SECTION III.

STATICS OF RIGID BODIES.

CHAPTER I.

GENERAL LAWS OF THE STATICS OF RIGID BODIES.

§ 83. *Transference of the Point of Application.*—Although every rigid body is changed in form by the action of forces upon it, *i. e.* becomes either compressed, extended, or bent, &c., it is nevertheless allowable for us to consider it for the most part as a rigid and invariable union of material points, partly because this change of form or displacement of parts is often very slight, and partly because it takes place in very short spaces of time. We shall, therefore, in the following, unless it be otherwise mentioned, regard every rigid body as a system of points, firmly connected, and we shall thereby essentially simplify the investigation.

A force P, Fig. 45, which acts upon a point A of a rigid body M, is transmitted in its proper direction $\overline{XX}$ uniformly throughout the body, and an equal and opposite force P_1 puts itself in equilibrium with it, then only when the point of application A_1 lies in the direction $\overline{XX}$ of the first force. The distance of A and A_1 is without influence on this condition of equilibrium. The two opposite forces hold themselves in equilibrium at every distance if the two points be rigidly connected. We may, therefore, assert *that the action of a force* P, Fig. 46, *remains the same at whatever point* A_1, A_2, A_3, *&c. of its direction it may be applied or may act directly upon the body.*

Fig. 45.

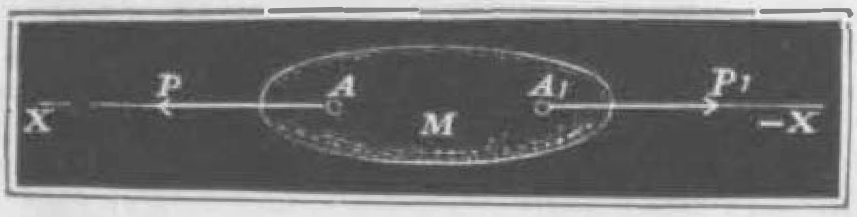
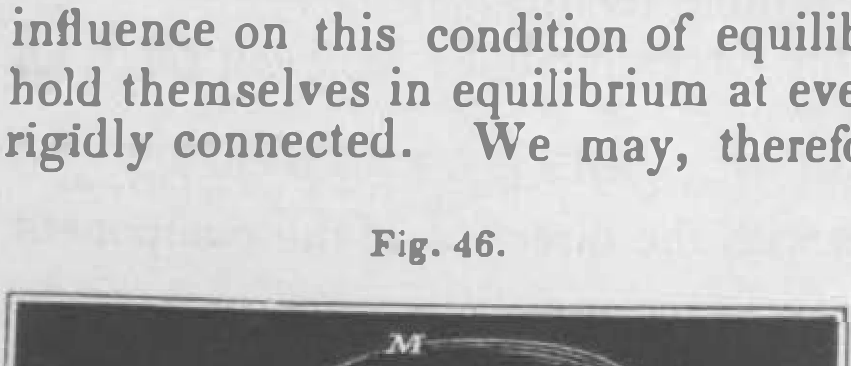
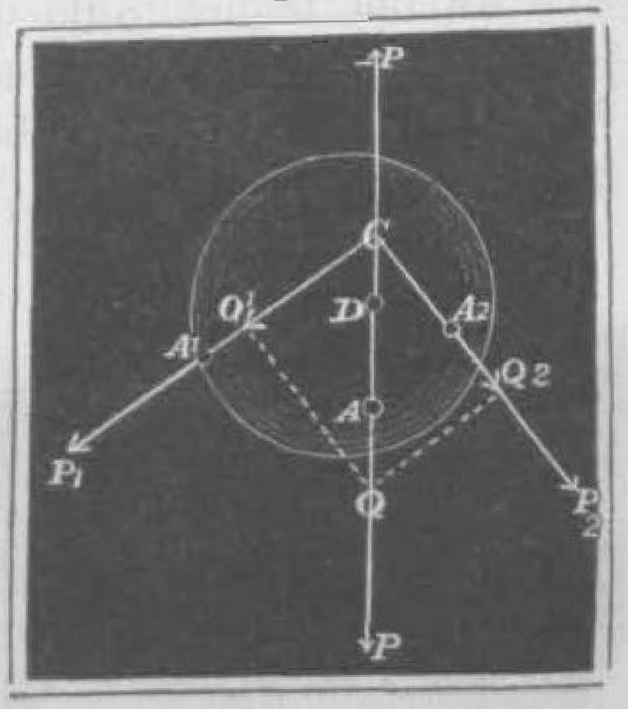


