# Optimal Portfolio Selection with Fixed Transactions Costs in the presence of Jumps and Random Drift

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**Abstract**

In this paper, we study the general problem of optimal portfolio selection with fixed transactions costs in the presence of jumps. We extend the analysis of Morton and Pliska to this setting by modeling the return processes of the risky assets in the investor's portfolio as jump-diffusion processes and derive the expression for the related optimal stopping time problem of a Markov process with jumps and explicitly solve it in the situation when the portfolio consists only of one risky asset. We also provide an asymptotic analysis of our model with one risky asset following the ideas of Wilmott and Atkinson.In the process, we also obtain a solution for the "Merton problem" generalized to the situation when there is credit risk. Finally, we consider the case where the drift of the stock price process is random and unobservable and obtain expressions for the optimal trading policies.

## 1 Introduction

In their paper, Morton and Pliska (1995) investigated the problem of optimal portfolio selection in the presence of fixed transactions costs. In this paper, we extend the Morton-Pliska model to study the problem of optimal portfolio selection with fixed transactions costs in the presence of jumps. From an economic standpoint, this is important in understanding the behavior of an investor in a general, imperfect market. Several authors have studied both pricing and portfolio selection problems when asset price returns are discontinuous. Some papers concerned with this issue are Aase(1993), Colwell and Elliott(1993), Jarrow and Madan(1995), Mercurio and Runggaldier(1993), and the references cited therein among others.

Within the framework of the model we propose, we shall see that the solution of the problem involves the mathematical investigation of a certain optimal stopping time problem of a Markov process with jumps which has been extensively studied in the mathematical literature.(see Bensoussan(1982)).

Morton and Pliska analyze the situation where the drift of the stock price process is constant which is a crude approximation from an economic standpoint especially since we are studying an infinite horizon optimality criterion. In this paper, we demonstrate how to tackle the situation when the drift is a random, unobservable process driven by a continuous time Markov chain.

In section 2, we present our extension of the Morton-Pliska model to the situation when there are jumps in the stock price process. In section 3, we study the *generalized* "Merton problem" of optimal portfolio selection without transactions costs but in the presence of jumps. In section 4, following Morton and Pliska, we describe the admissible trading strategies and derive the equations satisfied by the *optimal* trading policies. In section 5, we recast our problem as a stopping-time problem of the so-called "risky fraction" process. We derive the form of the variational inequalities satisfied by the value function of the optimal stopping time problem and show that the value function is, in fact, a viscosity solution of these variational inequalities, where we give a precise definition of viscosity solutions in our setting. In section 6, we study the situation where the investor's portfolio consists of one risky asset and obtain analytical expressions for the optimal trading strategy which we compare with the results obtained by Morton and Pliska when there is no credit risk. In section 7, we use the ideas of Atkinson and Wilmott([2]) to provide an asymptotic analysis of our model in the general setting. In section 8, we study the situation when the drift is a random, unobservable process driven by a continuous time Markov chain. We make a crucial, but economically reasonable assumption about the behavior of the investor in this situation and obtain a solution to the problem in this case.

In order to facilitate comparison with the results of Morton and Pliska we have tried to retain their notation throughout.

# 2 The Model

We assume that the investor's portfolio consists of one risk-free asset or bond with a deterministic, constant rate of return $r$ and $m$ risky stocks whose return processes are *jump diffusion* processes with constant drift and volatilities.

## 2.1 Price processes for the bond and stocks

The price process for the bond $Z_t^0$ is given by

$$dZ_t^0 = rZ_t^0 dt \tag{2.1}$$

and the price processes for the $m$ stocks are given by

$$dZ_t^k = Z_t^k(\mu^k dt + \Lambda_j^k dW_t^j + dM_t) \tag{2.2}$$

where $M_t$ is a *continuous time branching process* with $n$ branches at every node and the *jump probability distribution* at every node is identical and is such that $M_t$ is a martingal+e and has the *strong Markov property.* The *jump times* $T_n$ of $M_t$ are *Poisson distributed.* Thus, we have basically modeled the return processes for the stocks as continous diffusions between jump times of $M_t$. At a jump time $T_n$ of $M_t$, if the state of $M_t$ jumps from $\alpha$ to $\beta$, the price of every stock in the portfolio suddenly changes by a factor of $\beta - \alpha$. This expresses the sudden fluctuations in stock prices due to external factors like governmental intervention, interest rate fluctuations, etc. The price processes $Z_t^0$ and $Z_t^k$ are *left continuous* processes with *right limits.* The choice we have made for the jump process allows us to extend the result of proposition 3.1 in Morton and Pliska(1995) to our setting. More precisely, with the above form of the price processes, there exists a *stationary* optimal policy.

## 2.2 The Portfolio Value Process

We shall now derive the expression for the portfolio value $V_t$ in terms of the fractions of the portfolio value invested in the various stocks. Following the notation of Morton and Pliska, $b_t^k$ is the fraction of portfolio value invested in stock '$k$' at time '$t$', and $1 - \sum_{k=1}^m b_t^k$ is the fraction of portfolio value invested in the bond. Thus,

$$dV_t = (1 - \sum_{k=1}^m b_t^k) r V_t dt + \sum_{k=1}^m b_t^k V_t(\mu_k dt + \sum_{l=1}^m \lambda_l^k dW_t^l + dM_t) \tag{2.3}$$

We see that $V_t$ is an *exponential semimartingale* given by

$$V_t = exp(X_t - (1/2) < X_c, X_c >_t) \Pi_{s \leq t}(1 + \Delta X_s) exp(-\Delta X_s) \tag{2.4}$$

provided $\Delta X_s > -1$ and

$$dX_t = (1 - \mathbf{1}^{\mathbf{T}} \mathbf{b}_t) r dt + \mathbf{b^T}_t(\boldsymbol{\mu} dt + \Lambda \mathbf{dW}_t + \mathbf{1} dM_t) \tag{2.5}$$

where $\mathbf{b}_t$ is the *risky fraction* vector and $X_c$ is the *continuous* part of the semimartingale $X_t$.

