

# **On the virtual crack extension method for calculating Rates of Energy Release Rate**

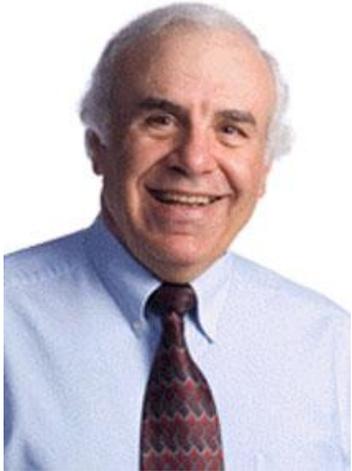
**Presented by Changyu Hwang (PhD '99)**

**Computer Simulation and Physical Testing of Complex Fracturing Processes**

**A Symposium in Honor of Anthony R. Ingraffea  
on the occasion of his retirement from Cornell University**

**September 27, 2014, Ithaca, NY**

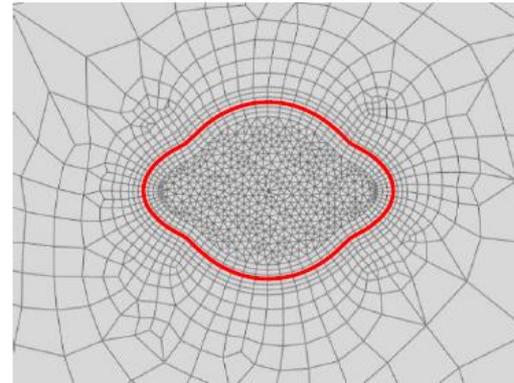
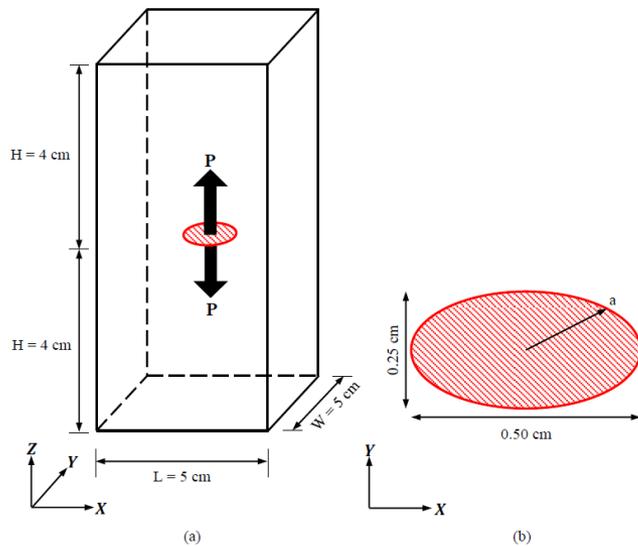
# Important Problems in Fracture Mechanics Analysis



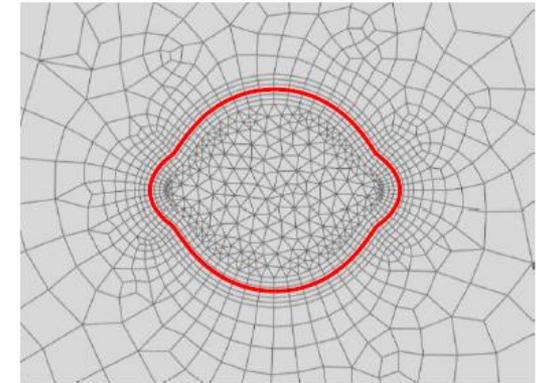
(Prof. Ingraffea)

- **Initiation: Will the crack grow ?**
- **Trajectory: Where does it grow ?**
- **Shape prediction: Into what shape will it grow ?**
- **Stability: How fast will it grow ?**

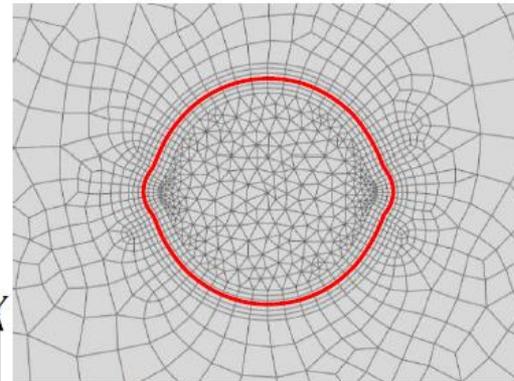
# Mission of 1994: Solve evolution problem of elliptical crack to circular crack



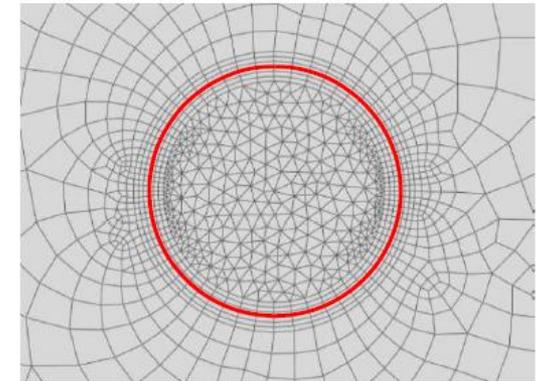
(a)



(b)



(c)



(d)

It needs a Solid Modeling and a Calculator.

# To answer those questions and solve the evolution problem, we need

The energy release rate at crack tip (=crack driving force)

$$G_i = -\frac{\delta \Pi}{\delta a_i} = -\frac{1}{2} u^T \frac{\delta K}{\delta a_i} u + u^T \frac{\delta f}{\delta a_i} \quad \text{where} \quad \Pi = \frac{1}{2} u^T K u - u^T f$$

and its first order derivatives

$$\frac{\delta G_i}{\delta a_j} = -u^T \frac{\delta K}{\delta a_i} \frac{\delta u}{\delta a_j} - \frac{1}{2} u^T \frac{\delta^2 K}{\delta a_i \delta a_j} u + \frac{\delta u^T}{\delta a_j} \frac{\delta f}{\delta a_i} + u^T \frac{\delta^2 f}{\delta a_i \delta a_j}$$

or its higher order derivatives

# Papers co-authored by me with Professor Ingrassia

G and  $dG/da$  for 2D single crack: (1988) by Lin and Abel

Generalized to

- **G and  $dG_i/da_j$  for 2D multiple crack system** (EFM 1998)
- **G and  $dG_i/da_j$  for 3D mode-I planar cracks** (EFM 2001)
- **Shape prediction and stability analysis of 3D planar cracks** (EFM 2004)
- **Derivatives of stress intensity factors for 2D multiple cracks** (EFM 2005)
- **2<sup>nd</sup> order derivatives of energy release rates for 2D multiple cracks** (EFM 2007)
- **Decomposition of 3D mixed-mode energy release rates** (EFM 2014, accepted)



Pergamon

1998

*Engineering Fracture Mechanics* Vol. 59, No. 4, pp. 521–542, 1998

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Printed in Great Britain

0013-7944/98 \$19.00 + 0.00

PII: S00137-944(97)00103-3

## ON THE VIRTUAL CRACK EXTENSION METHOD FOR CALCULATION OF THE RATES OF ENERGY RELEASE RATE

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**Abstract**—This paper generalizes the analytical virtual crack extension method presented by Lin and Abel by providing the higher order derivatives of energy release rate due to crack extension for multiply cracked bodies. It provides derivations and verifications of the following: extension to the general case of multiple crack systems, extension to the axisymmetric case, inclusion of crack-face and thermal loading, and evaluation of the second derivative of energy release rate. The salient feature of this method is that the energy release rate and its higher order derivatives for multiple crack systems are computed in a single analysis. It is shown that the number of rings of elements surrounding the crack tip that are involved in the mesh perturbation due to the virtual crack extension has an effect on the solution accuracy. Maximum errors for the mesh density used in the examples are about 0.2% for energy release rate, 2–3% for its first derivative, and 5–10% for its second derivative. © 1997 Elsevier Science Ltd. All rights reserved

**Keywords**—virtual crack extension method, rates of energy release rate, multiple crack systems, crack stability.



# 2001

PERGAMON

Engineering Fracture Mechanics 68 (2001) 925–947

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## On the virtual crack extension method for calculating the derivatives of energy release rates for a 3D planar crack of arbitrary shape under mode-I loading

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Received 1 December 1999; received in revised form 15 December 2000; accepted 20 December 2000

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### Abstract

This paper generalizes the analytical virtual crack extension method presented by Lin and Abel [Int. J. Fract. 38 (1988) 217] by providing the energy release rates and their derivatives at all points along a three-dimensional (3D), planar crack front of arbitrary shape. It is shown that the local variation of curvature along the crack front and interaction between crack front perturbations at adjacent crack points must be considered to properly calculate the derivatives of the energy release rates. The main advantage of the method is that the energy release rates and their derivatives at all points along the crack front in a multiply cracked, 3D body can be accurately calculated by the present virtual crack extension method in a single analysis. Comparisons of the energy release rates and their derivatives with exact solutions show that the present method can achieve sufficient accuracy for calculation of the energy release rates and their derivatives. All the advantages and accuracy of the two-dimensional virtual crack extension method presented by Hwang, [Engng. Fract. Mech. 59 (4) (1998) 521] are maintained for the 3D case. The present method has immediate application to the following and related problems: the shape prediction and stability analysis of an evolving 3D crack front in brittle fracture, configurational stability in fatigue crack propagation prediction, investigation of bifurcation in brittle fracture. © 2001 Elsevier Science Ltd. All rights reserved.

*Keywords:* Virtual crack extension method; Derivatives of energy release rate; 3D planar crack of arbitrary shape; Crack stability

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Engineering Fracture Mechanics 71 (2004) 1751–1777

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# Shape prediction and stability analysis of Mode-I planar cracks

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Received 22 May 2002; received in revised form 8 June 2003; accepted 20 June 2003

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## Abstract

This paper presents a numerical technique for simulating stable growth of Mode-I cracks in two and three dimensions, using energy release rate and its derivatives. The crack growth model used in the numerical simulation is based on the concept of maximizing potential energy of the system released as cracks evolve. Therefore, a series of quadratic programming (QP) problems with linear constraints and bounds are solved to simulate stable growth of Mode-I planar cracks. The derivative of energy release rate provides a stability condition for crack growth in structures and can be regarded as a discretized influence function that represents the strength of the interaction among crack extensions at different crack tips in 2-D and different locations along a crack front in 3-D. The energy release rate and its derivative are accurately calculated by the analytical virtual crack extension method [Engng. Fract. Mech. 59 (1998) 521; 68 (2001) 925] in a single analysis. Numerical examples are presented to demonstrate the capabilities of the proposed approach. Examples include a central crack subjected to wedge forces in a 2-D finite plate, a system of interacting thermally induced parallel cracks in a two-dimensional semi-infinite plane and a 3-D penny-shaped crack embedded in a large cylinder, pressurized in a central circular region.

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*Keywords:* Virtual crack extension method; Crack stability; Derivatives of energy release rates; Shape prediction; Growth pattern analysis

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Engineering Fracture Mechanics 72 (2005) 1171–1196

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## On the calculation of derivatives of stress intensity factors for multiple cracks

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Received 5 December 2003; received in revised form 1 August 2004; accepted 10 August 2004

Available online 13 December 2004

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### Abstract

In this paper, the work of Lin and Abel [Lin SC, Abel JF. Variational approach for a new direct-integration form of the virtual crack extension method. *Int J Fract* 1988;38:217–35] is further extended to the general case of multiple crack systems under mixed-mode loading. Analytical expressions are presented for stress intensity factors and their derivatives for a multiply cracked body using the mode decomposition technique. The salient feature of this method is that the stress intensity factors and their derivatives for the multiple crack system are computed in a single analysis. It is shown through two-dimensional numerical examples that the proposed method gives very accurate results for the stress intensity factors and their derivatives. It is also shown that the variation of mode I and II displacements at one crack-tip influence the mode I and II stress intensity factors at any other crack. The computed errors were about 0.4–3% for stress intensity factors, and 2–4% for their first order derivatives for the mesh density used in the examples.

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*Keywords:* Virtual crack extension method; Derivative of stress intensity factor; Mode decomposition

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## Virtual crack extension method for calculating the second order derivatives of energy release rates for multiply cracked systems

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Received 29 November 2005; received in revised form 23 July 2006; accepted 17 August 2006

Available online 20 October 2006

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### Abstract

In this paper, we further generalize the work of Lin and Abel [Lin SC, Abel JF. Variational approach for a new direct-integration form of the virtual crack extension method. *Int J Fract* 1988;38:217–35.] to the case of higher order derivatives of energy release rates for two-dimensional, multiply cracked systems. The direct integral expressions are presented for the energy release rates and their first and second order derivatives. The salient feature of this numerical method is that the energy release rates and their first and second order derivatives can be computed in a single analysis. It is demonstrated through a set of examples that the proposed method gives expectedly decreasing, but acceptably accurate results for the energy release rates and their first and second order derivatives. The computed errors were approximately 0.5% for the energy release rates, 3–5% for their first order derivatives and 10–20% for their second order derivatives for the mesh densities used in the examples. Potential applications of the present method include a universal size effect model and a probabilistic fracture analysis of cracked structures.

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2014

**Title:**        **Decomposition of 3-D Mixed-Mode Energy Release Rates  
Using the Virtual Crack Extension Method**

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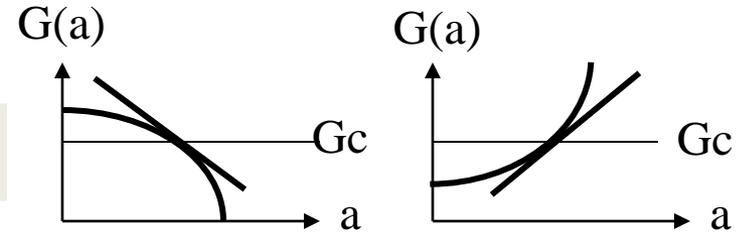
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**Abstract**

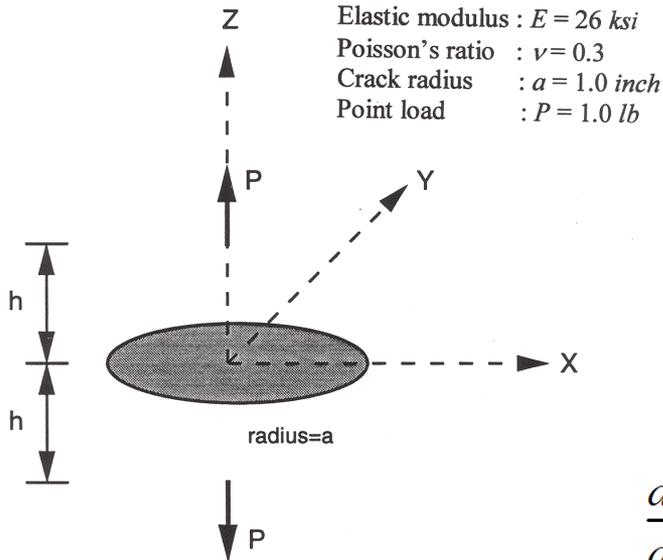
A technique was implemented for decomposing 3-D mixed-mode energy release rates using the Virtual Crack Extension (VCE) method. The technique uses a symmetric/anti-symmetric approach to decompose local crack-front displacements that are substituted into the global VCE energy release rate form. The subsequent expansion leads to the mixed-mode energy release rate expressions. As a result of the expansion, previously unaddressed modal-interaction coupling terms are found to impact the mixed-mode energy release rates. This development expands the VCE method's advantages over similar procedures when simulating arbitrary crack growth by providing the means to calculate both mixed-mode energy release rates and their variations.

