750$$
$$2s = 110000, \text{ consequently,}$$

$p = \frac{36 \times 750}{110000} = .2454$, or a little less than one-quarter of an inch.

It must, however, be evident that the *minimum* tenacity of any particular description of metal, is that on which all the calculations ought to be made when there is any probability that the actual pressure will, in practice, ever reach the limit assigned as the value of f in the calculation.

If we had plates of different metals, or of different known degrees of tenacity in the same kind of metal, and were desirous of ascertaining how strong a kind we must employ under a limited *thickness*, *diameter*, and *pressure*, we should decide the point by transforming the formula $p = \frac{xf}{2s}$, into $ps = \frac{xf}{2}$, and then into $s = \frac{xf}{2p}$. In other terms, in order to know the strength of the metal required, or the direct strain which an inch square bar of the same ought to be capable of sustaining, we must *multiply the diameter of the boiler in inches by the pressure per square inch in pounds, and divide the product by twice the intended thickness in parts of an inch.*

Thus, how strong a metal ought to be employed to sustain a pressure of 1000 lbs. to the square inch, in a boiler thirty inches in diameter, and one-fourth of an inch thick?

Here $s = \frac{30 \times 1000}{2 \times .25} = 60.000$. Hence we see that the metal must be capable of sustaining *sixty thousand pounds* to the inch bar, or in that proportion for any other size. This formula enables us to determine whether among the metals of known tenacity, *any* one can be found to fulfil the conditions under the thickness assigned.]

DIVISION II.

APPLICATION OF MECHANICS TO MACHINERY.

INTRODUCTION.

§ 38. *Machines.*—Machines are artificial arrangements, by which forces are applied to produce mechanical effect. *Tools* or *instruments* differ from machines chiefly in their being applied *immediately* to the work to be done, whilst machines are *intermediate.*

In every machine we have to distinguish between the *power* and the *resistance. Power* is the cause of the motion of the machine, and *resistance* is that which opposes the motion, and which it is the object of the machine to overcome. The *powers* applied to machines are modifications of those supplied by nature in the expansive force of heat, the action of gravity, the physical force of men and animals, &c. (Vol. I. § 60). The *resistances* to be overcome are the *transport,* and *the change of form* and *texture of materials.*

There are in every machine three principal parts. One which *receives* the power, a second *transmitting, communicating,* or *modifying* the power, and a third *applying* it. In the common flour mill, considered as a machine, a water *wheel* receives the power of a water *fall;* the spur wheel and pinion, or a *train of gear,* communicates the motion of the water wheel to a pair of stones revolving in a different *plane,* and at quite *different speed* it may be, from that of the water wheel, and these stones grind the corn, or *do the work* desired.

Remark. This sub-division is not always manifest; for there are machines, in which the power is transmitted so directly to the work to be done, that the communicators above mentioned are not apparent. The sub-division is, however, convenient, though it would, perhaps, be equally so to apply to recipients of power, the generic term *engine* or machine; to the communicators of the motion, the general term *mechanism;* and to the parts doing the work, the general term of *operators,* and in this manner to consider each separately, as they are, in fact, perfectly distinct. On this subject, there are excellent observations in Willis' "*Principles of Mechanism,* 1840," and in Ampère's "*Philosophie des Sciences.*"—Tr.

§ 39. *Mechanical Effect.*—The mechanical effect produced by a machine, is measured by the *work* done in a given time, or by the product of the force exerted, and the distance gone through in a unit of time in the direction of that force. If P be the force exerted, and s the distance passed through in a second, then is Ps a true measure of the effect of the machine $L = Ps$ ft. lbs.

It is very usual to assume a somewhat arbitrarily chosen, but now pretty generally adopted measure, termed *horse power*, as the unit of mechanical effect of engines or machines. The *horse power* is in England 33,000 lbs. avoird. raised 1 foot high in a minute. This is the *cheval vapeur* of the French, and which in French measures is 75 *kilogrammes* raised 1 *metre* high in a *second*. It is the *Pferdekraft* of the Germans, or 510 lbs. Prussian, raised 1 foot high in a second.