Here, we make our first simplifying assumption : we assume that $|\Delta X_s|$ is "small" so that terms of order higher than $\Delta X_s^2$ can be neglected. Effectively, we are assuming that $\Delta X_s << 1$. Hence,

$$V_t = V_0 exp[\int_0^t [(1 - \mathbf{1^T b}_s) r + \mathbf{b^T}_s \boldsymbol{\mu} - (1/2)\mathbf{b^T}_s \Lambda\Lambda^\mathbf{T} \mathbf{b}_s]$$
$$+ \int_0^t \mathbf{b^T}_s \Lambda \mathbf{dW}_s + \int_0^t \mathbf{b^T}_s \mathbf{1} dM_s - (1/2)\sum_{s\leq t}(\mathbf{b^T}_s \mathbf{1}\Delta M_s)^2] \quad (2.6)$$

Thus, in our model we assume that the *jumps* in prices $\Delta Z_t^k$ are such that $\Delta Z_t^k << Z_t^k$. The processes $\mathbf{b}_t$ are also *left continuous* with *right limits.*

# 3 Optimal Policies without Transaction Costs- the generalized Merton problem

From the expression for the portfolio value process $V_t$, we have

$$lnV_t = lnV_0 + [\int_0^t [(1 - \mathbf{1^T b}_s) r + \mathbf{b^T}_s \boldsymbol{\mu} - (1/2)\mathbf{b^T}_s \Lambda\Lambda^\mathbf{T} \mathbf{b}_s]$$
$$+ \int_0^t \mathbf{b^T}_s \Lambda \mathbf{dW}_s + \int_0^t \mathbf{b^T}_s \mathbf{1} dM_s - (1/2)\sum_{s\leq t}(\mathbf{b^T}_s \mathbf{1}\Delta M_s)^2] \quad (3.1)$$

The objective of the investor is to maximize $E[lnV_T]$ for some finite horizon $T$. The integrals with respect to $W_s$ and $M_s$ are martingales with zero expectation. Since the jump times of $M_t$ are Poisson distributed with intensity $\delta$, we can rewrite the expected value of the last term in the above expression as

$$-(1/2)\int_0^t \delta(\mathbf{b^T}_s \mathbf{1 1^T b}_s) E[(\Delta M_s)^2 | \Delta M_s \neq 0] ds \quad (3.2)$$

If the possible values of $\Delta M_s$ are $\alpha_1, \alpha_2, ..., \alpha_n$ with probabilities $p_1, p_2, ..., p_n$, then

$$E[(\Delta M_s)^2 | \Delta M_s \neq 0] = \Delta^2 = \sum_{i=1}^n p_i(\alpha_i)^2 \quad (3.3)$$

Hence, just as in the original Merton problem, we see that the investor should continuously rebalance his portfolio to maintain it in the proportions $\tilde{\mathbf{b}}$ , where $\tilde{\mathbf{b}}$ is the solution of

$$\max_{\mathbf{b}} (1 - \mathbf{1^T b}) r + \boldsymbol{b}^T \boldsymbol{\mu} - (1/2)\mathbf{b^T}(\Lambda\Lambda^\mathbf{T} + \mathbf{1 1^T} \delta(\Delta)^2)\mathbf{b} \quad (3.4)$$

such that

$$\mathbf{1^T b} \leq 1$$

and

$$\mathbf{b} \geq 0$$

If we make the assumption that all components of $\tilde{\mathbf{b}}$ are strictly positive and $\mathbf{1^T b} < 1$, then we have

$$\tilde{\mathbf{b}} = (\Lambda\Lambda^{\mathbf{T}} + \delta\Delta^2\mathbf{1}\mathbf{1}^{\mathbf{T}})^{-1}(\mu - r\mathbf{1}) \tag{3.5}$$

The optimal asymptotic growth rate which $\tilde{\mathbf{b}}$ induces is given by

$$\tilde{R} = (1 - \mathbf{1^T}\tilde{\mathbf{b}})r + \tilde{\mathbf{b}}^{\mathbf{T}}\boldsymbol{\mu} - (1/2)\tilde{\mathbf{b}}^{\mathbf{T}}(\Lambda\Lambda^{\mathbf{T}} + \mathbf{1}\mathbf{1}^{\mathbf{T}}\delta\Delta^2)\tilde{\mathbf{b}} \tag{3.6}$$

We notice that the above expressions can be obtained from the corresponding expressions in the original Merton problem by replacing $\Lambda\Lambda^T$ by $\Lambda\Lambda^T + \delta\Delta^2\mathbf{1}\mathbf{1}^{\mathbf{T}}$.

# 4 Optimal Portfolios with Transactions Costs in the presence of credit risk

We shall now generalize the Morton-Pliska model to our setting by incorporating the presence of credit risk, or, more generally, discontinuities in asset price returns.

## 4.1 The admissible trading strategies

The admissible trading strategies for the investor are exactly as described by Morton and Pliska . At time $t_0 = 0$, the investor allocates his portfolio $V_0$ in proportions $\mathbf{b_0}$. He holds this portfolio until some stopping time $t_1$ where he re-allocates the portfolio in the proportions $\mathbf{b_1}$ after paying the *transaction fee*, so that $V_{t_1+} = \alpha V_{t_1}$. However, in our situation, *unlike* the case in Morton-Pliska the *risky fraction* process $\mathbf{b}_t$ has jumps not only at portfolio re-allocation times, but also at jump times of the process $M_t$.