# Some fracture mechanics problems requiring $G$ & $\delta G/\delta a$

## (1) Stability and Arrest of A Single Crack



$G(a)=G_c$  Griffith's fracture criterion



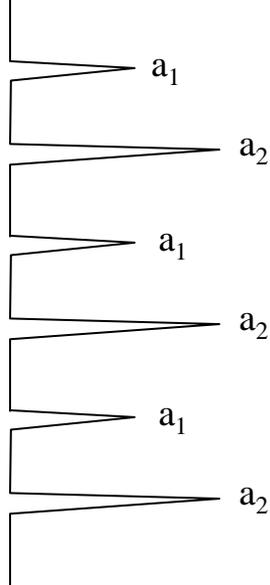
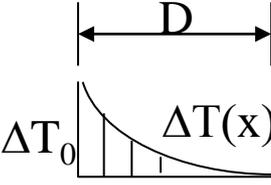
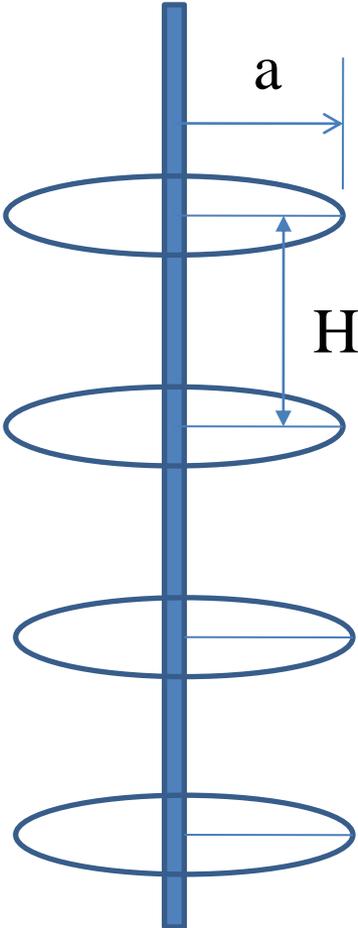
$$G = \frac{(1 - \nu^2) P^2 \alpha (\kappa + \alpha^2)}{E (\pi h)^3 (1 + \alpha^2)^4} \quad \text{where, } \alpha = \frac{a}{h} \text{ and } \kappa = \frac{2 - \nu}{1 - \nu}$$

$$\frac{dG}{d\alpha} < 0 \quad (\alpha > \alpha_m), \quad \frac{dG}{d\alpha} > 0 \quad (0 < \alpha < \alpha_m), \quad \frac{dG}{d\alpha} = 0 \quad (\alpha = \alpha_m)$$

$$\text{with } \alpha_m^2 = \frac{\sqrt{16\nu^2 - 72\nu + 105} - 2\nu + 9}{2(2 - \nu)}$$

# (2) Growth Pattern Analysis, Stability, Bifurcation of Multiple Crack Systems

Hydraulic fracturing  
"Fracking"



This critical state of crack propagation bifurcation corresponds to vanishing diagonal terms in the matrix

$$\begin{bmatrix} \frac{\partial(K_I)_1}{\partial a_1} & \frac{\partial(K_I)_1}{\partial a_2} \\ \frac{\partial(K_I)_2}{\partial a_1} & \frac{\partial(K_I)_2}{\partial a_2} \end{bmatrix}$$

$\uparrow$   
 0

Thermally induced parallel edge cracks having a periodic pattern in a semi-infinite plane

### (3) Probabilistic fracture mechanics

$$P_F \stackrel{\text{def}}{=} P_r [g(X) < 0] \stackrel{\text{def}}{=} \int_{g(x) < 0} f_x(x) dx$$

If the propagation of a mixed crack constitutes a failure condition and the maximum circumferential stress theory is used, the performance function is

$$g(x) = K_{Ic} - \left( K_I \cos^2 \frac{\theta}{2} - \frac{3}{2} K_{II} \sin \theta \right) \cos \frac{\theta}{2}$$

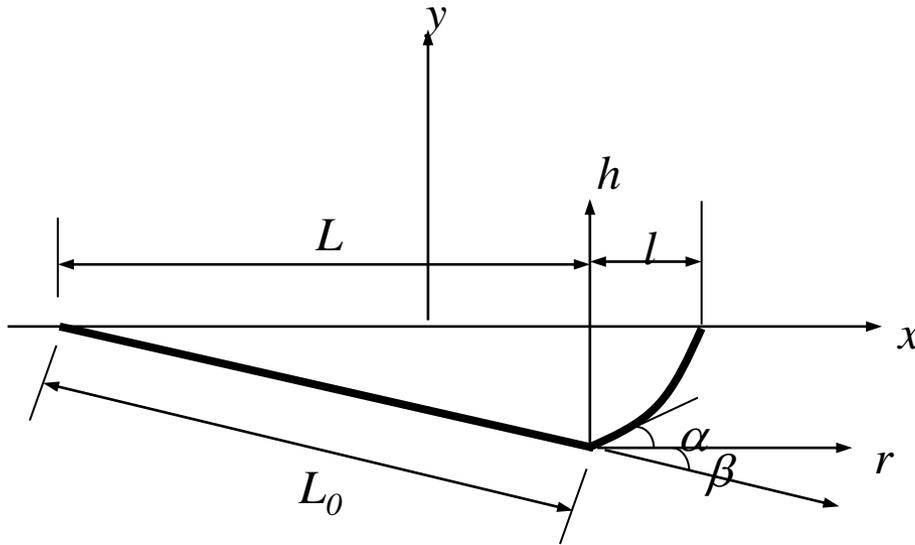
To compute the failure probability, use the first order reliability method (FORM).

It leads to nonlinear constrained optimization, requiring the derivative of the performance function with respect to crack size ( $a$ ) as

$$\frac{\partial g}{\partial a} = - \left[ K_I \left( -\frac{3}{2} \cos^2 \frac{\theta}{2} \sin \frac{\theta}{2} \frac{\partial \theta}{\partial a} \right) + \cos^3 \frac{\theta}{2} \frac{\partial K_I}{\partial a} \right] + \frac{3}{2} \left[ K_{II} \left( -\frac{1}{2} \sin \theta \sin \frac{\theta}{2} + \cos \theta \cos \frac{\theta}{2} \right) \frac{\partial \theta}{\partial a} + \sin \theta \cos \frac{\theta}{2} \frac{\partial K_{II}}{\partial a} \right],$$

$$\frac{\partial \theta}{\partial a} = \frac{1}{K_{II}^2} \left[ K_{II} \frac{\partial K_I}{\partial a} - K_I \frac{\partial K_{II}}{\partial a} \right] \frac{\sin^2 \theta}{3 - \cos \theta}$$

## (4) Slightly out-of-plane growth of the straight cracks



Karihaloo and his co-workers derived the condition for the deviation of the crack from straightness and showed that the curvature of the crack path depends not only on the in-plane stress, but also upon the derivatives of individual stress intensity factors with respect to the length of the main crack

Geometry of the straight crack and its kinked-curved extension

$$\lambda(r) = \begin{cases} h(r) - h(l), & 0 < r \leq l \\ -\beta(L + r), & -L \leq r \leq 0 \end{cases}$$

$$h(r) = (\alpha - \beta)r + \eta r^{3/2} + \chi r^2$$

$$\beta = h(l) / L$$

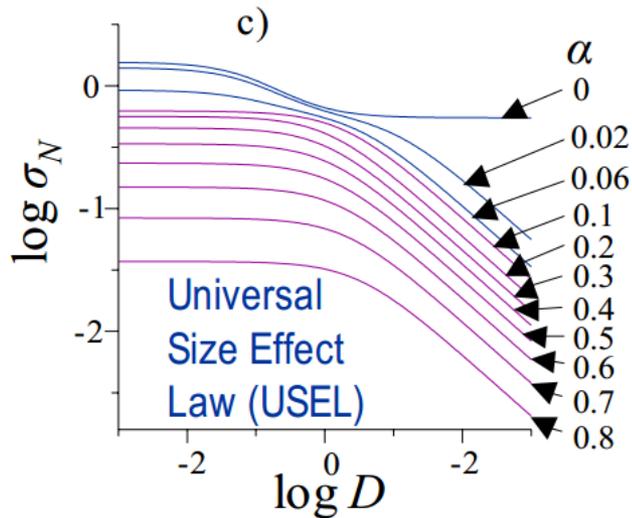
$$\alpha \approx -2K_{II} / K_I$$

$$\eta \approx \frac{8}{3} \left( \frac{2}{\pi} \right)^{1/2} \alpha \frac{T}{K_I}$$

$$\chi \approx \alpha \left[ 4 \frac{T^2}{K_I^2} + \frac{1}{2K_I} \frac{\partial K_I}{\partial L_0} + \frac{1}{2K_{II}} \frac{\partial K_{II}}{\partial L_0} \right]$$

## (5) Size Effect Model

$$\sigma_N = \left( \frac{E' G_f}{g_0' c_f + g_0 D} \right)^{1/2} \left( 1 - \frac{rc_f^2 g_0'' e^{-k\alpha^2}}{4(l_0 + D)(g_0 D + g_0' c_f)} \right)^{1/r}$$



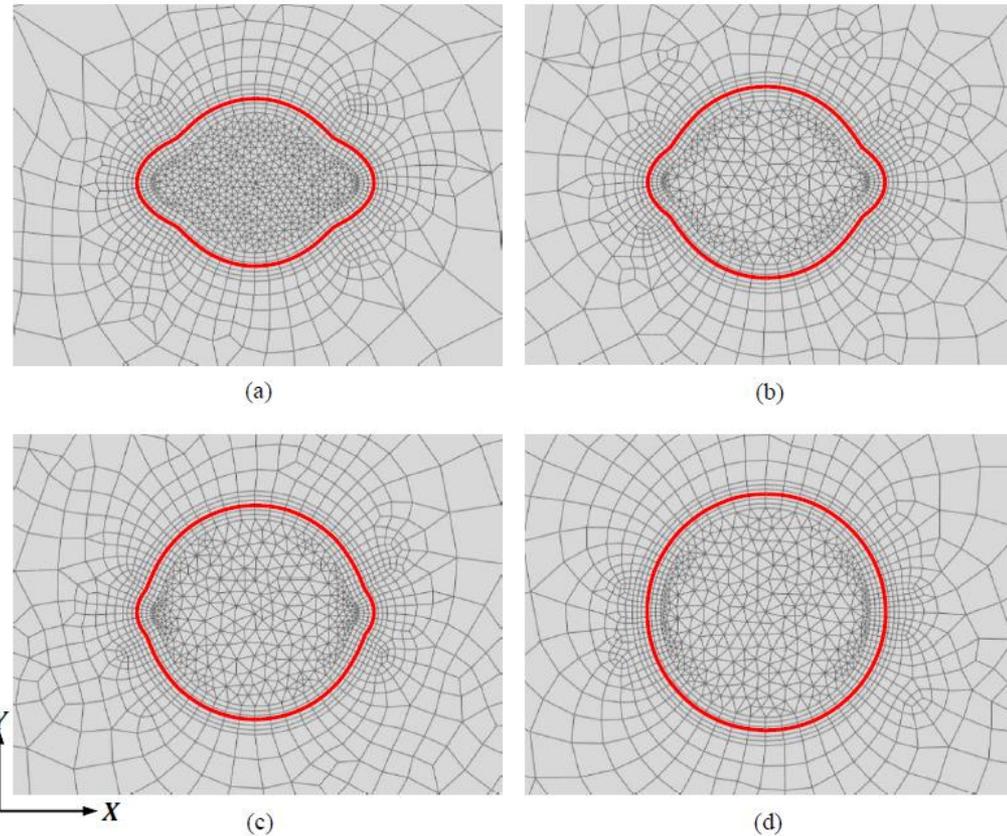
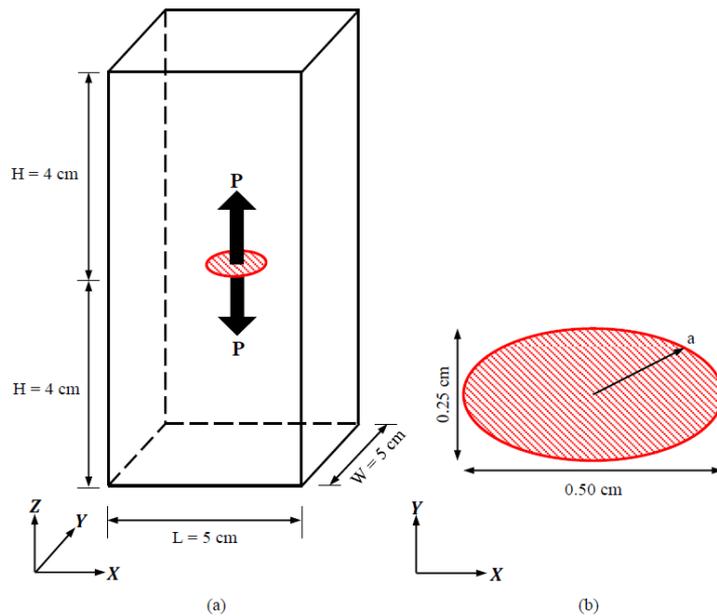
D : Structure size

**Bažant, Z.P., and Yu, Q. (2004).** "Size effect in concrete specimens and structures: New problems and progress." *Fracture Mechanics of Concrete Structures (Proc., FraMCoS-5, 5th Int. Conf. on Fracture Mech. of Concrete and Concr. Structures, Vail, Colo.)*, Vol. 1, V.C. Li, K.Y. Leung, Willam, K.J., and Billington, S.L., eds., IA-FraMCoS, 153--162.