We have to distinguish the *useful effect*, the lost effect, and the total effect of machines. The useful effect is the work done, the lost effect is that consumed in overcoming the friction of the parts of the machine lost in shocks, &c., and the total effect is the sum of these—the effect inherent in the *power*, or the effect taken out of it. An engine or machine is so much the more perfect, the smaller the lost effect compared with the total effect, or the less loss there arises in adapting and transmitting the power. The ratio of the useful effect, produced to the total effect, has been termed the *efficiency* of the machine. If $L =$ the total effect $L_1 =$ the useful effect, and $L_2 =$ the lost effect, the efficiency $\eta = \frac{L_1}{L} = \frac{L - L_2}{L}$. Thus, the more perfect the machine, the more nearly its efficiency approaches to unity; but as there is always friction, and other resistances and losses, that degree of perfection cannot be attained.

Example. An ore stamping mill consists of 20 *stampers*, each of which weighs 250 lbs., and each is raised 40 times per minute, 1 foot high. The machine driving these is a water wheel, taking on 260 cubic feet per minute, and the *fall* is 20 feet high—required the efficiency of this machine. The useful effect is:

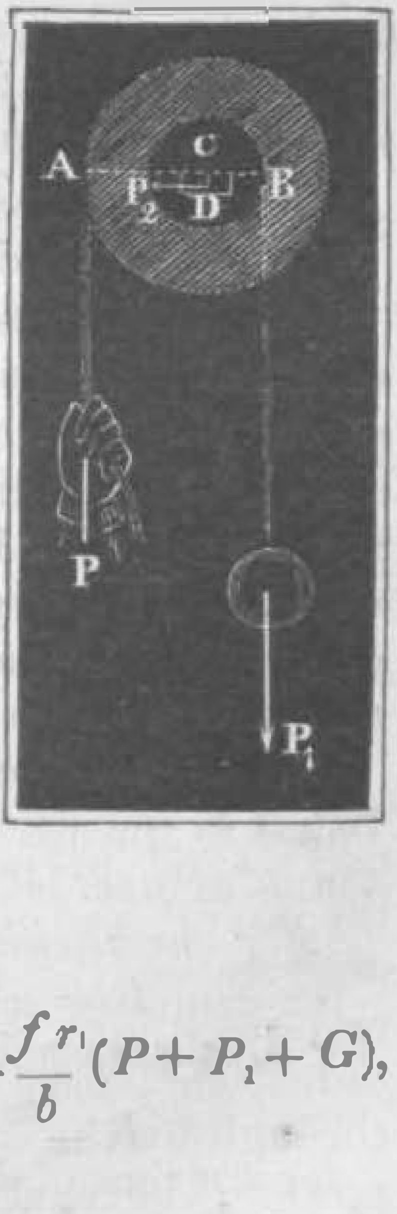
$20 \cdot \frac{250 \cdot 40 \cdot 1}{60} = 3333\frac{1}{3}$ ft. pounds per second $=$ 6 horse power; the total effect, however, is: $\frac{260 \times 62{,}25}{60}$ 270 pounds water through 20 feet per second $=$ 5400 feet lbs. per second $=$ 9,8 horse power; the *lost effect* $= 5400 - 3333\frac{1}{3} = 2066\frac{2}{3}$ feet lbs. $=$ 3,75 horse power; and the efficiency of the whole arrangement $= \frac{3333\frac{1}{3}}{5400} = 0{,}62$.