If $V_{t_n}$ represents the portfolio value after the $n$th fee has been paid, we have

$$V_{t_n} = \alpha V_{t_{n-1}}\{(1 - \mathbf{1^T b_{n-1}})exp(r(t_n - t_{n-1}) + b^1_{n-1}Z^1_{t_n}/Z^1_{t_{n-1}} + ... + b^m_{n-1}Z^m_{t_n}/Z_{t^m_{n-1}}\} \tag{4.1}$$

Thus

$$V_{t_n} = \alpha^n V_0 \Pi^n_{i=1}\{(1 - \mathbf{1^T b_{i-1}})exp(r(t_i - t_{i-1})) + b^1_{i-1}Z^1_{t_i}/Z^1_{t_{i-1}} + ... + b^m_{i-1}Z^m_{t_i}/Z^m_{t_{i-1}}\} \tag{4.2}$$

Thus, for $t$ between rebalance times, we have,

$$lnV_t = nln\alpha + lnV_0 + \sum_{i=1}^{n} ln\{(1 - \mathbf{1^T b_{i-1}})exp(r(t_i - t_{i-1})) + b_{i-1}^1 Z_{t_i}^1/Z_{t_{i-1}}^1 + ... + b_{i-1}^m Z_{t_i}^m/Z_{t_{i-1}}^m\} + ln\{(1 - \mathbf{1^T b_n}exp(r(t - t_n)) + b_n^1 Z_t^1/Z_{t_n}^1 + ... + b_n^m Z_t^m/Z_{t_n}^m\} \quad (4.3)$$

The investor's *admissible trading strategies* are sequences $\{(t_0, \mathbf{b_0}), (t_1, \mathbf{b_1}), (t_2, \mathbf{b_2}), ...\}$ where each $t_n$ is a *stopping time* and each $\mathbf{b_n} \in \mathcal{F}_{t_n}$ has strictly positive components with $\mathbf{1^T b_n} < 1$. The investment criterion is to choose an admissible strategy to maximize the asymptotic growth rate

$$\liminf E[lnV_T]/T \text{ as T } \to \infty$$

We let $R$ denote the maximum value.

In the next lemma we shall explicitly demonstrate that the *optimal* trading strategies do have the above form, i.e. the optimal trading strategies form a subset of the class of *admissible* trading strategies.

**Lemma 1** *If the price processes for the bond and stocks are almost surely finite on any compact time interval, then the transaction times $\{t_n\}$ for the optimal trading policy are almost surely increasing and any finite time interval almost surely contains only a finite number of transactions.*

*Proof.*

Let $\tau$ be the collection of transaction times. The first assertion in the proposition is trivial. Let us now prove the second part of the proposition.

Let $(\Omega, \mathcal{F}, P)$ be the underlying probability space. Thus, for $\omega \in \Omega$, the set $\tau(\omega)$ is a sequence of transaction times. Suppose there exists a time interval $[0, T]$ containing an infinite subcollection of the above set. Then, clearly the set $\tau(\omega)$ has an *accumulation point* in $[0, T]$. Therefore, there exists an increasing sequence $t_{n_k} \in \tau$ such that $t_{n_k} \to t$ where $t \in [0, T]$. Since the price processes are bounded almost surely, there exists $M < \infty$ such that $V_s < M$ for any $s \in [0, T]$. Since $V_{t_{n+}} = \alpha V_{t_n}$ for a transaction time $t_n$ the above clearly implies that $V_t = 0$. Equation (2.3) then implies that $V_s = 0$ for $s > t$ which implies that the optimal asymptotic growth rate is zero ! This contradiction proves the proposition.

Hence, the optimal trading policy is certainly an admissible trading policy and cannot involve continuous trading.

We now prove the following proposition which is analogous to the one proved by Morton and Pliska.

**Proposition 1** *Suppose* $(\tau^*, \mathbf{b}^*)$ *maximizes* $E[g(\tau, \mathbf{b})]/E[\tau]$ *over all stopping times* $\tau$ *and all strictly positive m-vectors* $\mathbf{b}$ *satisfying* $\mathbf{1^T b} < 1$. *Then*

$R = \ sup\ _{\tau,\mathbf{b}} E[g(\tau, \mathbf{b})]/E[\tau]$

*and, equivalently,*

$0 = \ sup\ _{\tau,\mathbf{b}}\{E[g(\tau, \mathbf{b})] - RE[\tau]\}$

*where* $g(\tau, \mathbf{b})$ *is the gain in log wealth over a period* $(t, t+\tau)$. *The optimal policy is to choose* $\mathbf{b_n} = \mathbf{b}^*$ *and* $t_{n+1} = t_n + \tau^*$ *for all n.*

*Proof.*

We apply the theory of semi-Markov decision processes as in Ross(1970). We can apply theorem 7.5 in Ross(1970) to conclude that for any *stationary policy,*

$$\liminf_{T\to\infty} E[lnV_T]/T = \liminf_{n\to\infty} E[lnV_{t_n}]/t_n = E[g(\tau, \mathbf{b})]/E[\tau]$$

The fact that the optimal policy is a stationary policy can be deduced from theorem 7.6 in Ross(1970) because the portfolio value process $V_t$ has stationary independent increments. The conclusions of the proposition thus follow as in the Morton-Pliska case.

We notice that the result of the above proposition depends crucially on the fact that the process $ln(V_t)$ has stationary, independent increments which is the case when the drift and volatility parameters are constants and the jump process is a continuous time branching process with the strong Markov property. In general, in the presence of time-dependent drifts and volatilities and time-inhomogenous jump processes, there exist no *stationary* optimal policies and, in fact, the optimal policy need not even be unique, so that we do not have a well-defined characterization of the optimal trading strategy of the investor. In the last section, we shall see that in the special situation where the drift is a random, *unobservable* process driven by a continuous time Markov chain, we can extend the result of the above proposition, i.e. a unique stationary optimal policy exists. We also see that the choice of the logarithmic utility function is rather special, since the process $ln(V_t)$ has stationary, independent increments in this case, so that the result of proposition 1 can be used to find stationary optimal policies. With a different utility function $\mathcal{U}$, the process $\mathcal{U}(V_t)$ need not have *stationary* independent increments, so that a *stationary* optimal policy need not exist !

# 5 The Risky Fraction Process and Stopping Time Problem

Following Morton and Pliska, we now recast our problem as an optimal stopping time problem of the *risky fraction* process $\mathbf{B}_t$. Since the number of shares of each stock held changes only when the portfolio is rebalanced,

$$(1 - \mathbf{1^T B}_\tau)V_\tau/exp(r\tau) = (1 - \mathbf{1^T b})/V_0 \tag{5.1}$$

Using the expression derived earlier for the *portfolio value process* $V_t$ and using Ito's lemma for a twice continuously differentiable function of a general semimartingale, we can derive the expression for the *risky fraction* process $\mathbf{B_t}$.