# (6) Augmented Energy Based Growth Formulation : Planar cracks

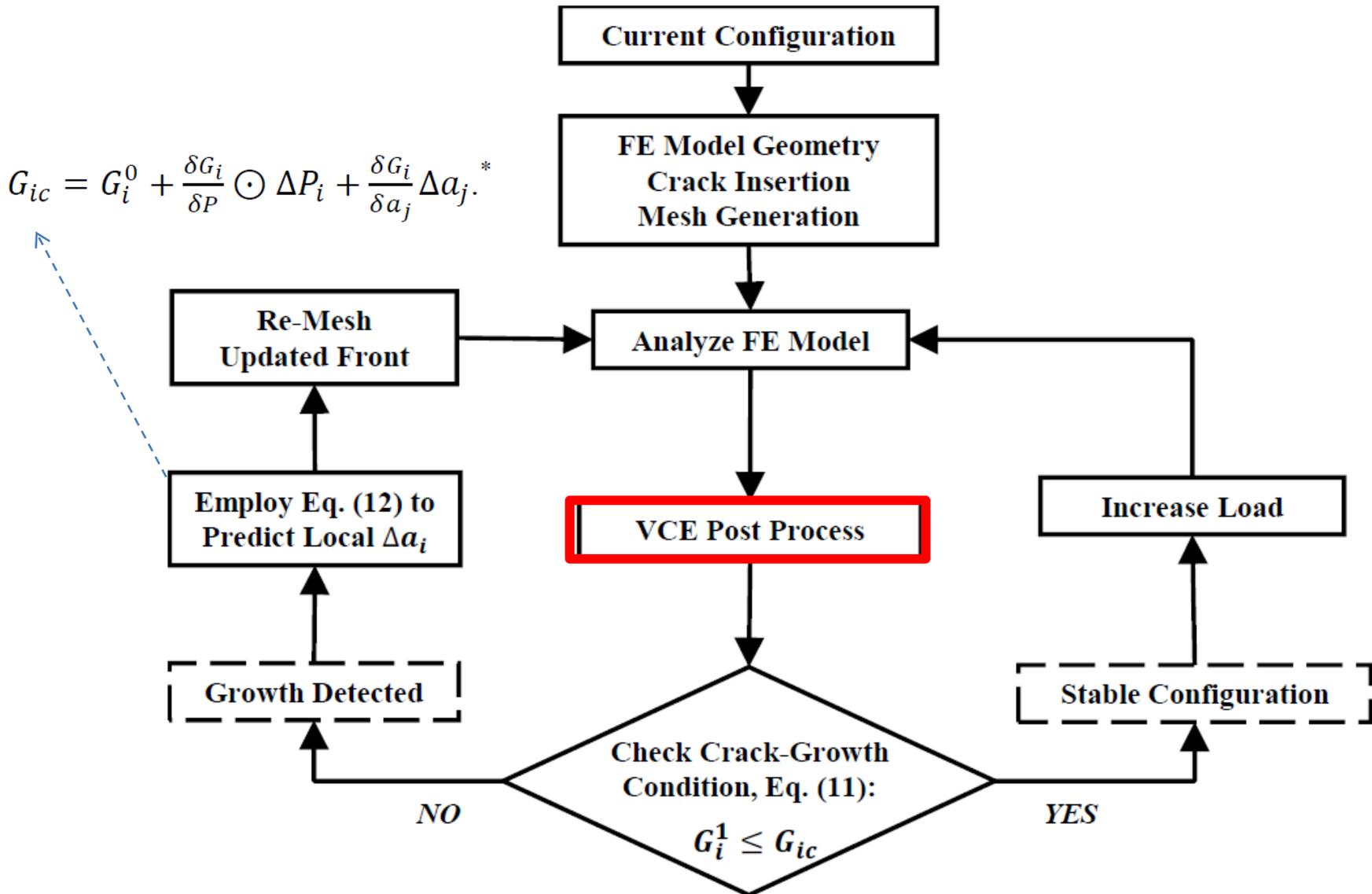
$$G_{ic} = G_i^0 + \frac{\delta G_i}{\delta P} \odot \Delta P_i + \frac{\delta G_i}{\delta a_j} \Delta a_j. *$$

(Brett Davis Dissertation, 2014 Cornell)



# Iterative Crack Growth Simulation Algorithm (Brett Davis Thesis, Cornell 2014)

“Cornell Engine for Crack Growth Simulation”



# Virtual Crack Extension Method

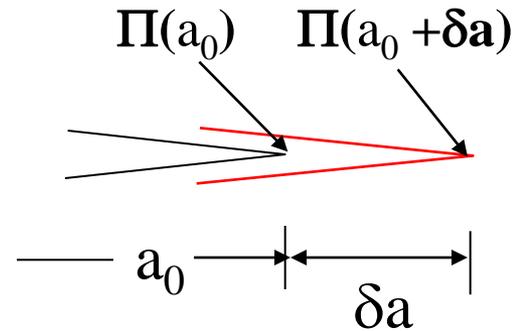
**Watwood [1968]**

$$G = -\frac{\delta\Pi}{\delta a} = -\frac{[\Pi(a_0) - \Pi(a_0 + \delta a)]}{\delta a}$$

: 2 Complete F.E. Analyses, Finite  $\delta a$

$$\delta G = -\frac{\delta^2\Pi}{\delta a^2} = \frac{[\Pi(a_0 + \delta a) - 2\Pi(a_0) + \Pi(a_0 - \delta a)]}{\delta a^2}$$

: 3 Complete F.E. Analyses, Finite  $\delta a$

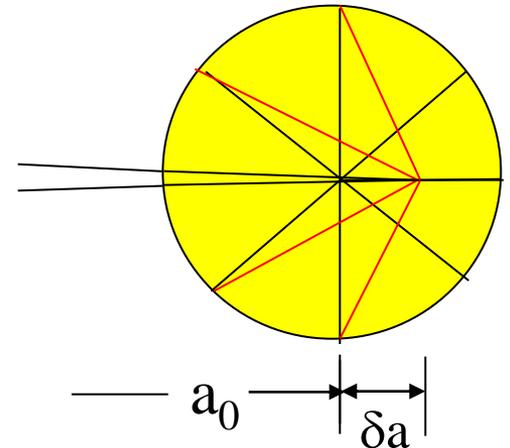


**Parks [1975]**

$$G_i = -\frac{\delta\Pi}{\delta a_i} = -\frac{1}{2} u^T \frac{\delta K}{\delta a_i} u + u^T \frac{\delta f}{\delta a_i}$$

:  $[\delta\mathbf{K}] = [\mathbf{K}(a+\delta a)] - [\mathbf{K}(a)]$ , Finite  $\delta a$

: 1 Complete F.E. Analysis +  $[\delta\mathbf{K}]$

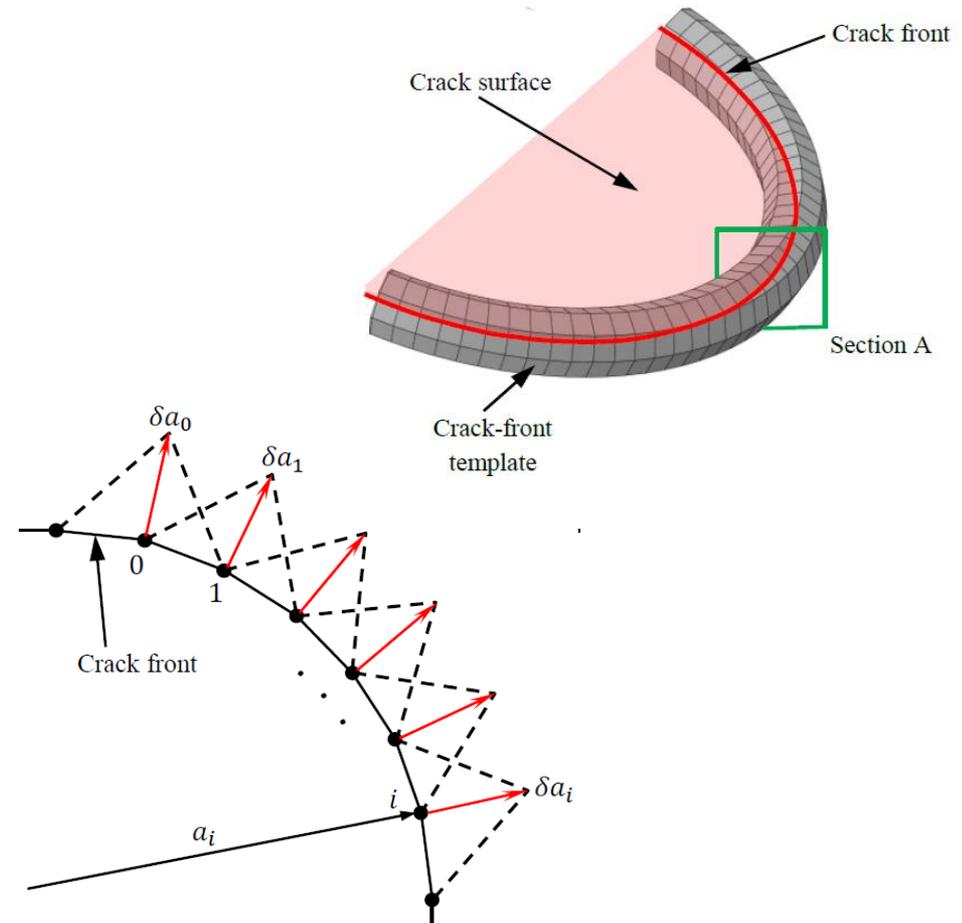
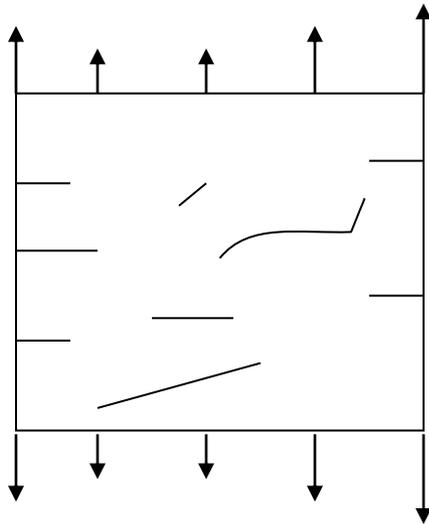


# Problems in computation of 2D multiple cracks or 3D crack

-Solution inaccuracy due to **finite difference approximation**

-**Multiple FE analyses** required for computing  $G$  (even more for  $dG/da$ )

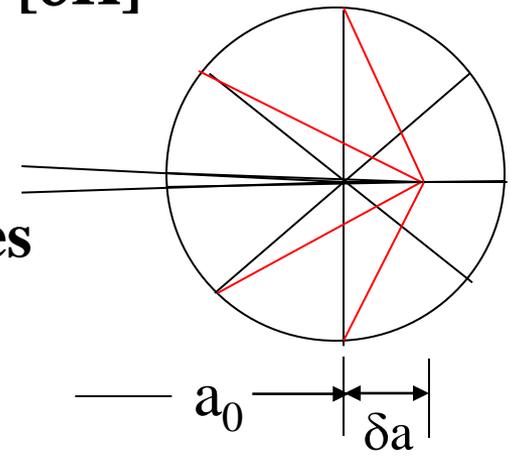
Ex)  $N$  cracks :  $N+1$  FE analyses for  $G$ , Many ( $>2N+1$ ) FE analyses for  $dG_i/da_j$



**Lin and Abel [1988]**

**Direct Integral Forms for  $[\delta K]$   
No need to specify the  
Finite  $\delta a$**

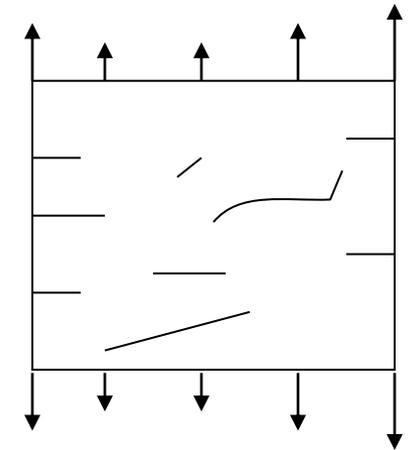
**G and its higher derivatives  
for a 2D Single Crack**



**Generalize** ↓

**Hwang & Ingraffea,  
Co-workers  
[1998,2001,2004,2005, 2007,2014]**

**G and  $\delta G/\delta a$  for multiple crack systems  
in 2D/3D in a single analysis**



# Present Virtual Crack Extension Method

Provides  $G_i$  and  $\delta G_i / \delta a_i$  for Multiple Crack Systems

Subjected to Arbitrary Thermal loading, Crack-face loading, Body forces in 2D and 3D Problems.

$$G_i = -\frac{\delta \Pi}{\delta a_i} = -\frac{1}{2} u^T \frac{\delta K}{\delta a_i} u + u^T \frac{\delta f}{\delta a_i}$$

Non-Zero for these loadings

$$\frac{\delta G_i}{\delta a_j} = -u^T \frac{\delta K}{\delta a_i} \frac{\delta u}{\delta a_j} - \frac{1}{2} u^T \frac{\delta^2 K}{\delta a_i \delta a_j} u + \frac{\delta u^T}{\delta a_j} \frac{\delta f}{\delta a_i} + u^T \frac{\delta^2 f}{\delta a_i \delta a_j}$$

Null for 2D

Non-Zero for 3D

when  $i \neq j$

# Mesh perturbation due to virtual crack extension

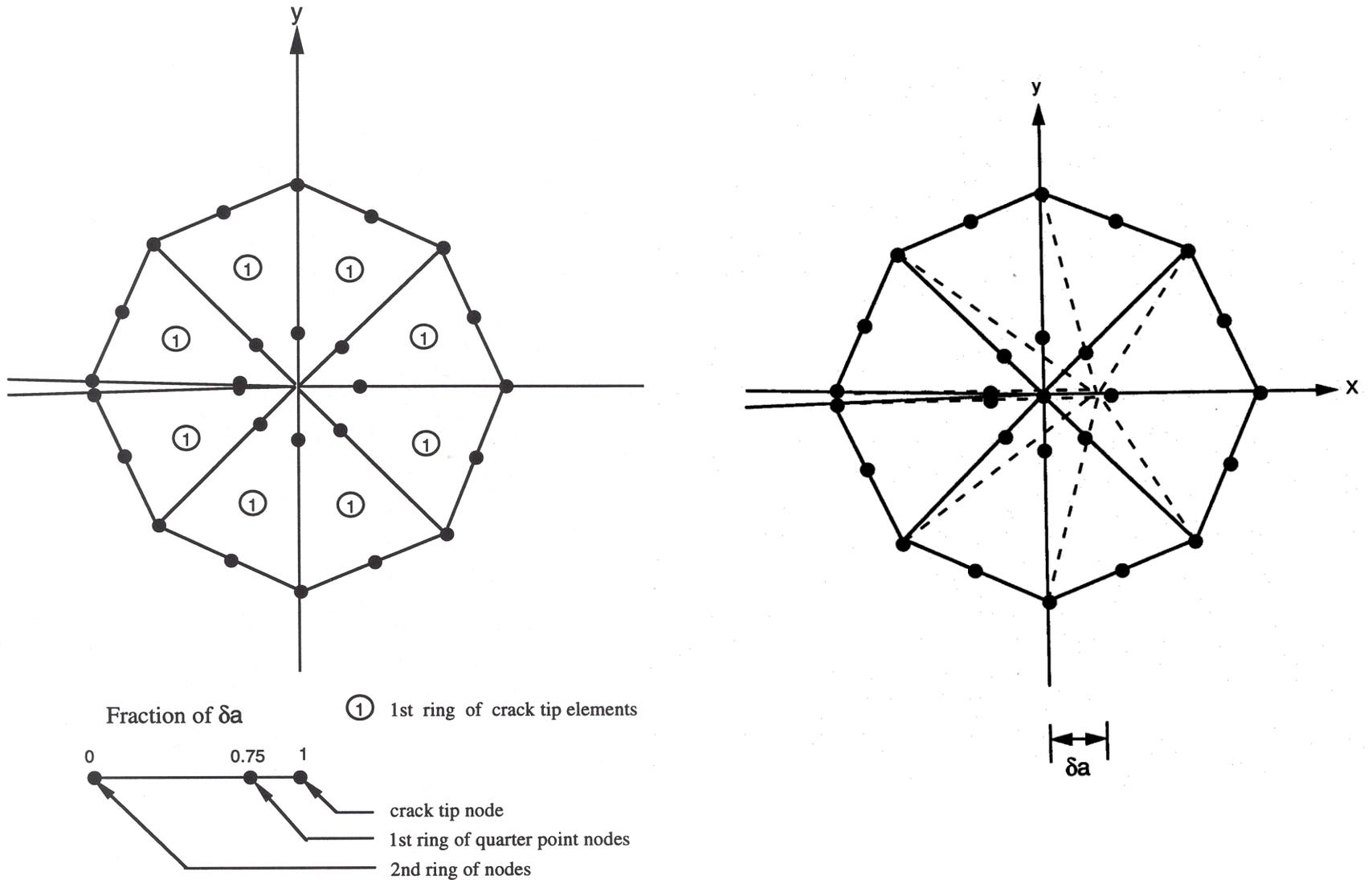
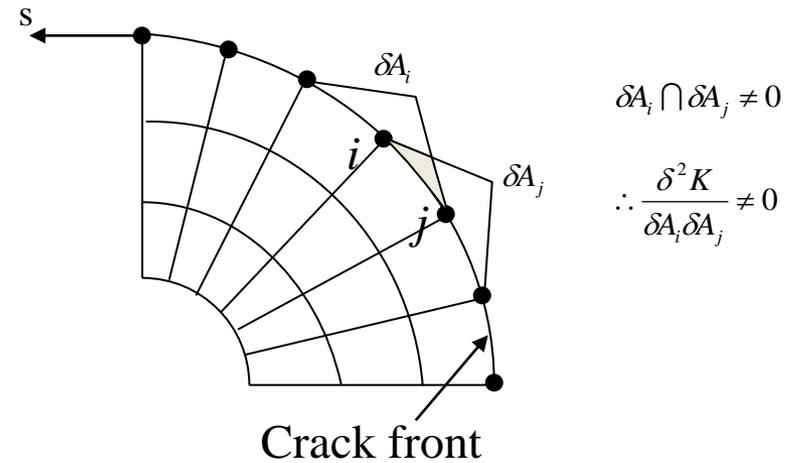
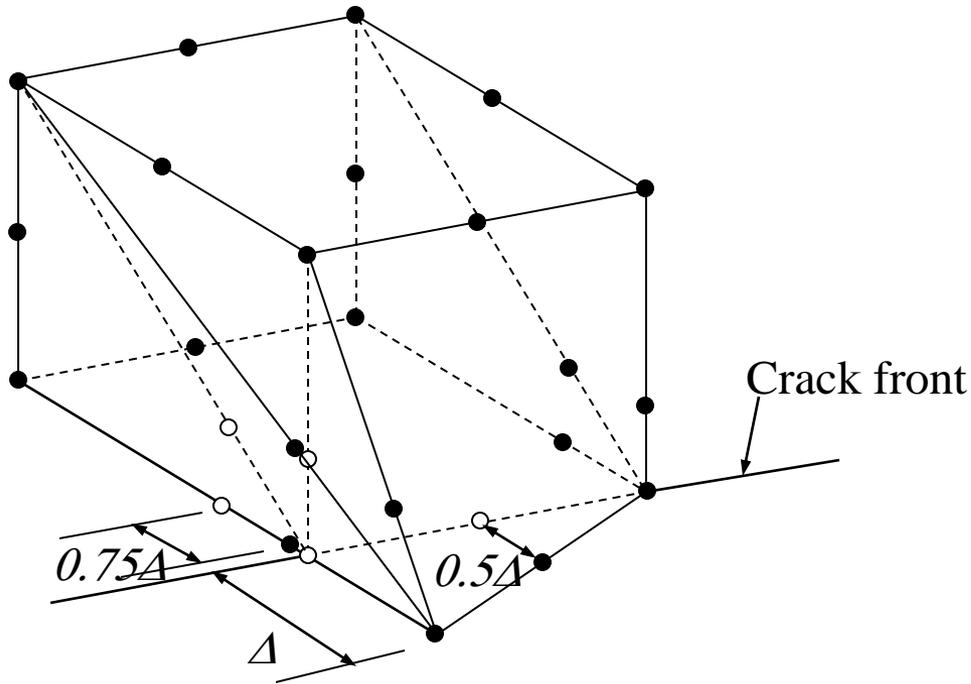