Fig. 46.



§ 84. When two forces P_1 and P_2 acting in the same plane are applied to a body at different points A_1 and A_2, their action upon the body is the same as if they had the point C, where the directions of the two forces intersect, for their common point of application, for from the proposition enunciated above, each of these points of application may be transferred to C without thereby producing any change in their effects. If, therefore, we make $CQ_1 = A_1P_1 = P_1$ and $CQ_2 = A_2P_2 = P_2$, and then complete the parallelogram $CQ_1\ QQ_2$, its diagonal will give us the resultant force $CQ = P$ of CQ_1 and CQ_2, and, therefore, also of the forces P_1 and P_2, and whose point of application may be any other point A in the direction of this diagonal.

Fig. 47.



If to the resultant force so found $AP = P$, there be put an opposite force $DP = -P$ equally great at any point D of the direction of the diagonal C, the two forces P_1 and P_2 will be thereby held in equilibrium; P_1, P_2, and $-P$ are, therefore, three forces in equilibrium.

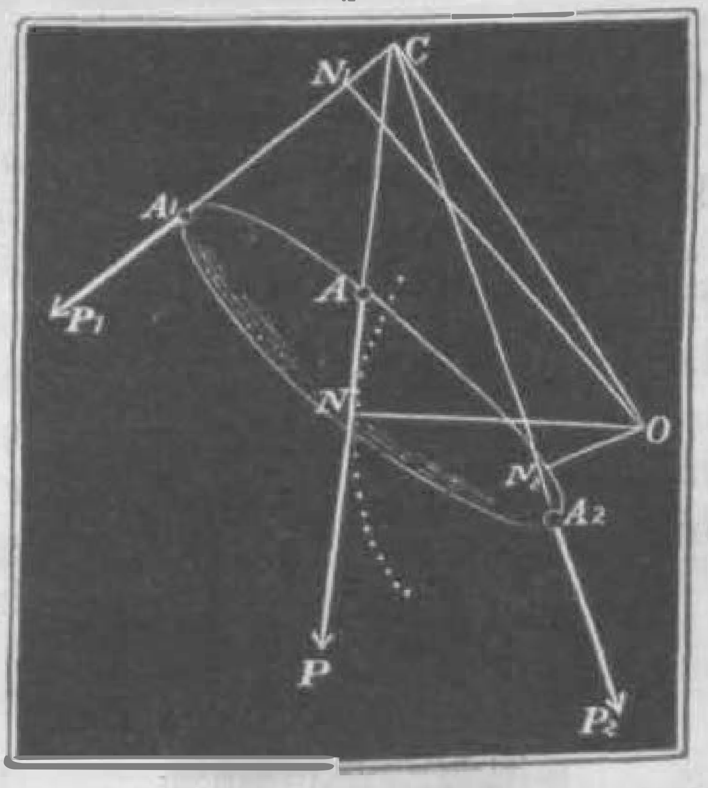
§ 85. If there be let fall from any point O, Fig. 48, in the plane of the forces perpendiculars ON_1, ON_2, and ON upon the directions of the component forces P_1 and P_2, and their resultant P, we have, according to § 79,

$$P \cdot ON = P_1 \cdot ON_1 + P_2 \cdot ON_2,$$

and the distance ON of the resultant force may be found from the perpendiculars or distances ON_1 and ON_2 of the component forces, if we put:

$$ON = \frac{P_1 \cdot ON_1 + P_2 \cdot ON_2}{P}.$$

Fig. 48.