**Proposition 2** *The risky fraction process is given by*

$$\mathbf{dB_t} = \; Diag\ (\mathbf{B}_t)(1 - \mathbf{1B}_t^T)((\boldsymbol{\mu} - r\mathbf{1} - \Lambda\Lambda^{\mathbf{T}}\mathbf{B}_t)dt + \Lambda\mathbf{dW_t}) + \Delta\mathbf{B}_t$$

*Proof.*
From the preceding equation for the risky fraction process, we have

$$1 - \mathbf{1^T B_t} = (1 - \mathbf{1^T b})exp[\int_0^t (\mathbf{1^T b_s}r - \mathbf{b_s^T}\boldsymbol{\mu} + (1/2)\mathbf{b_s^T}\boldsymbol{\Lambda}\Lambda^T b_s)ds$$
$$- \int_0^t \mathbf{b_s^T}\boldsymbol{\Lambda}dW_s - \int_0^t \mathbf{b_s^T 1}dM_s + (1/2)\sum_{s<t}(\mathbf{b_s^T 1}\Delta M_s)^2] \tag{5.2}$$

Ito's lemma for a twice continuously differentiable function of a general, left continuous semimartingale $X_t$ is given by

$$F(X_t) = F(X_0) + \int_0^t F'(X_s)dX_s + (1/2)\int_0^t F''(X_s)d < X_c, X_c >_s +$$
$$\sum_{0<s<t}(F(X_{s+}) - F(X_s) - F'(X_s)\Delta X_s) \tag{5.3}$$

Therefore, we have

$$- \mathbf{dB_t} = (1 - \mathbf{1^T b})(1 - \mathbf{1^T b})^{-1}(1 - \mathbf{1^T B_t})[(\mathbf{1^T B_t}r - \mathbf{B_t^T}\boldsymbol{\mu} + (1/2)\mathbf{B_t^T}\Lambda\Lambda^{\mathbf{T}}\mathbf{B_t})dt$$
$$- \mathbf{B_t^T}\Lambda\mathbf{dW_t} - \mathbf{B_t^T 1}dM_t + (1/2)(\mathbf{B_t}\Lambda\Lambda^{\mathbf{T}}\mathbf{B_t})dt + (1/2)(\mathbf{B_t^T 1}\Delta M_t)^2] +$$
$$(1 - \mathbf{1^T b})[(1 - \mathbf{1^T B_{t+}})/(1 - \mathbf{1^T b}) - (1 - \mathbf{1^T B_t})/(1 - \mathbf{1^T b}) -$$
$$(1 - \mathbf{1^T B_t})/(1 - \mathbf{1^T b})(-\mathbf{B_t^T 1}\Delta M_t - (1/2)(\mathbf{B_t 1}\Delta M_t)^2)] \tag{5.4}$$

Therefore

$$\mathbf{dB_t} = Diag(\mathbf{B_t})(1 - \mathbf{1^T B_t})((\boldsymbol{\mu} - r\mathbf{1} - \Lambda\Lambda^{\mathbf{T}}\mathbf{B_t})dt + \Lambda\mathbf{dW_t}) + \Delta\mathbf{B_t} \tag{5.5}$$

where

$$\Delta\mathbf{B_t} = \mathbf{B_{t+}} - \mathbf{B_t}$$

This completes the derivation.

## 5.1 Infinitesimal Generator for Risky Fraction Process

The infinitesimal generator A for the risky fraction process is given by :-

$$\mathcal{A}f(\mathbf{b}) = \lim_{h\to 0} E_{t\mathbf{b}} f(\mathbf{b}(t+h) - f(\mathbf{b}(t))/(h)$$

Using the expression for the risky fraction process, it is easy to show that

$$\begin{aligned}\mathcal{A}f(\mathbf{b}) = (1/2)\textstyle\sum_i \sum_j f_{ij} b^i b^j ((\mathbf{e_i^T} - \mathbf{b^T})\Lambda\Lambda^{\mathbf{T}}(\mathbf{e_j} - \mathbf{b})) \\ + \textstyle\sum_i f_i b_i ((\mathbf{e_i^T} - \mathbf{b}^T)(\boldsymbol{\mu} - r\mathbf{1} - \Lambda\Lambda^{\mathbf{T}}\mathbf{b})) \\ + \delta \textstyle\sum_i p_i [f(\mathbf{b^i}) - f(\mathbf{b})] \quad (5.6)\end{aligned}$$

where $\mathbf{b^i}$ is the value of the risky fraction process when $M_t$ jumps to state '$i$' and $p_i$ is the transition probability of the continuous time branching process and $\delta$ is the intensity of the *Poisson process* of jump times of $M_t$.
**Note**: The possible non-zero values of $\Delta M_t$ are $\alpha_1, ..., \alpha_n$.

## 5.2 Optimal Stopping Time Problem

By the result of **Proposition 1**, we need to solve

$$0 = \sup_{\mathbf{b}}[ln\alpha + ln(1 - \mathbf{1^T b}) + f_R(\mathbf{b})] \quad (5.7)$$

where

$$f_R(\mathbf{b}) = \sup_{\tau}[-E_{\mathbf{b}}(ln(1 - \mathbf{1^T b}_\tau)) - (R - r)E_{\mathbf{b}}(\tau)] \quad (5.8)$$

We therefore see that $f_R(\mathbf{b})$ is the *value function* of the optimal stopping time problem of the Markov process $\mathbf{b_t}$ with initial value $\mathbf{b}$, *continuation fee* $R - r$ and final *reward for stopping* equal to $-ln(1 - \mathbf{1^T b}_\tau)$.

Let us now rewrite the stopping time problem in "standard form".
Let $f'_R(\mathbf{b_t}) = -f_R(\mathbf{b_t})$. Therefore,

$$f'_R(\mathbf{b}) = \inf_{\tau}[E_{\mathbf{b}}(ln(1 - \mathbf{1^T b}_\tau)) + (R - r)E_{\mathbf{b}}[\tau]] \quad (5.9)$$

Let $\mathcal{O}$ be the Polish space $(0, 1 - \epsilon)^n$ where $\epsilon$ can be any arbitrarily small but nonzero positive number. We begin by assuming that $\mathbf{b_t} \in \mathcal{O}$ for any $t \geq 0$. This assumption is motivated by mathematical reasons but we shall later show that the solution of our problem is unaffected by this assumption provided we choose $\epsilon$ sufficiently small. We shall prove this by showing that the *continuation region* for the optimal stopping problem is an open subset of $(0,1)$ and that the optimal rebalance point $b^*$ lies in the continuation region so that the values of the optimal stopping time and rebalance point are independent of $\epsilon$ provided it is chosen sufficiently small.