Figure 2.1a. Standard arrangement of rosette of single ring of quarter point crack tip elements

# Virtual crack extensions



. Interaction between crack front perturbations.

Mesh perturbation  $\Delta$  in 15-noded wedge elements

O : nodes before mesh perturbation

The second variation, too.

$$\begin{aligned}
 \frac{\delta^2 G_i}{\delta a_j \delta a_k} &= -\frac{1}{2} u^T \frac{\delta^3 K}{\delta a_i \delta a_j \delta a_k} u - u^T \frac{\delta^2 K}{\delta a_i \delta a_j} \frac{\delta u}{\delta a_k} \\
 &- u^T \frac{\delta^2 K}{\delta a_i \delta a_k} \frac{\delta u}{\delta a_j} - u^T \frac{\delta K}{\delta a_i} \frac{\delta^2 u}{\delta a_j \delta a_k} - \frac{\delta u^T}{\delta a_k} \frac{\delta K}{\delta a_i} \frac{\delta u}{\delta a_j} \\
 &+ u^T \frac{\delta^3 f}{\delta a_i \delta a_j \delta a_k} + \frac{\delta u^T}{\delta a_j} \frac{\delta^2 f}{\delta a_i \delta a_k} + \frac{\delta u^T}{\delta a_k} \frac{\delta^2 f}{\delta a_i \delta a_j} + \frac{\delta^2 u^T}{\delta a_j \delta a_k} \frac{\delta f}{\delta a_i}
 \end{aligned}$$

In case of  $i \neq j \neq k$ , it is reduced to

$$\frac{\delta^2 G_i}{\delta a_j \delta a_k} = -u^T \frac{\delta K}{\delta a_i} \frac{\delta^2 u}{\delta a_j \delta a_k} - \frac{\delta u^T}{\delta a_k} \frac{\delta K}{\delta a_i} \frac{\delta u}{\delta a_j} + \frac{\delta^2 u^T}{\delta a_j \delta a_k} \frac{\delta f}{\delta a_i}$$

## Stiffness Variations

$$\delta k_e = \int_v \left[ \delta B^T D B + B^T D \delta B + \text{Tr}(\tilde{\varepsilon}) B^T D B \right] dV$$

$$\begin{aligned} \delta^2 k_e = & \int_v \left[ \delta^2 B^T D B + 2 \delta B^T D \delta B + B^T D \delta^2 B \right] dV \\ & + \int_v \left[ 2 |\tilde{\varepsilon}| B^T D B + 2 \text{Tr}(\tilde{\varepsilon}) \left( \delta B^T D B + B^T D \delta B \right) \right] dV \end{aligned}$$

$$\begin{aligned} \delta^3 k = & \int_v \left[ \delta^3 B^T D B + 3 \delta^2 B^T D \delta B + 3 \delta B^T D \delta^2 B + B^T D \delta^3 B \right] dV \\ & + 3 \int_v \left[ \delta^2 B^T D B + 2 \delta B^T D \delta B + B^T D \delta^2 B \right] \text{Tr}(\tilde{\varepsilon}) dV \\ & + \int_v \left[ 2 |\tilde{\varepsilon}| B^T D B + 6 |\tilde{\varepsilon}| \left( \delta B^T D B + B^T D \delta B \right) + 2 |\tilde{\varepsilon}| \text{Tr}(\tilde{\varepsilon}) B^T D B \right] dV \end{aligned}$$

# Variation of B matrix

$$B = \begin{bmatrix} \frac{\partial \mathcal{N}}{\partial x^1} & & \\ & \frac{\partial \mathcal{N}}{\partial x^2} & \\ & & \frac{\partial \mathcal{N}}{\partial x^3} \\ \frac{\partial \mathcal{N}}{\partial x^2} & \frac{\partial \mathcal{N}}{\partial x^1} & \\ \frac{\partial \mathcal{N}}{\partial x^3} & \frac{\partial \mathcal{N}}{\partial x^3} & \frac{\partial \mathcal{N}}{\partial x^2} \\ \frac{\partial \mathcal{N}}{\partial x^3} & \frac{\partial \mathcal{N}}{\partial x^1} & \frac{\partial \mathcal{N}}{\partial x^1} \end{bmatrix}$$

$$\frac{\delta B}{\delta a_i} = - \begin{bmatrix} \left[ \tilde{\varepsilon}_1 \right] & & \\ \left[ \tilde{\varepsilon}_2 \right] & \left[ \tilde{\varepsilon}_2 \right] & \\ \left[ \tilde{\varepsilon}_3 \right] & \left[ \tilde{\varepsilon}_3 \right] & \left[ \tilde{\varepsilon}_3 \right] \\ & \left[ \tilde{\varepsilon}_2 \right] & \left[ \tilde{\varepsilon}_1 \right] \\ & \left[ \tilde{\varepsilon}_3 \right] & \left[ \tilde{\varepsilon}_1 \right] \end{bmatrix} \begin{bmatrix} \frac{\partial \mathcal{N}}{\partial x^1} \\ \frac{\partial \mathcal{N}}{\partial x^2} \\ \frac{\partial \mathcal{N}}{\partial x^3} \\ \frac{\partial \mathcal{N}}{\partial x^1} \\ \frac{\partial \mathcal{N}}{\partial x^2} \\ \frac{\partial \mathcal{N}}{\partial x^3} \end{bmatrix}$$

$$\frac{\delta^2 B}{\delta a_i \delta a_j} = - \begin{bmatrix} \left[ \tilde{\varepsilon}_1'' \right] & & \\ & \left[ \tilde{\varepsilon}_2'' \right] & \\ \left[ \tilde{\varepsilon}_2'' \right] & \left[ \tilde{\varepsilon}_1'' \right] & \left[ \tilde{\varepsilon}_3'' \right] \\ & \left[ \tilde{\varepsilon}_3'' \right] & \left[ \tilde{\varepsilon}_2'' \right] \\ \left[ \tilde{\varepsilon}_3'' \right] & & \left[ \tilde{\varepsilon}_1'' \right] \end{bmatrix} \begin{bmatrix} \frac{\partial \mathcal{N}}{\partial x^1} \\ \frac{\partial \mathcal{N}}{\partial x^2} \\ \frac{\partial \mathcal{N}}{\partial x^3} \\ \frac{\partial \mathcal{N}}{\partial x^1} \\ \frac{\partial \mathcal{N}}{\partial x^2} \\ \frac{\partial \mathcal{N}}{\partial x^3} \end{bmatrix}$$

$$\tilde{\varepsilon} = J^{-1} \begin{Bmatrix} \frac{\partial \mathcal{N}}{\partial \xi^1} \\ \frac{\partial \mathcal{N}}{\partial \xi^2} \\ \frac{\partial \mathcal{N}}{\partial \xi^3} \end{Bmatrix} \begin{bmatrix} \Delta_n^1 & \Delta_n^2 & \Delta_n^3 \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{bmatrix}^{-1} \begin{Bmatrix} \frac{\partial \mathcal{N}}{\partial \xi^1} \\ \frac{\partial \mathcal{N}}{\partial \xi^2} \\ \frac{\partial \mathcal{N}}{\partial \xi^3} \end{Bmatrix} \begin{bmatrix} \Delta_n^1 & \Delta_n^2 & \Delta_n^3 \end{bmatrix}$$

$$\tilde{\varepsilon}'' = \begin{Bmatrix} \left[ \tilde{\varepsilon}_1'' \right] \\ \left[ \tilde{\varepsilon}_2'' \right] \\ \left[ \tilde{\varepsilon}_3'' \right] \end{Bmatrix} = \left[ \varepsilon_j \right] \left[ \varepsilon_i \right] + \left[ \varepsilon_i \right] \left[ \varepsilon_j \right]$$

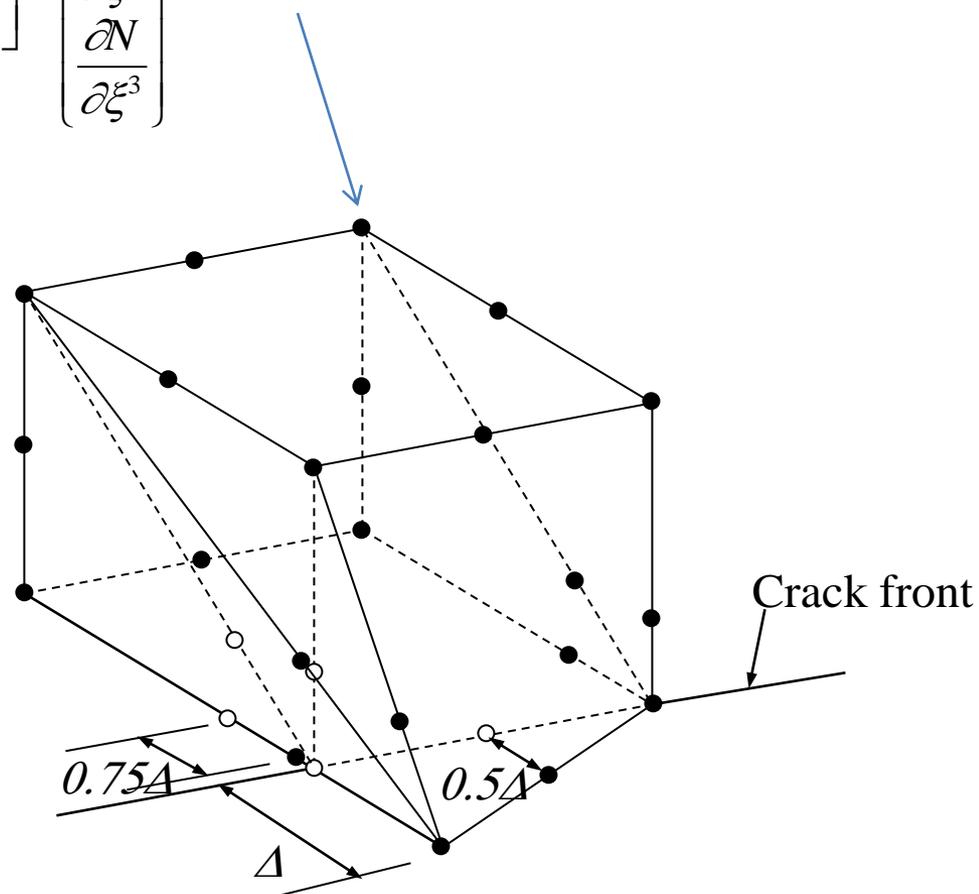
# Introducing virtual strain like matrix

$$\tilde{\varepsilon} = J^{-1} \begin{Bmatrix} \frac{\partial \mathcal{N}}{\partial \xi^1} \\ \frac{\partial \mathcal{N}}{\partial \xi^2} \\ \frac{\partial \mathcal{N}}{\partial \xi^3} \end{Bmatrix} \begin{bmatrix} \Delta_n^1 & \Delta_n^2 & \Delta_n^3 \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{bmatrix}^{-1} \begin{Bmatrix} \frac{\partial \mathcal{N}}{\partial \xi^1} \\ \frac{\partial \mathcal{N}}{\partial \xi^2} \\ \frac{\partial \mathcal{N}}{\partial \xi^3} \end{Bmatrix} \begin{bmatrix} \Delta_n^1 & \Delta_n^2 & \Delta_n^3 \end{bmatrix}$$

$$\frac{\delta |J|}{\delta a_i} = \text{Tr}(\tilde{\varepsilon}) |J|$$

$$\frac{\delta^2 |J|}{\delta a_i^2} = \left( \text{Tr}^2(\tilde{\varepsilon}) - \text{Tr}(\tilde{\varepsilon}^2) \right) |J|$$

$$\frac{\delta^2 |J|}{\delta a_i \delta a_j} = \left( \text{Tr}(\tilde{\varepsilon}_i) \text{Tr}(\tilde{\varepsilon}_j) - \text{Tr}(\tilde{\varepsilon}_j \tilde{\varepsilon}_i) \right) |J|$$



Mesh perturbation  $\Delta$  in 15-noded wedge elements

O : nodes before mesh perturbation

# Papers co-authored by me with Professor Ingrassia

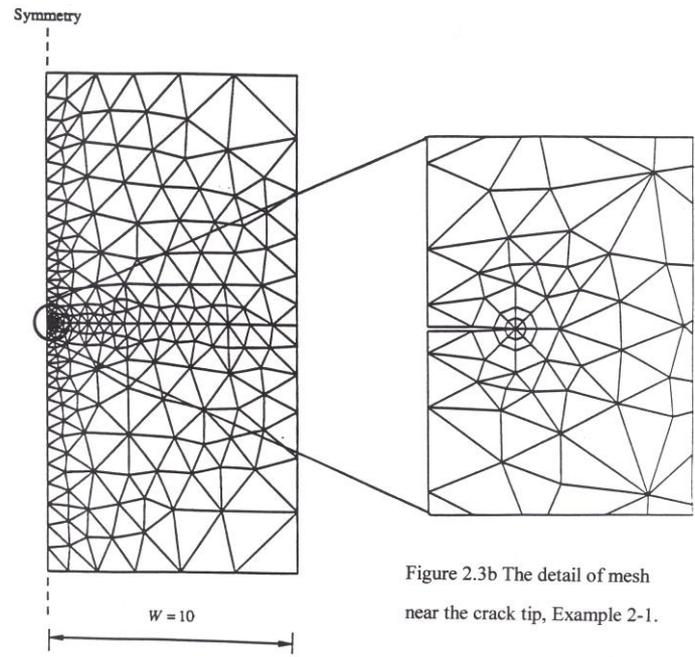
G and  $dG/da$  for 2D single crack: (1988) by Lin and Abel