Whilst we find the direction and magnitude of the resultant by the application of the parallelogram of forces, its position is given with the help of the last formula, by determining its distance ON.

If the prolonged direction of the forces includes between them an angle $P_1\ CP_2 = \alpha$, we then have:

1. The magnitude of the resultant $P = \sqrt{P_1^2 + P_2^2 + 2P_1\ P_2 \cos. \alpha}$.

Further, if the resultant makes with the direction of the component P_1 the angle $PCP_1 = \phi$, then:

$$2.\ \sin. \phi = \frac{P_2 \sin. \alpha}{P}.$$

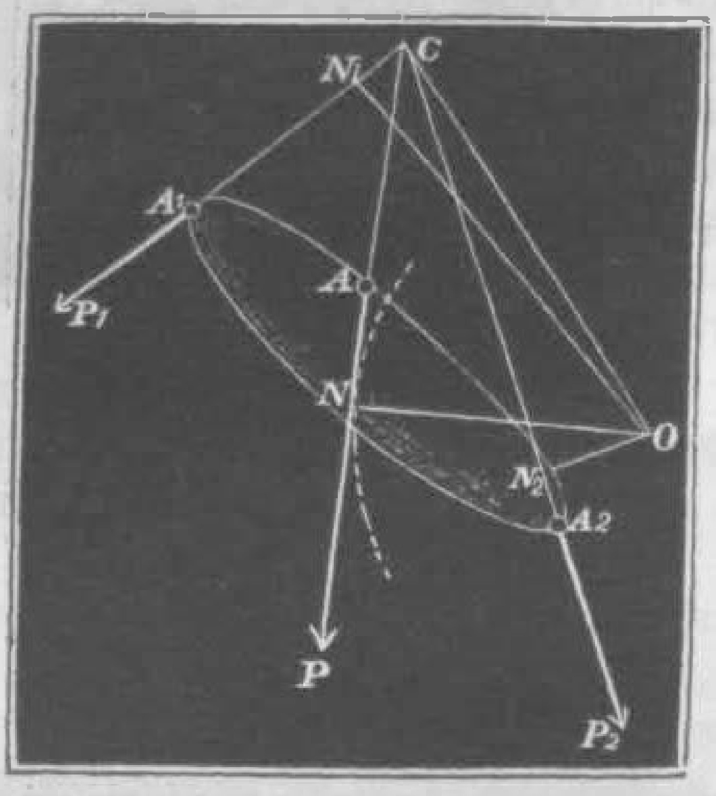
If the directions CP_1 and CP_2 of the given forces are distant $ON_1 = a_1$ and $ON_2 = a_2$ from an arbitrary point O, the distance $ON = a$ of the direction CP of the resultant from this point is:

$$3.\ a = \frac{P_1 a_1 + P_2 a_2}{P}.$$

With the help of this distance a, the position of the resultant is given without regard to the point C, if we describe a circle from O as a centre with radius a, and to this draw a tangent NP, whose direction is determined by the angle ϕ.

Example. There act upon a body the forces $P_1 = 20$ lbs. and $P_2 = 34$ lbs. whose directions meet under an angle $P_1 CP_2 = a = 70^\circ$, and are distant from a certain point $O = ● N_1 = a_1 = 4$ feet, and $O\ N_2 = a_2 = 1$ foot; what is the magnitude, direction, and position of the resultant? The magnitude of the resultant is:

Fig. 49.



$$P = \sqrt{20^2 + 34^2 + 2 \times 20 \times 34\ \cos.\ 70^\circ}$$
$$= \sqrt{400 + 1156 + 1360 \times 0{,}34202}$$
$$= \sqrt{2021{,}15} = 44{,}96 \text{ lbs.};$$ further, for its direction, $sin.\ \phi = \frac{34 \times sin.\ 70^\circ}{44{,}96}$,