§ 40. *Useful and prejudicial Resistance.*—The resistance to be overcome by machines may be subdivided, in like manner, into useful and prejudicial resistance, but as the power is applied to the useful and prejudicial resistances at different points, we cannot directly set the power equal to the sum of the useful and prejudicial resistance, but there must be a preliminary *reduction*. This reduction is made by means of the spaces simultaneously passed through by the different points of resistance of the machine. If the power P be exerted for a space s, and the useful resistance P_1 for a space s_1, and the prejudicial resistance P_2 for a space s_2, we have $Ps = P_1 s_1 + P_2 s_2$, hence $P = \frac{s_1}{s} P_1 + \frac{s_2}{s} P_2$.

The point in the machine or system at which P is applied, is termed the point of application of the power, and the points at which P_1 and P_2 act, are the points of application of the resistances; we have in $\frac{s_1}{s} P_1$ the useful resistance reduced to the point of application of power, and in $\frac{s_2}{s} P_2$, the prejudicial resistance *reduced* to the same point. *The power is, therefore, equal to the sum of the useful and prejudicial resistances, reduced to the point of application of the power.* Again $P_1 = \frac{s}{s_1} P - \frac{s_2}{s_1} P_1$, or the useful resistance is equal to the difference of the power reduced to the point of application of that resistance, and the prejudicial resistance reduced to the same point. Hence the efficiency of a machine: $\mu = \frac{P_1 s_1}{P s} = \frac{s_1}{s} P_1 : P = P_1 : \frac{s}{s_1} P$, that is, the quotient of the useful resistance reduced to the *power-point* and the power, or the quotient of the useful resistance, and the power reduced to the point of application of the useful resistance.

Very many machines are adaptations of the wheel and axle (Vol. I. § 152), and hence the *reductions* may often be accomplished as for a lever. If in the wheel and axle ABC, Fig. 90, the radius of the wheel $CA = a$, the drum's radius $CB = b$, then the statical moment of the power $P, = Pa$, and that of the useful resistance $P_1 = P_1 b$, and therefore the useful resistance reduced to the power-point $A = \frac{b}{a} P_1$, and the power reduced to the point of application b of the resistance $= \frac{a}{b} P$. If the prejudicial resistance P_2, consist in the axle friction $f(P + P_1 + G)$, and if $r =$ the radius DC of the axle, the moment of it is $= P_2 r$, and therefore the prejudicial resistance reduced to the application of power $= \frac{P_2 r}{a} = \frac{f r}{a}(P + P_1 + G)$, the prejudicial resistance reduced to the point of application of the resistance $= \frac{P_2 r}{b} = \frac{f r}{b}(P + P_1 + G)$.

Fig. 90.



Hence $P = \frac{b}{a} P_1 = \frac{f r}{a}(P + P_1 + G)$, also $P_1 = \frac{a}{b} P - \frac{f r}{b}(P + P_1 + G)$, lastly, $\eta = \frac{b}{a} P_1 : P = P_1 : \frac{a}{b} P = \frac{P_1 b}{P a}$.

Example. For a *wheel* and *drum* weighing 250 lbs, the wheel being 30 inches radius, and the drum 6 inches radius, the axle ½ inch radius—the useful resistance being 500 lbs., the co-efficient of axle friction $\frac{1}{10}$, then the useful resistance reduced to the point of application of the power $= \frac{b}{a} P_1 = \frac{6}{30} 500 = 100$ lbs, and the prejudicial resistance reduced to the same point:

$$= \frac{fr}{a}(P + P_1 + G) = \frac{1}{10} \cdot \frac{1}{2 \cdot 30}(750 + P) = \frac{5}{4} + \frac{P}{600},$$

and hence we have to put the power:

$$P = 100 + \frac{5}{4} + \frac{P}{600}, \text{ i.e. } P = 101{,}25 . \frac{600}{599} = 101{,}42 \text{ lbs.},$$

and the *efficiency* of the machine: $\eta = \frac{100}{101{,}42} = 0{,}986.$

ON THE RIGIDITY OF CORDAGE.

Amontons, and, after him, Coulomb, experimented on the rigidity of hemp ropes and cords: and Weisbach, adopting Coulomb's method, has recently experimented on the rigidity of hemp and wire rope, such as are used in the drawing-shafts of mines.