Generalized to

- **G and  $dG_i/da_j$  for 2D multiple crack system** (EFM 1998)
- **G and  $dG_i/da_j$  for 3D mode-I planar cracks** (EFM 2001)
- **Shape prediction and stability analysis of 3D planar cracks** (EFM 2004)
- **Derivatives of stress intensity factors for 2D multiple cracks** (EFM 2005)
- **2<sup>nd</sup> order derivatives of energy release rates for 2D multiple cracks** (EFM 2007)
- **Decomposition of 3D mixed-mode energy release rates** (EFM 2014, accepted)

# Verification

# A Center Cracked Infinite Plate Subjected to a Uniform Remote Tensile Stress



$$K_I = \sigma_0 \sqrt{\pi a}$$

$$\frac{\delta K_I}{\delta a} = \frac{\sigma_0}{2} \sqrt{\frac{\pi}{a}}$$

$$\frac{\delta^2 K_I}{\delta a^2} = -\frac{\sigma_0}{4a} \sqrt{\frac{\pi}{a}}$$

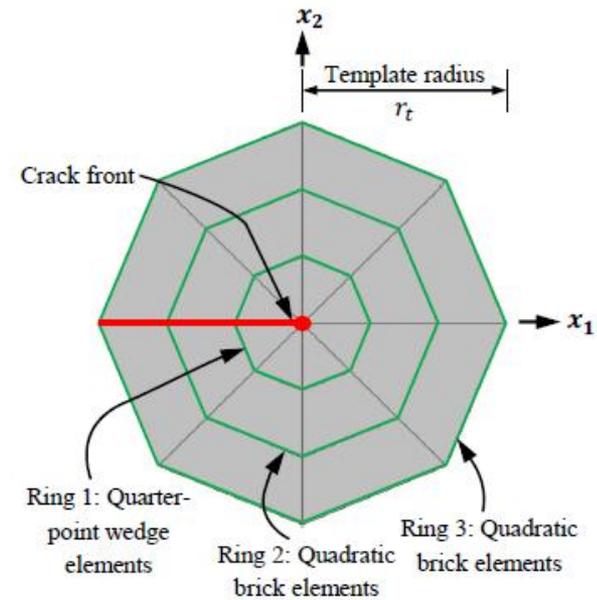
$$K_I = \sigma_0 \sqrt{\pi a}$$

$$\frac{\delta K_I}{\delta a} = \frac{\sigma_0}{2} \sqrt{\frac{\pi}{a}}$$

a/W	Exact (KI)	computed KI (Error %)	Exact δKI	computed δKI (Error %)
0.010	0.5605	0.5610 (0.09)	2.802	2.739 (2.25)
0.011	0.5879	0.5884 (0.08)	2.672	2.617 (2.06)
0.012	0.6140	0.6145 (0.08)	2.558	2.561 (1.17)
0.013	0.6391	0.6395 (0.06)	2.458	2.404 (2.20)
0.014	0.6632	0.6627 (0.08)	2.369	2.313 (2.36)

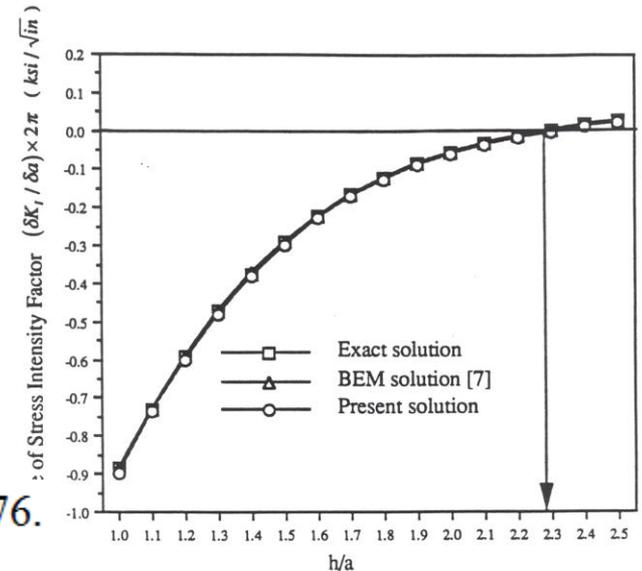
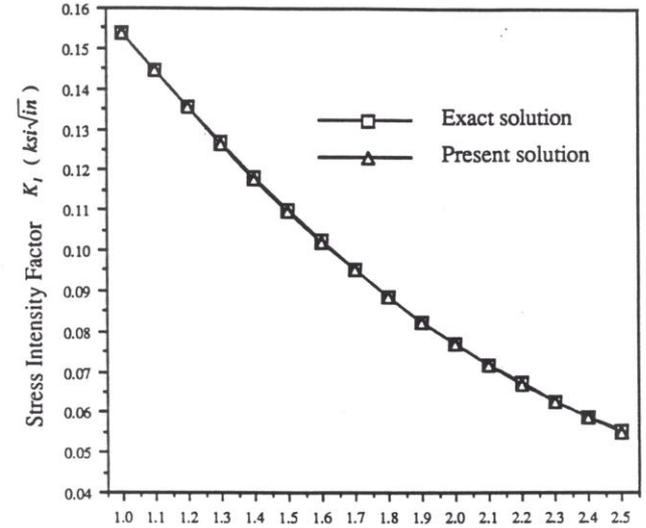
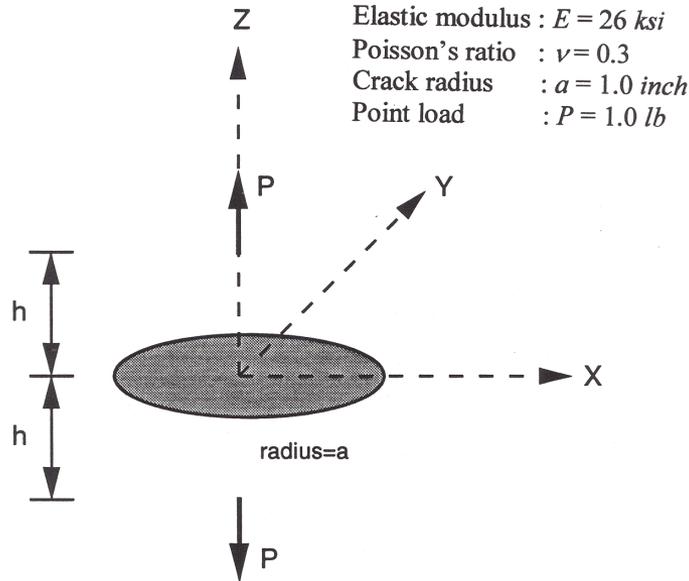
## Solutions for different rings of elements surrounding crack-tip

$$\frac{\delta^2 K_I}{\delta a^2} = -\frac{\sigma_0}{4a} \sqrt{\frac{\pi}{a}}$$



a/W	Exact	1st ring (Error %)	1st + 2nd rings (Error %)	1st+2nd+3rd (Error %)
0.010	-14.012	-52.373 (273.8)	-17.465 (24.6)	-15.044 (7.37)
0.011	-12.145	-58.599 (382.5)	-15.386 (26.7)	-11.637 (4.18)
0.012	-10.659	-40.801 (282.8)	-13.447 (26.2)	-11.915 (11.8)
0.013	-9.454	-46.023 (386.8)	-12.394 (31.1)	-9.881 (4.52)
0.014	-8.459	-22.607 (167.3)	-10.941 (29.3)	-8.911 (5.34)

# A Circular Crack under Two Symmetric Point Loads in



$$G = \frac{(1 - \nu^2) P^2 \alpha (\kappa + \alpha^2)}{E (\pi h)^3 (1 + \alpha^2)^4} \quad \text{where, } \alpha = \frac{a}{h} \text{ and } \kappa = \frac{2 - \nu}{1 - \nu}$$

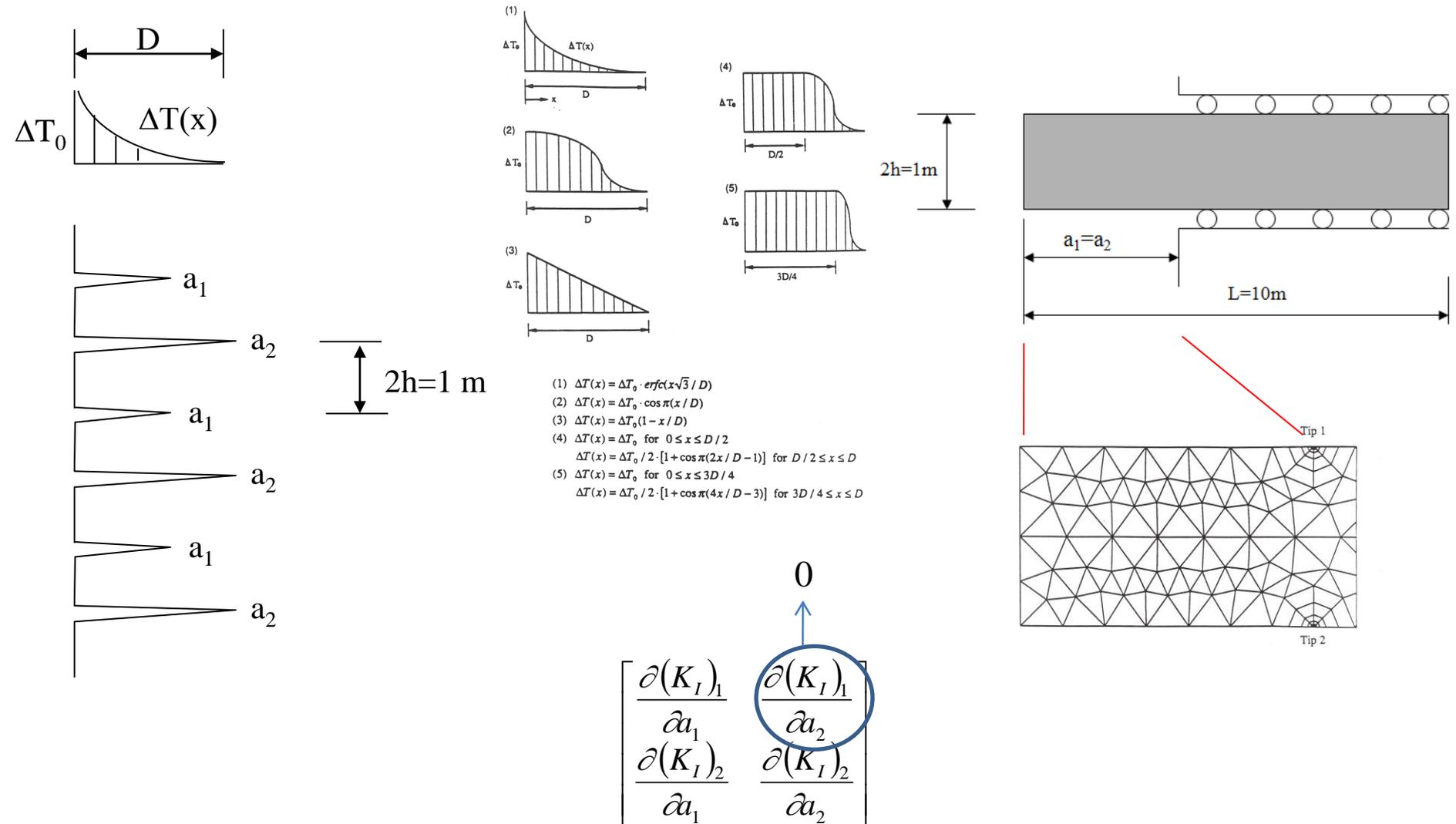
$$\frac{dG}{d\alpha} < 0 \quad (\alpha > \alpha_m), \quad \frac{dG}{d\alpha} > 0 \quad (0 < \alpha < \alpha_m), \quad \frac{dG}{d\alpha} = 0 \quad (\alpha = \alpha_m)$$

$$\text{with } \alpha_m^2 = \frac{\sqrt{16\nu^2 - 72\nu + 105} - 2\nu + 9}{2(2 - \nu)}$$

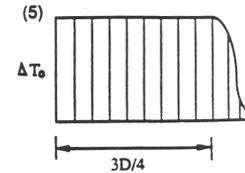
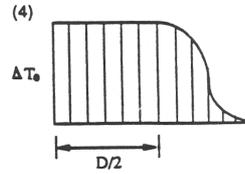
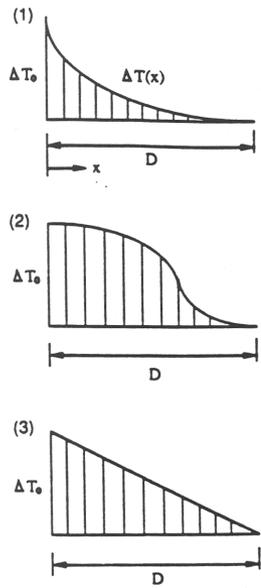
For a Poisson's ratio,  $\nu = 0.3$ , the critical value  $\alpha_m$  is about 2.276.