Log. sin. $\phi = 9{,}8516384$, therefore, $\phi = 45^\circ\ 17'$, the angle which this resultant makes with the direction of P_1. The position finally is determined by its distance $O\ N$ from O, which is:

$$a = \frac{20 \times 4 + 34 \times 1}{44{,}96} = \frac{114}{44{,}96} = 2{,}536 \text{ feet.}$$

§ 86. The normal distances $ON_1 = a_1$, $ON_2 = a_2$, &c., of the direction of the forces from an arbitrary point O, Fig. 50, are called the arms of the forces, because they form essential elements in the theory of the lever, to be treated of subsequently. The product Pa of the force and lever arm, is called the ***statical moment*** of the force. But since $Pa = P_1 a_1 + P_2 a_2$, the statical moment of the resultant is equivalent to the sum of the statical moments of the components.

In the addition of the moments, regard must be had to the signs plus and minus. If the forces P_1 and P_2, Fig. 50, act about the point O in like directions, and if the directions of force coincide with the direction of motion of the hands of a watch, these forces, as well as

Fig. 50.

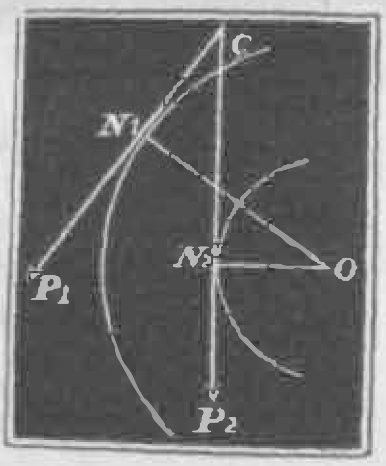
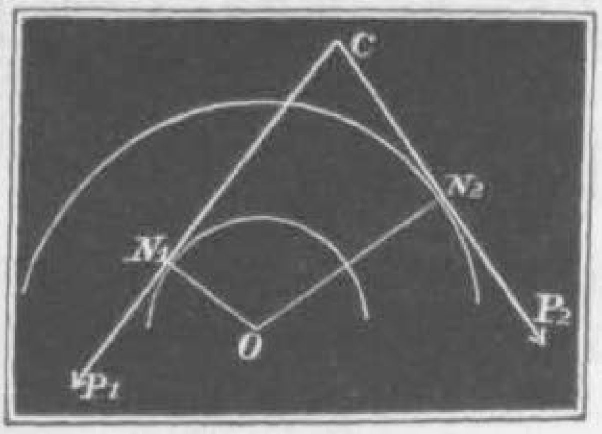
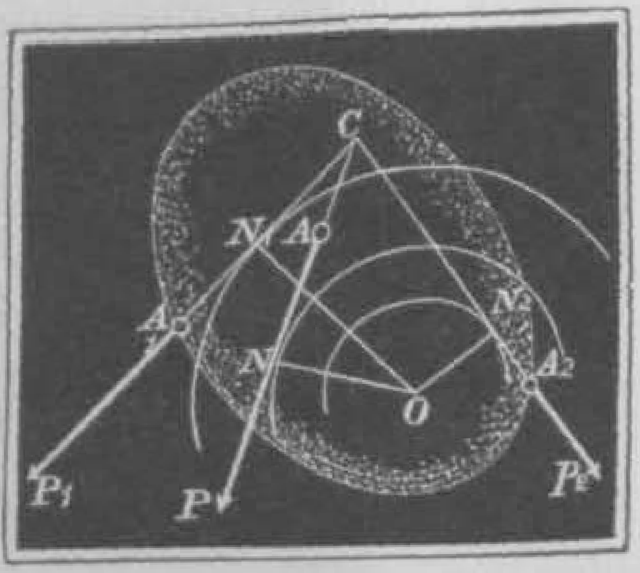


Fig. 51.