Let $\mathcal{C}$ be the space of uniformly continuous functions on $\mathcal{O}$. Let $\Phi(t)$ be the contraction semigroup associated with the Markov process $\mathbf{b_t}$. Then

**Proposition 3** *$f'_R$ is the maximum element of the set of functions $u$ satisfying :*

$$u \in \mathcal{C}, u \leq ln(1 - \mathbf{1^T b_t})$$

$$u \leq \int_0^t \Phi(s)(R-r)ds + \Phi(t)u, \ \textit{for any}\ t \geq 0$$

*The optimal stopping time $\tau$ is given explicitly by*

$$\tau = inf(t | f'_R(\mathbf{b}(t)) = ln(1 - \mathbf{1^T b_t}))$$

*Proof*. The set of functions $u$ satisfying the conditions of the proposition is not empty and has a maximum element by the result of theorem 5.3 on page 316 in Bensoussan(1982). This maximum element is the unique solution of the optimal stopping time problem by the result of theorem 6.1 on page 341 in Bensoussan(1982). This completes the proof.

**Corollary 1** *If there exists a twice-continuously differentiable function $h(\mathbf{b})$ satisfying :*

$$h(\mathbf{b}) \leq ln(1 - \mathbf{1^T b}) \ \textit{and}\ (\mathcal{A}h)(\mathbf{b}) \geq r - R \tag{5.10}$$

*and*

$$(h(\mathbf{b}) - ln(1 - \mathbf{1^T b}))((\mathcal{A}h)(\mathbf{b}) - r + R) = 0 \tag{5.11}$$

*where $\mathcal{A}$ is the infinitesimal generator of semigroup $\Phi(t)$ and hence the Markov process $\mathbf{B(t)}$, then $h$ is the unique solution of the optimal stopping time problem.*

*Proof.*

Since $h \in \mathcal{D}(\mathcal{A})$, we can apply Dynkin's formula to conclude that

$$\mathcal{A}h \geq r - R \implies \int_0^t \Phi(s)(R-r)ds + (\Phi(t) - 1)h \geq 0 \text{ for any } t \geq 0$$

Then, equation (5.10) and (5.11) therefore imply that $h$ satisfies the hypotheses of proposition 2 and is therefore the unique solution of the optimal stopping time problem.

In fact, in the next proposition, we shall show that if $f'_R \in C(\hat{\mathcal{O}})$, it is the *viscosity solution* of equations (5.10) and (5.11), where the notion of a *viscosity solution* is defined as follows :

**Definition 1** *$f'_R$ is a viscosity supersolution of (5.10) and (5.11) in $\mathcal{O}$ if for any $w \in \mathcal{D}(\mathcal{A})$ and $\bar{b} \in argmin\{f'_R - w\} \bigcap \mathcal{O}$ with $f'_R(\bar{b}) = w(\bar{b})$ then*

$$f'_R(b) \leq ln(1 - \mathbf{1^T b}), b \in \hat{\mathcal{O}}$$

*and*

$$f'_R(\bar{b}) < ln(1 - \mathbf{1^T \bar{b}}) \implies \mathcal{A}w(\bar{b}) \geq r - R$$

*$f'_R$ is a viscosity subsolution of (5.10) and (5.11) in $\mathcal{O}$ if for any $w \in \mathcal{D}(\mathcal{A})$ and $\bar{b} \in argmax\{f'_R - w\} \bigcap \mathcal{O}$ with $f'_R(\bar{b}) = w(\bar{b})$ then*

$$f'_R(b) \leq ln(1 - \mathbf{1^T b}), b \in \hat{\mathcal{O}}$$

*and*

$$f'_R(\bar{b}) < ln(1 - \mathbf{1^T \bar{b}}) \implies \mathcal{A}w(\bar{b}) \leq r - R$$

*$f'_R$ is a viscosity solution if it satisfies both of the above conditions.*

**Proposition 4** *If $f'_R \in C(\hat{\mathcal{O}})$, it is a viscosity solution of (22) and (23) in $\mathcal{O}$*

*Proof.*
(Viscosity supersolution) If $f'_R(\bar{b}) < ln(1 - \mathbf{1^T \bar{b}})$ then there exists $h > 0$ such that

$$w(\bar{b}) = f'_R(\bar{b}) = \int_0^{h'} \Phi(s)(R - r)ds + \Phi(h')f'_R(\bar{b}) \text{ for any } h' \leq h$$

Since $f'_R \geq w$ it follows that

$$w(\bar{b}) \geq \int_0^{h'} \Phi(s)(R - r)ds + \Phi(h')w(\bar{b}) \text{ for any } h' \leq h$$

Letting $h' \to 0$ and realizing that $w \in \mathcal{D}(\mathcal{A})$ we see that

$$\mathcal{A}w(\bar{b}) \leq r - R$$

Hence, $f'_R$ is a *viscosity supersolution*

(Viscosity subsolution) If $f'_R(\bar{b}) < ln(1 - \mathbf{1^T\bar{b}})$ then there exists $h > 0$ such that

$$w(\bar{b}) = f'_R(\bar{b}) = \int_0^{h'} \Phi(s)(R-r)ds + \Phi(h')f'_R(\bar{b}) \text{ for any } h' \leq h$$

Since $f'_R \leq w$

$$W(\bar{b}) \leq \int_0^{h'} \Phi(s)(R-r)ds + \Phi(h')w(\bar{b}) \text{ for any } h' \leq h$$

Letting $h' \to 0$ and realizing that $w \in \mathcal{D}(\mathcal{A})$ we see that

$$\mathcal{A}w(\bar{b}) \geq r - R$$

Hence, $f'_R$ is a viscosity subsolution. This completes the proof.