Present solution = 2.297

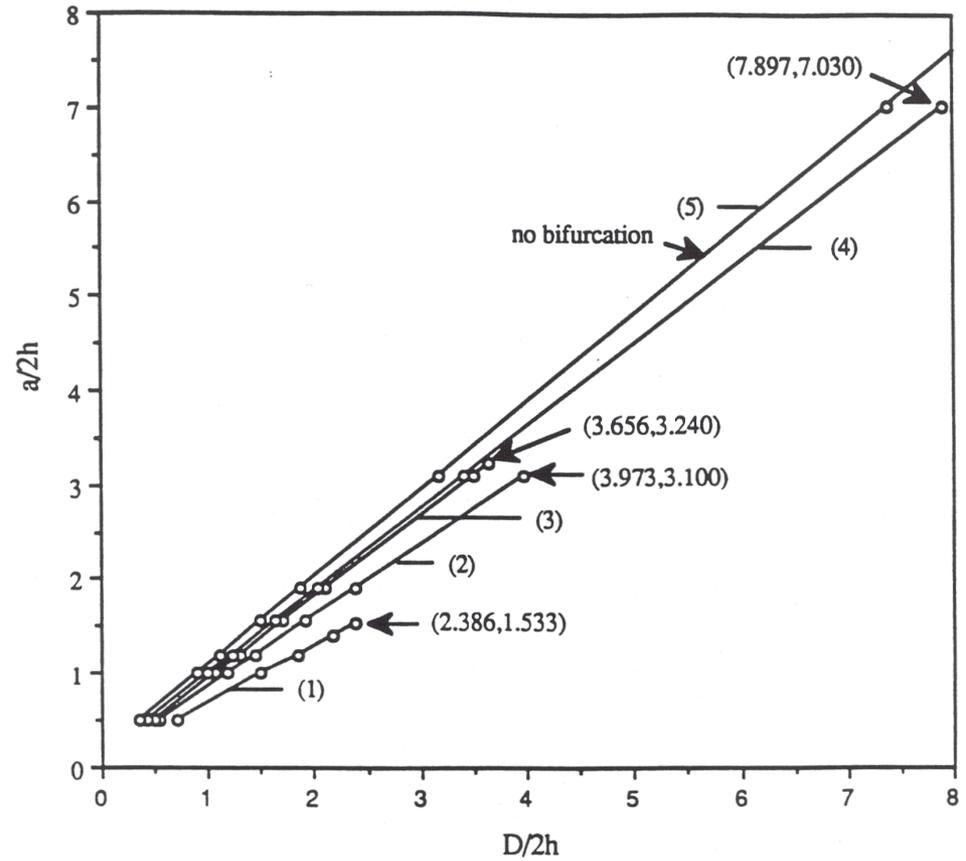
# Thermally induced parallel edge cracks having a periodic pattern in a semi-infinite plane



- (1)  $\Delta T(x) = \Delta T_0 \cdot \text{erfc}(x\sqrt{3}/D)$
- (2)  $\Delta T(x) = \Delta T_0 \cdot \cos(\pi x/D)$
- (3)  $\Delta T(x) = \Delta T_0(1-x/D)$
- (4)  $\Delta T(x) = \Delta T_0$  for  $0 \leq x \leq D/2$   
 $\Delta T(x) = \Delta T_0/2 \cdot [1 + \cos(\pi(2x/D - 1))]$  for  $D/2 \leq x \leq D$
- (5)  $\Delta T(x) = \Delta T_0$  for  $0 \leq x \leq 3D/4$   
 $\Delta T(x) = \Delta T_0/2 \cdot [1 + \cos(\pi(4x/D - 3))]$  for  $3D/4 \leq x \leq D$



- (1)  $\Delta T(x) = \Delta T_0 \cdot \operatorname{erfc}(x\sqrt{3}/D)$   
 (2)  $\Delta T(x) = \Delta T_0 \cdot \cos \pi(x/D)$   
 (3)  $\Delta T(x) = \Delta T_0(1 - x/D)$   
 (4)  $\Delta T(x) = \Delta T_0$  for  $0 \leq x \leq D/2$   
 $\Delta T(x) = \Delta T_0/2 \cdot [1 + \cos \pi(2x/D - 1)]$  for  $D/2 \leq x \leq D$   
 (5)  $\Delta T(x) = \Delta T_0$  for  $0 \leq x \leq 3D/4$   
 $\Delta T(x) = \Delta T_0/2 \cdot [1 + \cos \pi(4x/D - 3)]$  for  $3D/4 \leq x \leq D$



	Results from [Bazant 79a]			Results of present method		
Profile	a/2h	D/2h	a/D	a/2h	D/2h	a/D
1	1.53	2.342	<b>0.651</b>	1.53	2.386	<b>0.642</b>
2	3.07	3.907	<b>0.787</b>	3.10	3.973	<b>0.780</b>
3	3.20	3.594	<b>0.891</b>	3.24	3.656	<b>0.886</b>
4	6.97	7.806	<b>0.893</b>	7.03	7.897	<b>0.890</b>

# Inclined cracks under remote uniform stress

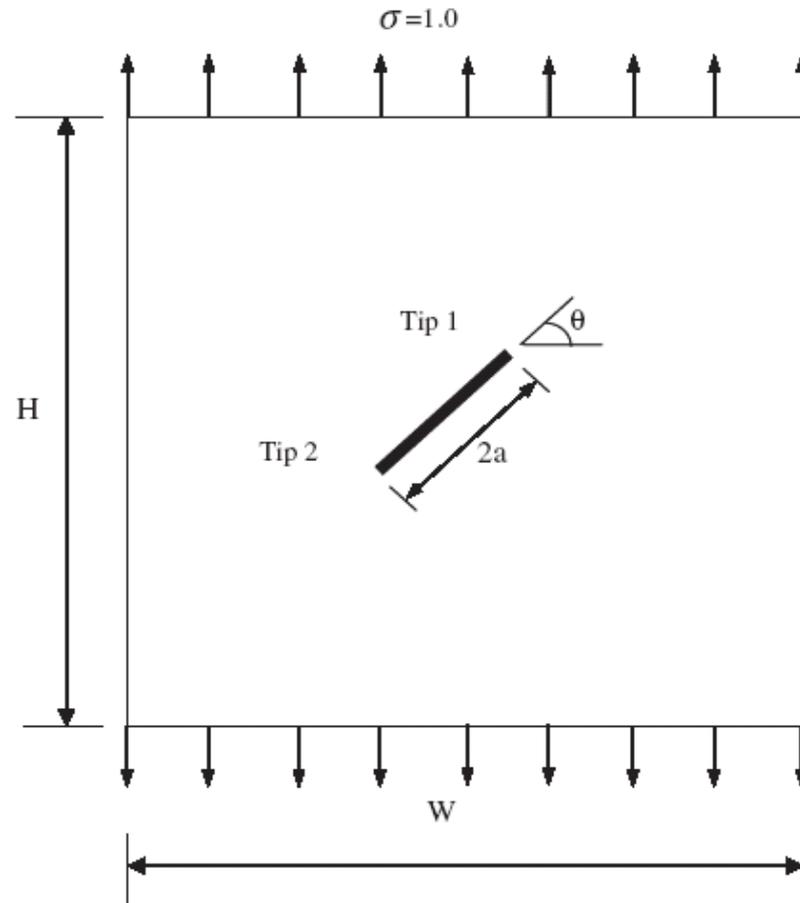


Table Fig. 3. An inclined crack in a simulated infinite plate under remote unit tensile stress ( $W=H=20$ ,  $2a=0.5$  or  $1.0$ ,  $\sigma=1.0$ ,  $\theta=45^\circ$ ) (not to scale, Example 1).

Crack	$2a/W = 0.1$		$2a/W = 0.2$	
	$K_I$	$K_{II}$	$K_I$	$K_{II}$
1	0.6285 (0.3%)	0.6294 (0.4%)	0.8981 (1.3%)	0.8962 (1.1%)
2	0.6285 (0.3%)	0.6293 (0.4%)	0.8978 (1.3%)	0.8964 (1.1%)

Analytical solutions for infinite plate:  $K_I = K_{II} = 0.6267$  for  $2a/W = 0.1$ ;  $K_I = K_{II} = 0.8862$  for  $2a/W = 0.2$ .

# Comparison of $dK/da$ with FDM solutions

Table 2

Comparison between the present numerical solution and a finite difference method (FDM) solution for derivatives of stress intensity factors: Example 1, crack-tip element size =  $2a/12$ ,  $2a/W = 0.1$

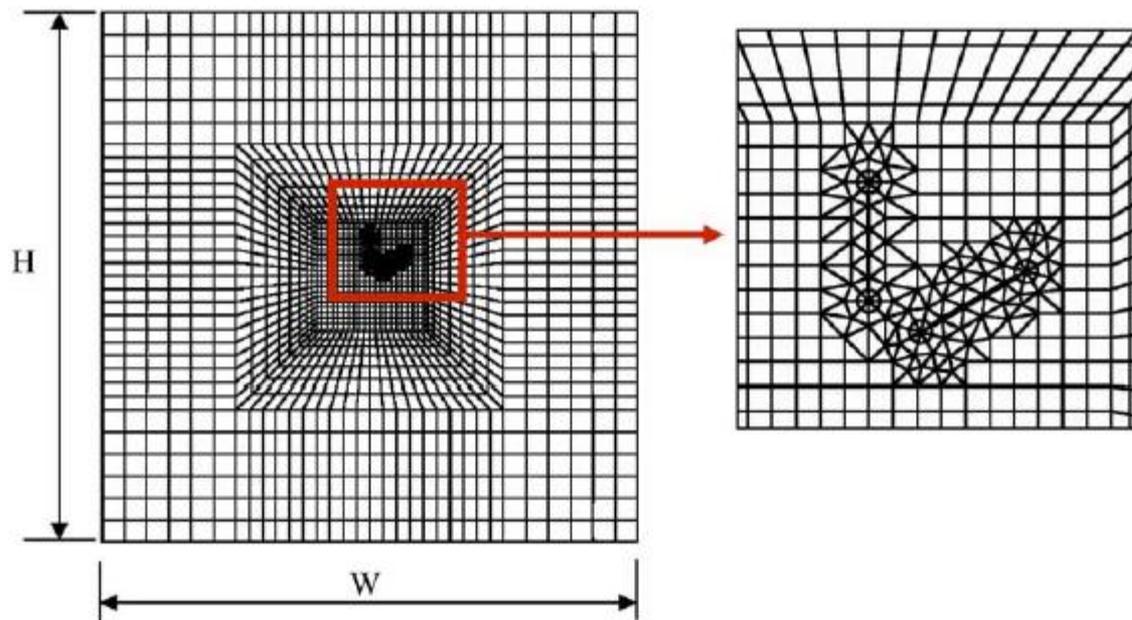
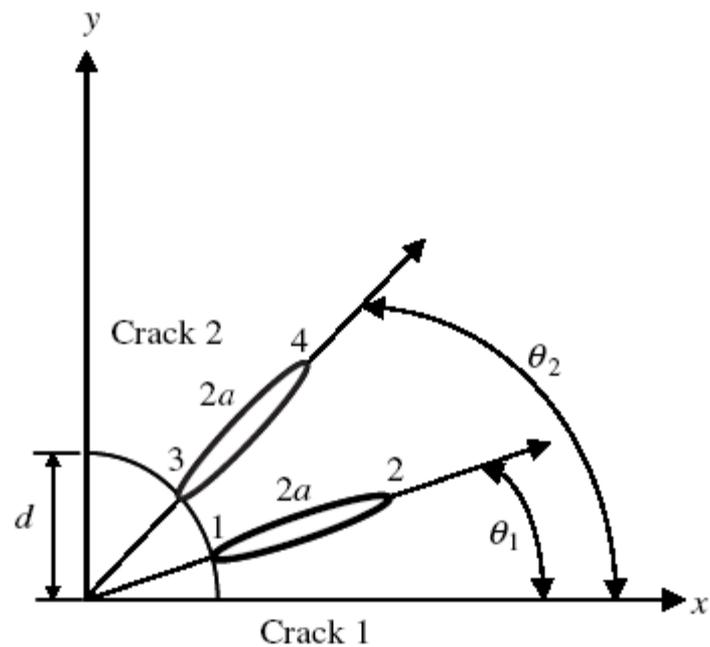
		FDM solution		Present solution		Difference (%)	
<i>Mode-I</i>							
$\delta(K_I)_1/\delta a_1$	$\delta(K_I)_1/\delta a_2$	0.3194	0.3188	0.3133	0.3186	-1.91	-0.06
$\delta(K_I)_2/\delta a_1$	$\delta(K_I)_2/\delta a_2$	0.3184	0.3183	0.3186	0.3137	0.06	-1.44
<i>Mode-II</i>							
$\delta(K_{II})_1/\delta a_1$	$\delta(K_{II})_1/\delta a_2$	0.3173	0.3173	0.3207	0.3081	1.07	-2.90
$\delta(K_{II})_2/\delta a_1$	$\delta(K_{II})_2/\delta a_2$	0.3176	0.3178	0.3082	0.3203	-2.96	0.79

Table 3

Comparison between the present numerical solution and finite difference method (FDM) solution for the derivatives of stress intensity factors: Example 1, crack-tip element size =  $2a/12$ ,  $2a/W = 0.2$

		FDM solution		Present method		Difference (%)	
<i>Mode-I</i>							
$\delta(K_I)_1/\delta a_1$	$\delta(K_I)_1/\delta a_2$	0.2360	0.2360	0.2337	0.2360	-0.97	0.00
$\delta(K_I)_2/\delta a_1$	$\delta(K_I)_2/\delta a_2$	0.2360	0.2360	0.2361	0.2339	0.04	-0.89
<i>Mode-II</i>							
$\delta(K_{II})_1/\delta a_1$	$\delta(K_{II})_1/\delta a_2$	0.2330	0.2330	0.2328	0.2994	-0.97	-1.54
$\delta(K_{II})_2/\delta a_1$	$\delta(K_{II})_2/\delta a_2$	0.2330	0.2330	0.2293	0.2320	-1.59	-0.43

# System of Radial Cracks (4 crack-tips)



e 3).

Fig. 8. Finite element discretization for radial multiple cracks embedded in the infinite plate ( $H = W = 40$ ) (Example 3).

## $\theta_1=30$ degree, GI, GII

Table 12

Comparison between numerical and analytical solutions for the normalized mode-I energy release rate,  $(G_I)/G_0$ ,  $\theta_1 = 30^\circ$

Crack tip	Present method			Analytical solution [23]		
	$\theta_2 = 45^\circ$	$\theta_2 = 60^\circ$	$\theta_2 = 90^\circ$	$\theta_2 = 45^\circ$	$\theta_2 = 60^\circ$	$\theta_2 = 90^\circ$
1	0.1555	0.3473	0.6096	0.1595	0.3494	0.6004
2	0.6541	0.6244	0.5932	0.6425	0.6198	0.5827
3	0.2421	0.0335	0.0068	0.2325	0.0323	0.0024
4	0.0202	0.0004	0.0106	0.0208	0.0004	0.0067

Table 13

Comparison between numerical and analytical solutions for the normalized mode-II energy release rate,  $(G_{II})/G_0$ ,  $\theta_1 = 30^\circ$

Crack tip	Present method			Analytical solution [23]		
	$\theta_2 = 45^\circ$	$\theta_2 = 60^\circ$	$\theta_2 = 90^\circ$	$\theta_2 = 45^\circ$	$\theta_2 = 60^\circ$	$\theta_2 = 90^\circ$
1	0.0904	0.1807	0.1894	0.0953	0.1863	0.1869
2	0.1512	0.1707	0.1760	0.1538	0.1751	0.1763
3	0.2988	0.2508	0.0003	0.3075	0.2567	0.0002
4	0.0629	0.0733	0.0054	0.0646	0.0752	0.0054

## θ<sub>1</sub>=30 degree, θ<sub>2</sub>=90 degree, Derivatives of GI, GII

Table 14

The calculated derivatives of normalized mode-I energy release rates,  $\delta(G_{Ij})/\delta a_j/G_0$ , by the present virtual crack extension method and the finite difference approximation,  $\theta_1 = 30^\circ$  and  $\theta_2 = 90^\circ$

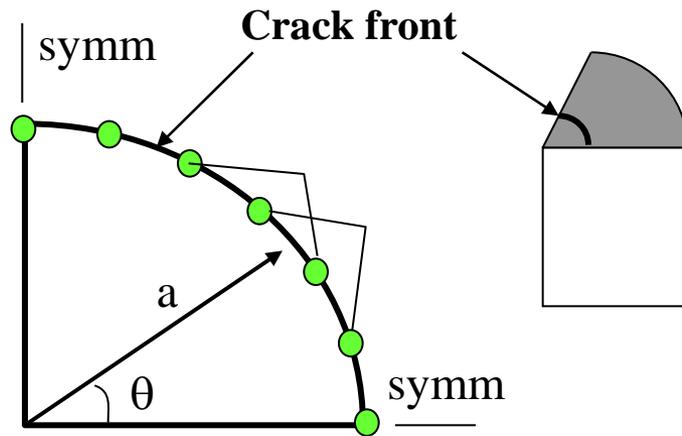
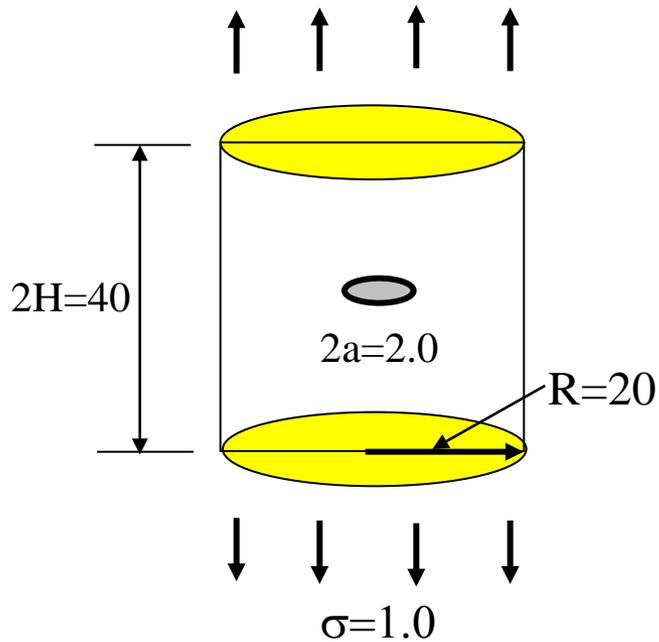
	Present numerical solution					Finite difference solution			
	1	2	3	4		1	2	3	4
1	0.368	0.324	0.027	0.021	1	0.367	0.318	0.032	0.023
2	0.324	0.286	0.004	0.017	2	0.336	0.291	0.009	0.023
3	0.027	0.004	-0.020	0.003	3	0.025	-0.001	-0.021	0.002
4	0.021	0.017	0.003	-0.003	4	0.013	0.012	0.002	-0.004