Coulomb deduced from his experiments, that the law of this *resistance to winding* may be represented by a formula composed of two terms; the one, *a constant for each* drum or pulley, which we may designate by a, and which the distinguished experimenter termed "*natural rigidity*," because it depends on the mode of manufacture of the rope, and on the degree of twist given to the threads and strands; the other, proportional to the tension T on the rope, and expressed by the product βT, in which β is a constant for any rope or drum. Thus the resistance to winding, $R = a + \beta T$.

Coulomb also deduced from his experiments that the resistance to winding *varies inversely as the diameter* d *of the drum or pulley;* so that $R = a + \frac{\beta T}{d}$.

Naviet, in using Coulomb's experiments to construct a formula, assumed that the co-efficients, a and β, are proportional to a certain power of the diameter, depending on the state of *wear* of the rope; but this assumption is not true. For it would lead to this, that a *worn* rope of 1 foot diameter has the same rigidity as a new one, which is evidently not true: and besides, the comparison of the values of a and β prove that the power to which the diameter has to be raised cannot be the same for the two terms of the resistance.

Coulomb's experiments, however, show that the *rigidity is proportional to the number* n *of threads in the rope*, for ropes of a given manufacture.

For new white ropes, the formula:

$$R = n[.0002 + .000171\,n + .000243\ Q] \text{ lbs.}$$

for drums or pulleys of 1 foot in diameter, and

$$R = \frac{n}{d}[.0002 + .000171\,n + .000243\ Q] \text{ lbs.}$$

for a drum of diameter d in feet, accords well with experiments.*

* For the complete discussion of this subject, see Morin, "Leçons de Mécanique pratique," lère partie.

For tarred ropes: $R = \frac{n}{d}[.001 + .000232\, n + .00028\, Q]$ lbs.

Whence it appears that *tarred ropes are rather more rigid* than white ropes.

Weisbach has deduced from his experiments on wire rope, (4 wires round a core in each strand, and 4 strands round a core in the rope,) weighing 3 lbs. to the fathom, the formula:

$$R = 0{,}72 + 0{,}0262 \frac{Q}{d} \text{ lbs.},$$

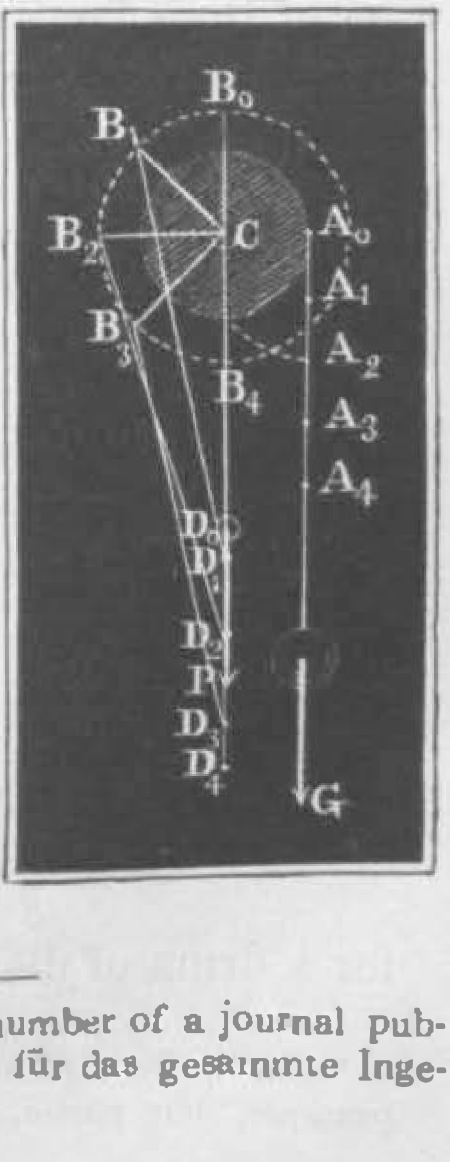
in which Q is the strain on the rope in cwts., and d the diameter of the pulley. Whereas, for the hemp ropes, *fit for the same uses*, or of the same strength, $R = 3{,}02 + 0{,}086 \frac{Q}{d}$: or the rigidity is considerably *greater*.