$f'_R$ is a *classical solution* of (5.10) and (5.11) if it is an element of $\mathcal{D}(\mathcal{A})$, i.e. it is twice-continuously differentiable in $\mathcal{O}$.

**Proposition 5** *If $f'_R \in \mathcal{D}(\mathcal{A})$, then it is a viscosity solution if and only if it is a classical solution.*

*Proof.*
Suppose $f'_R$ is a viscosity solution. Since $f'_R \in \mathcal{D}(\mathcal{A})$, $w = f'_R$ is a test function. Therefore every $\bar{b}$ is a maximizer and minimizer of $f'_R - w$. From the properties of viscosity solutions, it easily follows that (22) and (23) are satisfied at every $b \in \mathcal{O}$.

We shall now prove the converse. Suppose $w \in \mathcal{D}(\mathcal{A})$ and $\bar{b} \in \mathcal{O}$ be a maximizer of $f'_R - w$ with $f'_R(\bar{b}) = w(\bar{b})$. If $f'_R(\bar{b}) < ln(1 - \mathbf{1^T\bar{b}})$, then $\mathcal{A}f'_R(\bar{b}) = r - R$. Since $w \geq f'_R$, $\mathcal{A}w(\bar{b}) \geq r - R$. Hence, $f'_R$ is a viscosity subsolution. The viscosity supersolution property is proved similarly.

The viscosity solution is unique if it is continuous by the result of proposition 3 and corollary 1 therefore follows from the above proposition.

Since $f_R = -f'_R$, we can now conclude that

**Corollary 2** *If there exists a twice continuously differentiable function $g$ satisfying*

$$g(\mathbf{b}) \geq -ln(1 - \mathbf{1^T b}) \text{ and } (\mathcal{A}g)(\mathbf{b}) \leq R - r \tag{5.12}$$

*and*

$$(g + ln(1 - \mathbf{1^T b}))((\mathcal{A}g)(\mathbf{b}) - R + r) = 0 \tag{5.13}$$

*then*

$$f_R(\mathbf{b}) = g(\mathbf{b}) \tag{5.14}$$

*Moreover, the optimal stopping time $\tau$ is given by*

$$\tau = inf\{t \geq 0 : f_R(\mathbf{b_t}) = -ln(1 - \mathbf{1^T b_t})\} \tag{5.15}$$

*Proof.*
This follows directly from Proposition 2 and Corollary 1.

Just as in Morton and Pliska(1995), the values $b^*$ and $R$ are obtained as solutions of the following equations:

$$(\partial ln(1 - \mathbf{1^T b})/\partial b^i + \partial f_R(\mathbf{b})/\partial b^i)_{b^*} = 0, 1 \leq i \leq m \tag{5.16}$$

$$ln\alpha + ln(1 - \mathbf{1^T b^*}) + f_R(\mathbf{b^*}) = 0 \tag{5.17}$$

The continuation region $C$ is the set of values of the risky fraction process for which it is optimal not to trade.

$$C = \{b \in \mathcal{O} : f_R(b) > -ln(1 - 1^T b)\} \tag{5.18}$$

$$(\mathcal{A}f(b) = R - r \text{ for all } b \in C \tag{5.19}$$

and

$$\tau = \text{ inf } \{t \geq 0 : b_t \notin C\} \tag{5.20}$$

From equation (30) it is clear that the continuation region is an open *proper subset* of $(0,1)^n$. Hence, there exists some $\epsilon > 0$ such that $C \subset (0, 1 - \epsilon)^n$. This justifies the assumption that $b_t \in O$ by the argument in the paragraph preceding proposition 2. We shall now prove that the continuation region $C$ is not empty and contains the *generalized Merton point* $\tilde{b}$, the optimal rebalance point when trading is costless.

**Proposition 6** *For any transaction cost fraction $1 - \alpha > 0$, the continuation region $C$ is not empty and contains $\tilde{b}$, the generalized Merton point.*

*Proof.* Suppose $\tilde{b} \notin C$. Then $f_R(\tilde{b}) = -ln(1 - 1^T\tilde{b})$. Therefore

$$\mathcal{A}f_R(\tilde{b}) = (1/(2(1 - 1^T\tilde{b})^2))\tilde{b}^T(I - 1\tilde{b}^T)\Lambda\Lambda^T(I - \tilde{b}1^T)\tilde{b}$$
$$+ (1/(1 - 1^T b))\tilde{b}^T(I - 1\tilde{b}^T)(\mu - r - \Lambda\Lambda^T\tilde{b}) + \delta\sum_{i=1}^{n} p_i[f_R(\tilde{b}^i) - f(\tilde{b})] \quad (5.21)$$

Since we have assumed that "jumps" are small

$$f_R(\tilde{b}^i) - f_R(\tilde{b}) \approx Df_R(\tilde{b}).(\tilde{b}^i - \tilde{b}) + (\tilde{b}^i - \tilde{b}^T)D^2f(\tilde{b})(\tilde{b}^i - \tilde{b})$$

It is easy to see that

$$\tilde{b}^i - \tilde{b} = (1 + \alpha^i)\tilde{b}/(1 + \alpha^i 1^T\tilde{b}) - \tilde{b} = \alpha^i(\tilde{b})(1 - 1^T\tilde{b})/(1 + \alpha^i(1^T\tilde{b}))$$

The above is approximately equal to(neglecting terms higher than second order in $\alpha^i$) $\alpha^i(\tilde{b})(1 - 1^T\tilde{b}) - (\alpha^i)^2(\tilde{b})(1^T\tilde{b})(1 - 1^T\tilde{b})$.

It is now easy to see that

$$\mathcal{A}f_R(\tilde{b}) = (1 - 1^T\tilde{b})r + \tilde{b}^T\mu - (1/2)\tilde{b}^T(\Lambda\Lambda^T + 11^T\delta\Delta^2)\tilde{b} - r$$

which is equal to $\tilde{R} - r$ where $\tilde{R}$ is the asymptotic growth rate attained by $\tilde{b}$ when there are no costs. Since $R$ is the growth rate attainable with costs $R < \tilde{R}$. Hence, $\mathcal{A}f_R(\tilde{b}) > R - r$ which contradicts the fact that $\mathcal{A}f_R(\tilde{b}) \leq R - r$. This completes the proof.