Table 15

The calculated derivatives of normalized mode-II energy release rates,  $\delta(G_{IIj})/\delta a_j/G_0$ , by the present virtual crack extension method and the finite difference approximation,  $\theta_1 = 30^\circ$  and  $\theta_2 = 90^\circ$

	Present numerical solution					Finite difference solution			
	1	2	3	4		1	2	3	4
1	0.118	0.078	-0.005	0.011	1	0.110	0.096	-0.005	-0.001
2	0.078	0.096	-0.003	-0.002	2	0.071	0.089	-0.008	-0.007
3	-0.005	-0.003	0.003	0.000	3	-0.003	0.001	0.003	-0.001
4	0.011	-0.002	0.000	0.004	4	0.019	0.004	0.000	0.004

# A Penny-Shaped Crack Embedded in A Large Cylinder Under Remote Uniform Tensile Loading

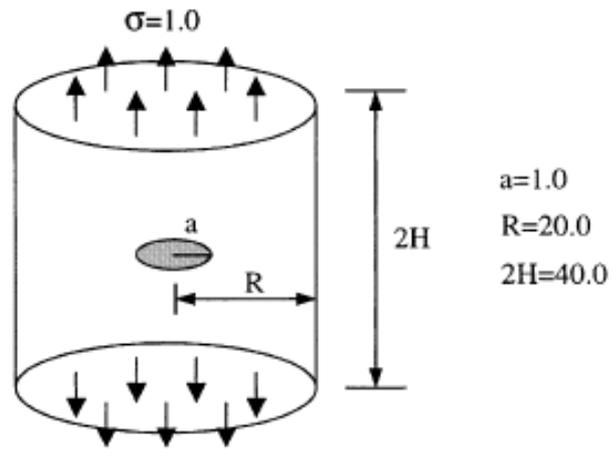


Axisymm. solution  
[Sneddon 46]

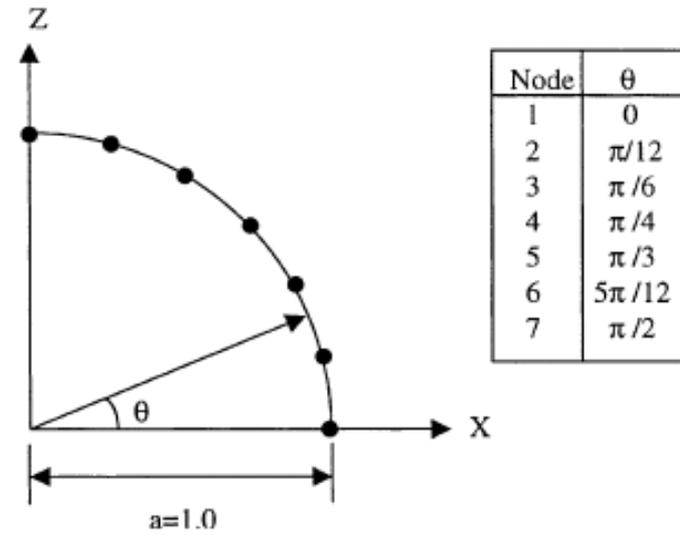
$$K_I = 2\sigma \sqrt{\frac{a}{\pi}}$$

$$\frac{\delta K_I}{\delta a} = \frac{\sigma}{\sqrt{a\pi}}$$

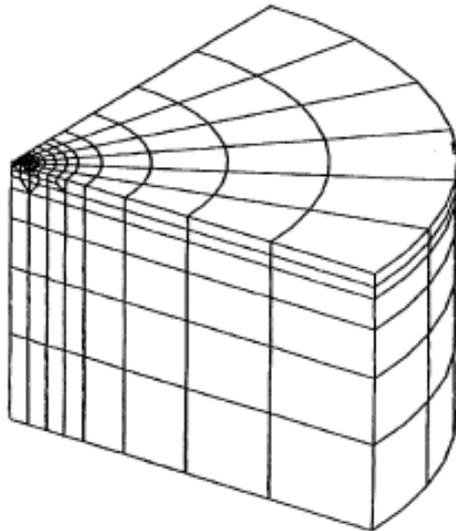
(a)



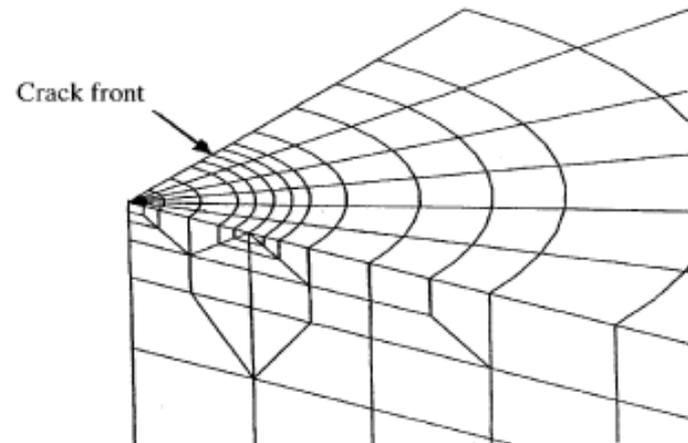
(b)



(c)



(d)



# RESULTS

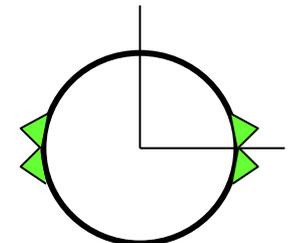
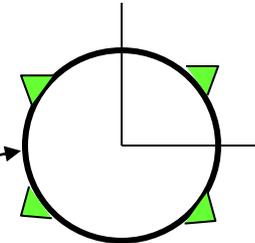
Analytical solution:  $K_I = 1.1284$ ,  $\delta K_I = 0.5642$

Present solution:  $K_I = 1.1302$  for all  $\theta$  (Error 0.2 %)  
 $\delta K_I = 0.5406$  (Error 4 %)

At node 4

At node 2

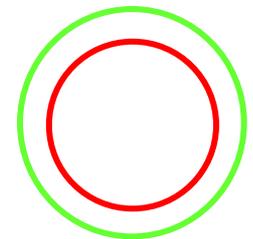
Crack front



$\delta K_i / \delta a_j$

Node	j:1	2	3	4	5	6	7	Row sum
i:1	<u>-1.9232</u>	1.4975	0.4725	0.2064	0.1347	0.1040	0.0487	0.5406
2	0.7487	<u>-1.6870</u>	0.8519	0.3036	0.1552	0.1161	0.0520	0.5406
3	0.2362	0.8519	<u>-1.8558</u>	0.8007	0.2850	0.1552	0.0674	0.5406
4	0.1032	0.3036	0.8007	<u>-1.8745</u>	0.8007	0.3036	0.1032	0.5406
5	0.0674	0.1552	0.2850	0.8007	<u>-1.8558</u>	0.8519	0.2362	0.5406
6	0.0520	0.1161	0.1552	0.3036	0.8519	<u>-1.6870</u>	0.7487	0.5406
7	0.0487	0.1040	0.1347	0.2064	0.4725	1.4975	<u>-1.9232</u>	0.5406

Uniform extension



# Decomposition of 3D mixed-mode energy release rates (EFM 2014, accepted)

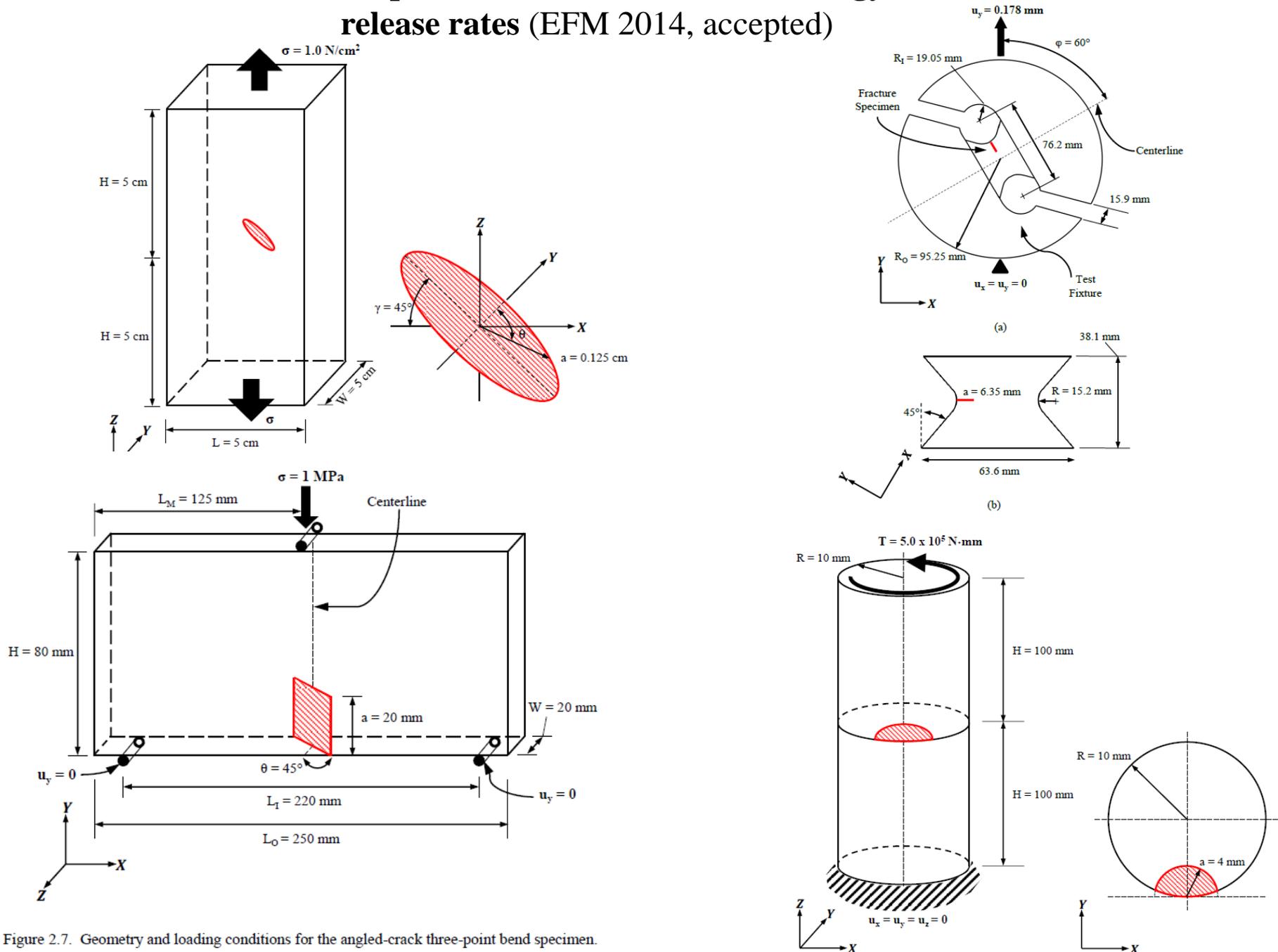


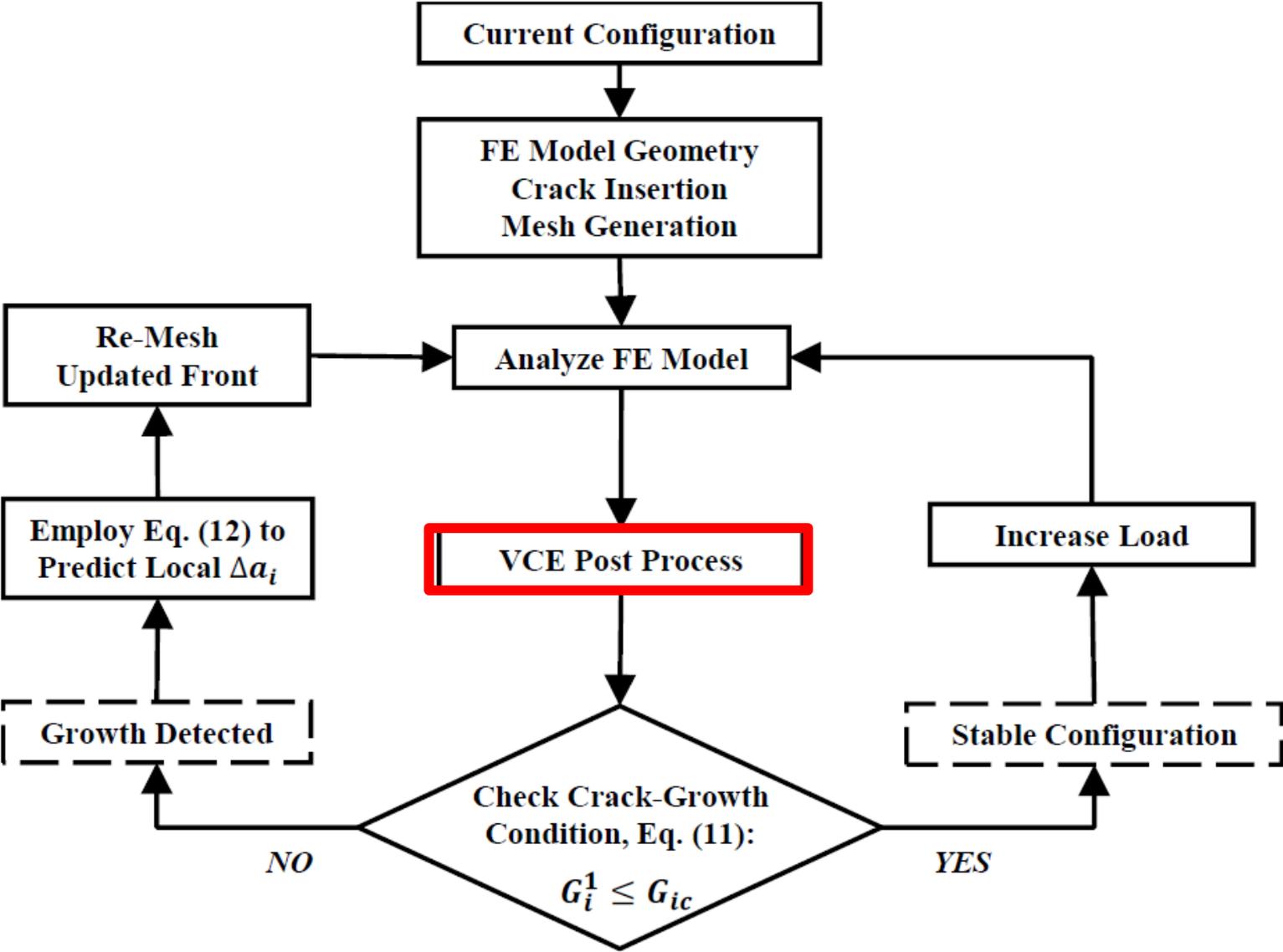
Figure 2.7. Geometry and loading conditions for the angled-crack three-point bend specimen.