Wire ropes, newly tarred or greased, have about 40 per cent. less rigidity than untarred ropes.*

§ 41. *Working Condition.*—When a machine is set in motion, it soon comes to its *working condition*, that is, there recur at regular periods the same relative position of the parts, the periodic motion becomes uniformly so. In this condition we assume machines to be in applying our principles, but their working condition may, according to circumstances, be either uniform or variable. The causes inducing irregularity are variations in the power or in the resistance, as also the proportions of, or construction of the machine, in reference to variations in the spaces described in a given time by the power and resistance, and the state of motion of inert masses.

In a steam-engine, the power is variable when the engine "works expansively," that is, when the steam is cut off during the progressive motion of the piston. In a mill for rolling iron, the power and resistance are continually varying, because the forge hammer is *out of gear* when *falling* on the blooms, and, therefore, the working condition of the machines is *irregular*. If the engine work expansively, then there would arise from the combination of the engine, and hammer, and rollers, three causes of irregularity. When a weight G, Fig. 91, is raised by a steam-engine with uniform pressure by means of a wheel CA_0, and crank CB_0, the machine has a variable working condition, because equal spaces A_0A_1, A_1A_2, A_2A_3, A_3A_4, of the resistance correspond to very unequal distances described by the power, and, therefore, the ratio during a half revolution is variable, but for *periods* of a half revolution it is uniform.

Fig. 91.



* Weisbach's paper on this subject is contained in the first number of a journal published at Freyberg, under the title "Der Ingenieur Zeitschrift für das gesammte Ingenieurwesen," 1846.

In the case of uniform working condition, the inert masses on a machine are without influence, because it is only at first, when the machine is still accelerating in motion, that they absorb mechanical effect, but later, when uniform motion has established itself, there is neither loss nor gain of mechanical effect (Vol. I. § 52). But if, on the other hand, a machine be subject to irregular working conditions, the inert masses of the parts have an essential influence on the motion of the machine, because they absorb mechanical effect at every acceleration of speed, and this they again give off at each retardation. If M be the sum of all the masses reduced to the power or resistance-point of a machine, v_1 and v_2, the minimum and maximum velocities of the power, or resistance-points, we have the mechanical effect which the inert masses absorb during their transition from the velocity v_1 to v_2, and which they again give out in passing from v_2 to $v_1 = \left(\frac{v_2^2 - v_1^2}{2}\right) M$. Thus, in each period, the inertia of the masses increases and diminishes the lost effect by the above amount, and, therefore, the total effect for the whole period, or the mean effect is the same as if these inert masses were not there. Hence, as a general formula, $Ps = P_1 s_1 + P_2 s_2$ holds good for a variable working condition, if by s, s_1, s_2, we understand the spaces described in a complete period, or if for P, P_1, P_2, we substitute the mean values of the power, and useful and prejudicial resistance, for a given period. For the case of accelerating motion: $Ps = P_1 s_1 + P_2 s_2 + \left(\frac{v_2^2 - v_1^2}{2}\right) M$, hence $v_2 - v_1 \frac{Ps - (P_1 s_1 + P_2 s_2)}{\left(\frac{v_2 + v_1}{2}\right) M}$. This formula shows that the variations of velocity of a machine are not only less, the less the difference between the effects of the power and the sum of the effects of the resistances, but also the greater the masses of the parts of the machine, and the greater their velocity.

Remark. It does not follow that because the mass of the parts do not affect the efficiency of a machine, but only its working condition, that it is a matter of indifference, whether the parts of a machine have more or less mass. Weight increases friction, gives rise to shocks, &c., which are prejudicial. But of this in the sequel.