# 6 Portfolio with one risky stock

We shall now carry out an explicit analytical calculation for the case where the portfolio has one risky stock. The infinitesimal generator is given by :

$$\mathcal{A}h(b) = (1/2)h''(b)\lambda^2 b^2(1 - b)^2 + h'(b)b(1 - b)(\mu - r - b\lambda^2)$$
$$+ \delta[\sum_i p_i[h(b^i) - h(b)]] \quad (6.1)$$

where $\lambda$ is the volatility and $\mu$ is the drift of the stock price process. and $\delta dt$ is the probability that a jump occurs at time '$t$'. Therefore

For simplicity, we assume that $M_t$ has two branches at every node. (The arguments can by trivially generalized to the situation where $M_t$ has any finite number of branches at every node.)

Therefore,

$$\mathcal{A}h(b) = (1/2)h''(b)\lambda^2 b^2(1-b)^2 + h'(b)b(1-b)(\mu - r - b\lambda^2) \\ + (\delta/2)p_1[h(b^1) - h(b)] + (\delta/2)p_2[h(b^2) - h(b)] \quad (6.2)$$

When $M_t$ jumps to state '1'

$$Z_t^1 \rightarrow (1+\alpha_1)Z_t^1 \quad (6.3)$$

When $M_t$ jumps to state '2'

$$Z_t^1 \rightarrow (1+\alpha_2)Z_t^1 \quad (6.4)$$

Therefore,

$$b^1 = (1+\alpha_1)b/(1+\alpha_1 b); b^2 = (1+\alpha_2)b/(1+\alpha_2 b) \quad (6.5)$$

where $p_1\alpha_1 + p_2\alpha_2 = 0$ so that $M_t$ is a martingale.

Since $\alpha_1 and \alpha_2$ are "small"

$$h(b^1) - h(b) \approx h'(b)(b^1 - b) + (h''(b)/2)(b^1 - b)^2 \quad (6.6)$$

and

$$h(b^2) - h(b) \approx h'(b)(b^2 - b) + (h''(b)/2)(b^2 - b)^2 \quad (6.7)$$

We shall now look for a twice continuously differentiable function $h(b)$ as the solution to the optimal stopping time problem by the result of corollary 2. We plug the above expressions into the expression for $\mathcal{A}h(b)$ neglecting terms of order higher than $\alpha_1^2$ and $\alpha_2^2$. In the *continuation region* for the optimal stopping time problem

$$\mathcal{A}h(b) = R - r \quad (6.8)$$

Therefore, we obtain

$$(1/2)b^2(1-b)^2(\lambda^2 + \delta\Delta^2)h''(b) + b(1-b)(\mu - r + -b\lambda^2 - \Delta^2 b)h'(b) = R - r \quad (6.9)$$

The above equation can be solved analytically and we obtain

$$h(b) = a + k(b/(1-b))^{1-2b'} + (2(R-r)/(\lambda^2+\Delta^2)(2b'-1))ln(b/(1-b)) \quad (6.10)$$

where

$$b' = (\mu - r)/(\lambda^2 + \Delta^2) \tag{6.11}$$

where $\Delta^2 = p_1\alpha_1^2 + p_2\alpha_2^2$. It is interesting to note that the above is the expression for the value function obtained by Morton and Pliska with $\lambda^2$ replaced by $\lambda^2 + \Delta^2$ which is what we expect from our earlier analysis of the generalized Merton problem. We therefore see the explicit dependence of the expression for the value function on the parameters of the credit risk process.

By the result of corollary 2, the value function $f_R$ is the smallest function $h$ in the domain of the generator $\mathcal{A}$ satisfying $\mathcal{A}h(b) \leq R - r$ and $h(b) \geq -ln(1-b)$. The continuation region $C$ is some interval $(b_l, b_u)$. Therefore, for some constants $a, k$,

$$f_R(b) = h(b) \text{ for } b \in C$$

$$f_R(b) = -ln(1-b) \text{ for } b \notin C$$

$$f_R(b_l) = -ln(1-b_l); f_R(b_u) = -ln(1-b_u)$$

$$f_R'(b_l) = -d/db(ln(1-b_l)) = 1/(1-b_l)$$

$$f_R'(b_u) = -d/db(ln(1-b_u)) = 1/(1-b_u)$$

If $b^*$ is the *optimal rebalance point*,

$$f_R'(b^*) = 1/(1-b^*)$$

and

$$ln(1-b^*) + f_R(b^*) + ln(\alpha) = 0$$

We therefore have, as in Morton's and Pliska's case, six nonlinear equations to be solved for six unknowns. In the next section , we shall use the ideas introduced by Atkinson and Wilmott to carry out an asymptotic analysis of our model with one risky asset to obtain explicit expressions for the continuation region $\mathcal{C}$ and the optimal rebalance point $b^*$.

# 7 Asymptotic Analysis of Model with one risky asset

In this section, we shall use the ideas of Atkinson and Wilmott to carry out an asymptotic analysis of the model with one risky asset in the limit when the transaction cost fraction $1-\alpha$ is "small". In the process, we obtain explicit expressions for the value function $f_R$, the continuation region $(b_l, b_u)$, and the optimal rebalance point $b^*$, and the *generalized Merton point* representing the optimal portfolio without transactions costs, but in the presence of credit risk.

Let us write

$$f_R(b) = -log(1-b) + G(b) \tag{7.1}$$

Therefore, we have

$$\mathcal{A}(f_R) = \mathcal{A}(G) + (\lambda'^2/2)(2(\mu - r)b/\lambda'^2 - b^2) \tag{7.2}$$

where

$$\lambda'^2 = \lambda^2 + \Delta^2 \tag{7.3}$$

We easily see that our results correspond exactly with those obtained by Atkinson and Wilmott with the parameter $\lambda^2$ replaced by $\lambda^2 + \Delta^2$ .