## Numerical Results

Numerical Example	$G_I$ (N/mm)			$G_{II}$ (N/mm)		
	VCE	Ref. Sol.	% Diff	VCE	Ref. Sol.	% Diff
Inclined Penny Crack	3.974E-07	3.979E-07	0.06	1.976E-07	1.989E-07	0.16
Arcan Specimen	1.845	1.835	0.25	2.039	2.018	0.53
Angled-Crack 3-Point Bend	3.012E-06	2.993E-06	0.35	2.316E-07	2.293E-07	0.04
Surface-Cracked Cylinder	2.146E-05	2.272E-05	0.00	1.358	1.361	0.14

Numerical Example	$G_{III}$ (N/mm)			G (N/mm)		
	VCE	Ref. Sol.	% Diff	VCE	Ref. Sol.	% Diff
Inclined Penny Crack	1.996E-07	1.989E-07	0.09	7.947E-07	7.958E-07	0.14
Arcan Specimen	0.0733	0.0821	0.22	3.957	3.935	0.56
Angled-Crack 3-Point Bend	2.170E-06	2.160E-06	0.19	5.414E-06	5.383E-06	0.58
Surface-Cracked Cylinder	0.7347	0.7419	0.34	2.093	2.102	0.43

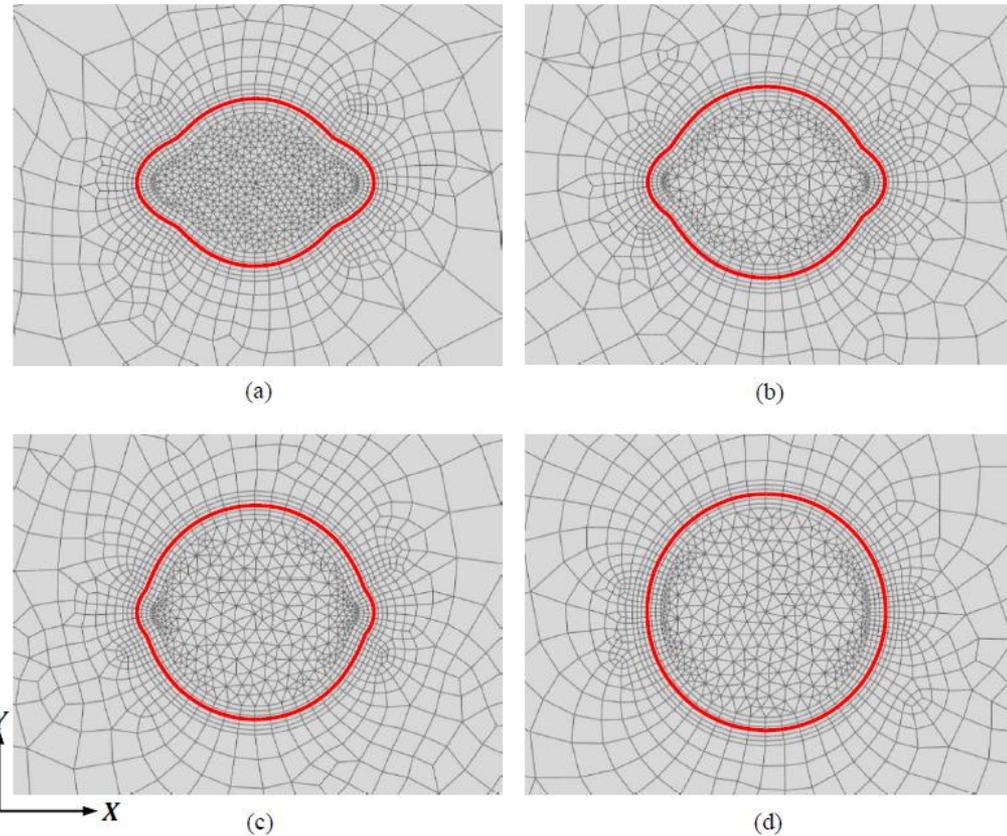
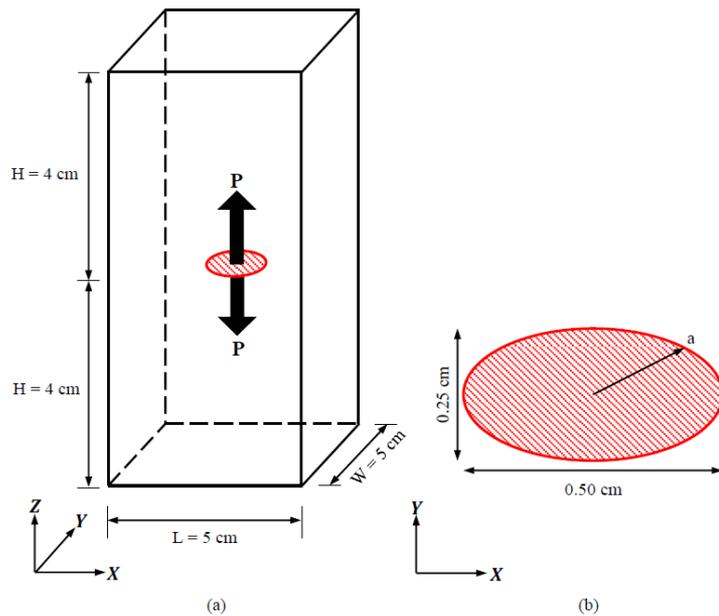
# Iterative Crack Growth Simulation Algorithm (Brett Davis Thesis, Cornell 2014)



# Augmented Energy Based Growth Formulation : Planar cracks

$$G_{ic} = G_i^0 + \frac{\delta G_i}{\delta P} \odot \Delta P_i + \frac{\delta G_i}{\delta a_j} \Delta a_j. *$$

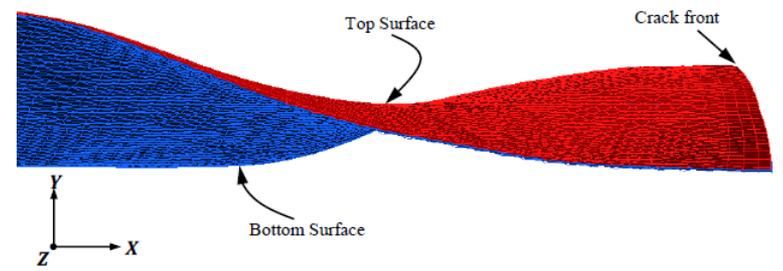
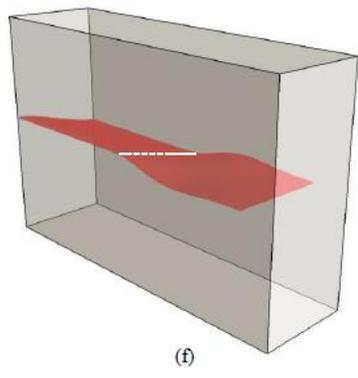
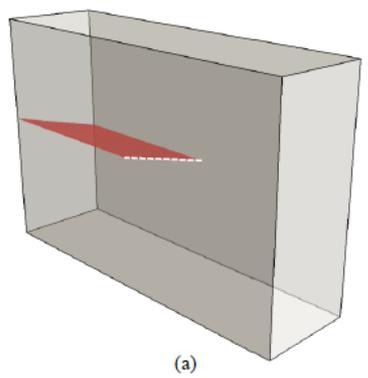
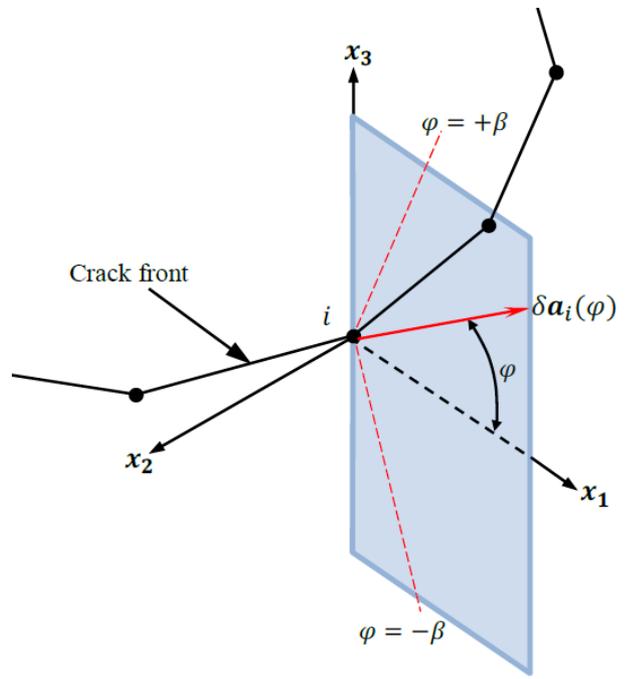
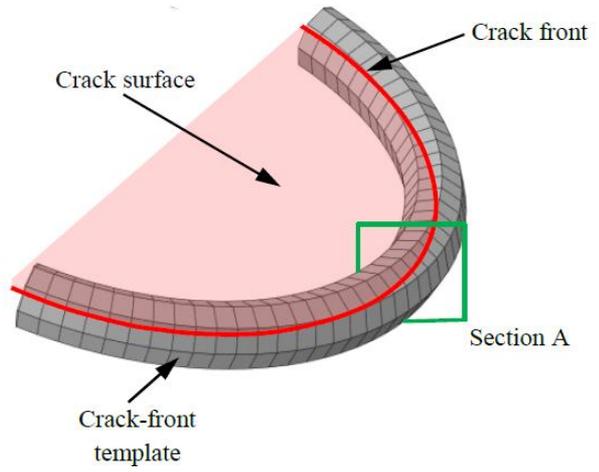
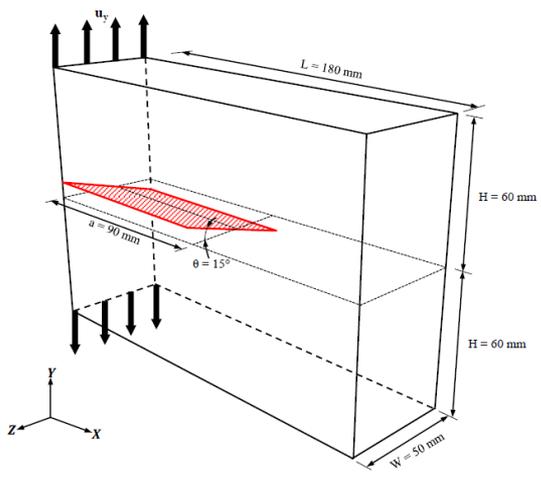
(Brett Davis Dissertation, 2014 Cornell)



# Augmented Energy Based Growth Formulation : Non-planar cracks

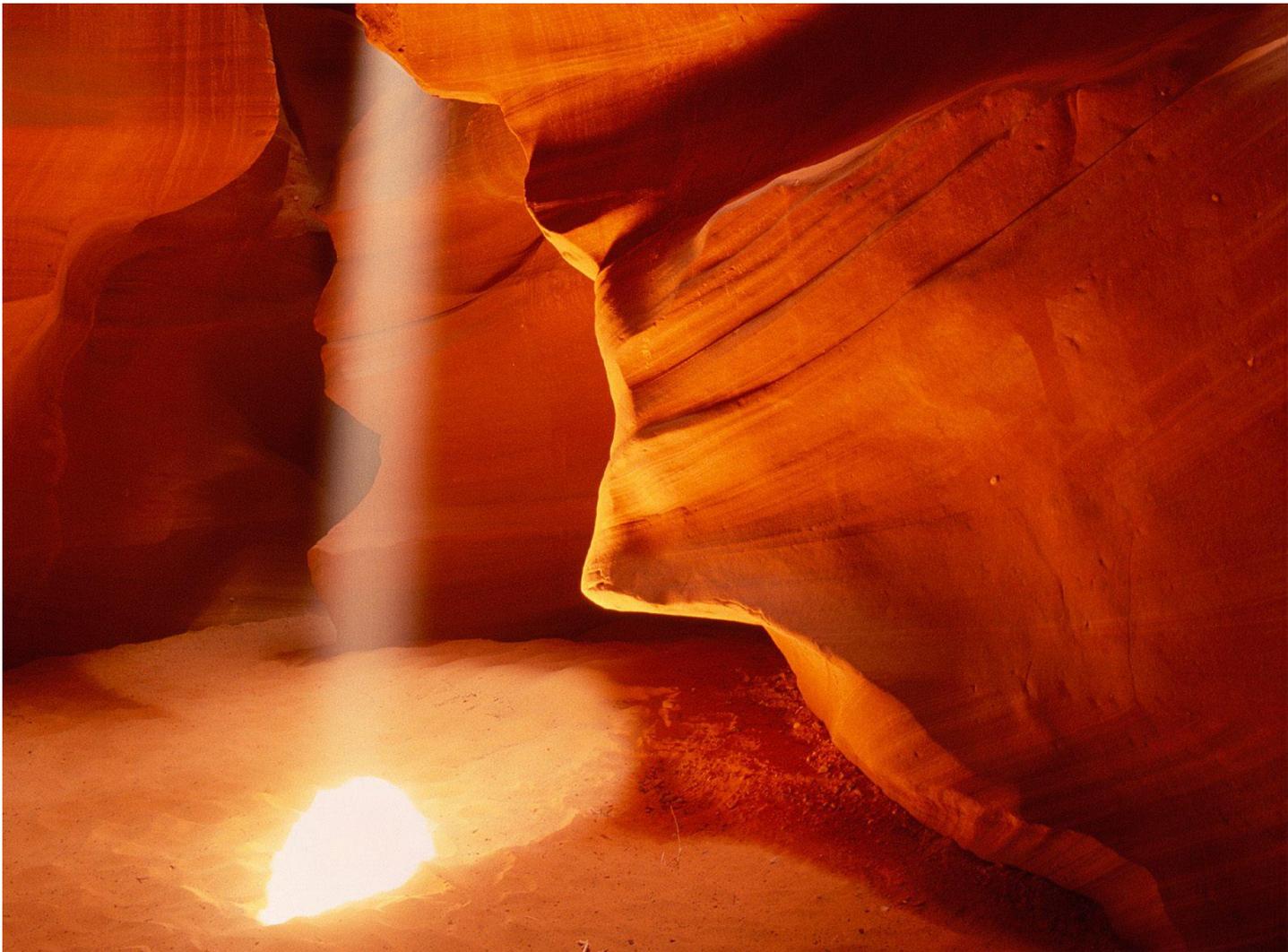
(Brett Davis Dissertation, 2014 Cornell)

$$G_{ic} = G_i^0 + \frac{\delta G_i}{\delta P} \odot \Delta P_i + \frac{\delta G_i}{\delta a_j} \Delta a_j^*$$



# Conclusions

- A decent method developed for calculating the energy release rates and their higher order derivatives for a 2D/3D multiply cracked body.
- The analytical virtual crack extension method by Lin and Abel is extended to the general case of a system of 2D interacting cracks, extension to the axisymmetric case, extension to 3D crack with an arbitrarily curved front under general mixed-mode loading conditions, inclusion of non-uniform crack-face pressure and thermal loading.
- The salient feature of this method is that the energy release rates and their higher derivatives for multiple cracks in two and three dimensions can be accurately computed in a single analysis.
- This method is essential to an iterative crack growth simulation algorithm developed by A.R. Ingraffea and co-workers.



“THERE IS A CRACK IN EVERYTHING.  
THAT’S HOW THE LIGHT GETS IN.”

— LEONARD COHEN

THANK YOU, TONY !