When $(\alpha = 1)$ (no transactions costs), we therefore see that $b_l$ and $b_u$ coincide with $\tilde{b}$, where

$$d/db[(\lambda'^2/2)(2(\mu - r)b/\lambda'^2 - b^2)]_{\tilde{b}} = 0$$

Thus,

$$\tilde{b} = (\mu - r)/\lambda'^2 \tag{7.4}$$

We therefore see that $\tilde{b}$ is the solution to the "Merton problem in the presence of credit risk" which we obtained in section 2. In this case, the optimal growth rate is given by

$$\tilde{R} = r + (\mu - r)^2/(2\lambda'^2) \tag{7.5}$$

Putting $\epsilon = 1 - \alpha$, we perform an analysis identical to the one by Atkinson and Wilmott to write

$$b = \tilde{b} + \epsilon^{1/4}\bar{b}$$

and

$$G(b) = \epsilon\bar{G}(\bar{b}) + o(\epsilon)$$

and

$$R = \tilde{R} + \epsilon^{1/2}\bar{R} + o(\epsilon^{1/2})$$

to obtain

$$\bar{G}(\bar{b}) = (\bar{b}^2 - \beta^2)/(12\tilde{b}^2(1 - \tilde{b}^2))$$

with

$$\bar{b}_u = -\bar{b}_l = \beta, \bar{R} = -\lambda'^2\beta^2/6$$

and

$$\beta = 3^{1/4}\sqrt{2}\tilde{b}(1 - \tilde{b})$$

Since

$$f_R(b) = -log(1-b) + \epsilon\bar{G}(\bar{b})$$

and

$$b = \tilde{b} + \epsilon^{1/4}\bar{b}$$

we now have an asymptotic solution of our model with one risky asset,i.e. in one dimension. Just as in the Morton-Pliska case, the "no-trade" region $(b_l, b_u)$ is symmetric about the *generalized Merton value* $\tilde{b}$, with a width $O(\epsilon^{1/4})$.

# 8 Optimal policies under random time-dependent drifts

in this section, we shall investigate the situation when the drift of the stock price process is a random process driven by a finite-state continuous time Markov chain $N_t$. Thus, the price processes for the bond and stocks is given by

$$dZ_t^0 = rZ_t^0 dt \tag{8.1}$$

$$dZ_t^k = \mu^k(t)Z_t^k dt + \lambda_l^k dW_t^l \tag{8.2}$$

The process $\mu_k(t)$ is a p-state continuous time Markov chain with Poisson-distributed jump times $S_n$. For simplicity, we have assumed here that the drifts for all the stocks jump at the same time so that the process $\boldsymbol{\mu}(t)$ is a *vector* continuous time Markov chain. We have also assumed the absence of credit risk, or more generally, the absence of jumps in the price processes. We can easily incorporate these situations using the results of the previous sections. Therefore, in this section, we analyze the extension of the Morton-Pliska model to the situation where the drifts of the stock prices are random and time-dependent but piecewise constant. Here , we make an important assumption about the behavior of the investor in this situation :

**Assumption**

As time unfolds, the investor is unable to derive information about the drift processes $\mu^k(t)$ from the knowledge of the price process $Z(t)$. Thus, when he enters the market, he makes the natural assumption that the drift process is distributed according to the *asymptotic* or *stationary* distribution of the Markov chain $\boldsymbol{\mu}(t)$. (of course, we need to assume here that $\boldsymbol{\mu}(t)$ does have a stationary distribution, which is very reasonable from the economic interpretation of $\boldsymbol{\mu}(t)$!. Therefore, at any time $t$

$$P(\boldsymbol{\mu}(t) = \boldsymbol{\beta}_j) = \Pi_j, j = 1...p$$

Under the above assumptions, we see that the portfolio value process has stationary, time-independent increments so that the result of proposition 1 directly extends to this situation. It is easy to see that the expressions for the value function of the optimal stopping time problem, the optimal rebalance point the time between transactions,the Merton point,etc can be obtained from the expressions we have already obtained earlier simply by replacing $\boldsymbol{\mu}$ by $\bar{\boldsymbol{\mu}} = \sum_{j=1}^{p} \Pi_j \boldsymbol{\beta}_j$

# 9 Conclusions

In this paper, we have investigated the general problem of optimal portfolio selection with fixed transactions costs in the presence of credit risk. Within the framework of the model we have proposed, the explicit results we obtain in the case when the portfolio has only one risky stock, indicate that the expressions for the value function of the optimal stopping time problem and therefore the optimal rebalance point $b*$ differ from the corresponding expressions in the Morton-Pliska case(i.e. when there is no credit risk), by replacing the parameter $\lambda^2$ by $\lambda^2 + \Delta^2$ . In the process, we see that the solution to the Merton problem generalized to the situation when there is credit risk is obtained by making the same substitutions in the expression for the "Merton point". The asymptotic analysis in the limit when transactions costs are "small" indicates that the *continuation region* $(b_l, b_u)$ is symmetric about the *generalized Merton point* when higher order terms are neglected. Just as in the Morton-Pliska case , the investor allocates his portfolio in the proportions $b*$(the optimal rebalance point) and holds the portfolio till it reaches the boundary of the *continuation region* $(b_l, b_u)$, when he makes trades to bring the portfolio back to the optimal rebalance point. The calculation of the mean time interval between transactions is, unfortunately, considerably more cumbersome than in the Morton-Pliska case, due to the discontinuities in the "risky fraction" process.

All our calculations have been carried out under the assumption that the jumps are small and the jump process is a martingale. It is straightforward to extend our analysis to the situation when the jump process is not a martingale, i.e. it has a drift. Retracing our arguments it is easy to see that this would effectively amount to replacing the parameter $\mu$ in all the expressions we have obtained by a parameter $\mu'$ in order to incorporate the effect of the drift introduced by the process $M_t$.

Finally, we have also investigated the situation when the drift of the stock price process is random and time-dependent. Under the assumption that the process is a continuous time Markov chain with a stationary distribution, we can easily extend our results to this setting by replacing the drift $\boldsymbol{\mu}$ by $\bar{\boldsymbol{\mu}} = \sum \Pi_j \boldsymbol{\beta}_j$ where $\boldsymbol{\beta}_j, j = 1, ..., p$ are the possible values of the drift process $\boldsymbol{\mu}(t)$.

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