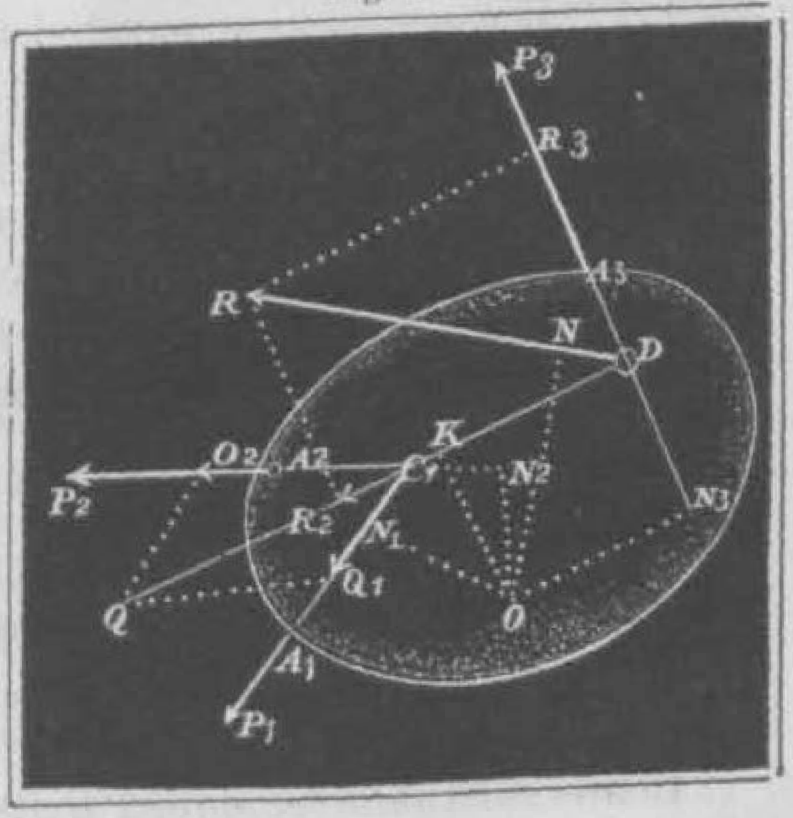
their statical moments, are said to have like signs; if the one be positive, the other must be positive likewise. If, on the other hand, Fig. 51, the directions of the forces about the point ● be opposite to each other, then the same, as well as their statical moments, are of contrary signs; if the one be negative, the other must be positive. In the composition of forces represented in Fig. 52, $Pa = P_1a_1 - P_2a_2$, because P_2 is opposed to the force P_1; its statical moment is, therefore, negative.

Fig. 52.



§ 87. *Composition of forces in a plane.*—If three forces, P_1, P_2, P_3, Fig. 53, act upon a body at the points A_1, A_2, A_3, two of these forces (P_1, P_2) by the last rule must be joined, and their resultant $CQ = Q$ found, this again joined to the third force (P_3), and the parallelogram DR_2RR_3 constructed from the forces $DR_2 = CQ$ and $DR_3 = A_3P_3$. The diagonal $DR = P$ is the required resultant of P_1, P_2, P_3. It is from this easy to see how the resultant might be found if a fourth force P_4 were to be introduced.

Fig. 53.



In this composition of the forces, the magnitude and direction of the resultant is as accurately found as if the forces acted in one single point (§ 77); the rules of calculation (§ 77) are, therefore, applicable for finding these two first elements of the resultant; but in order to find the third, viz., the position of the resultant or its line of action, we must make use of the equation between the statical moments. Here, also, $ON_1 = a_1$, $ON_2 = a_2$, $ON_3 = a_3$, and $ON = a$, are the arms of the three components P_1, P_2, P_3, and of their resultant P, with reference to an arbitrary point O. So that:

$Pa = Q \, . \, OK + P_3a_3$, and

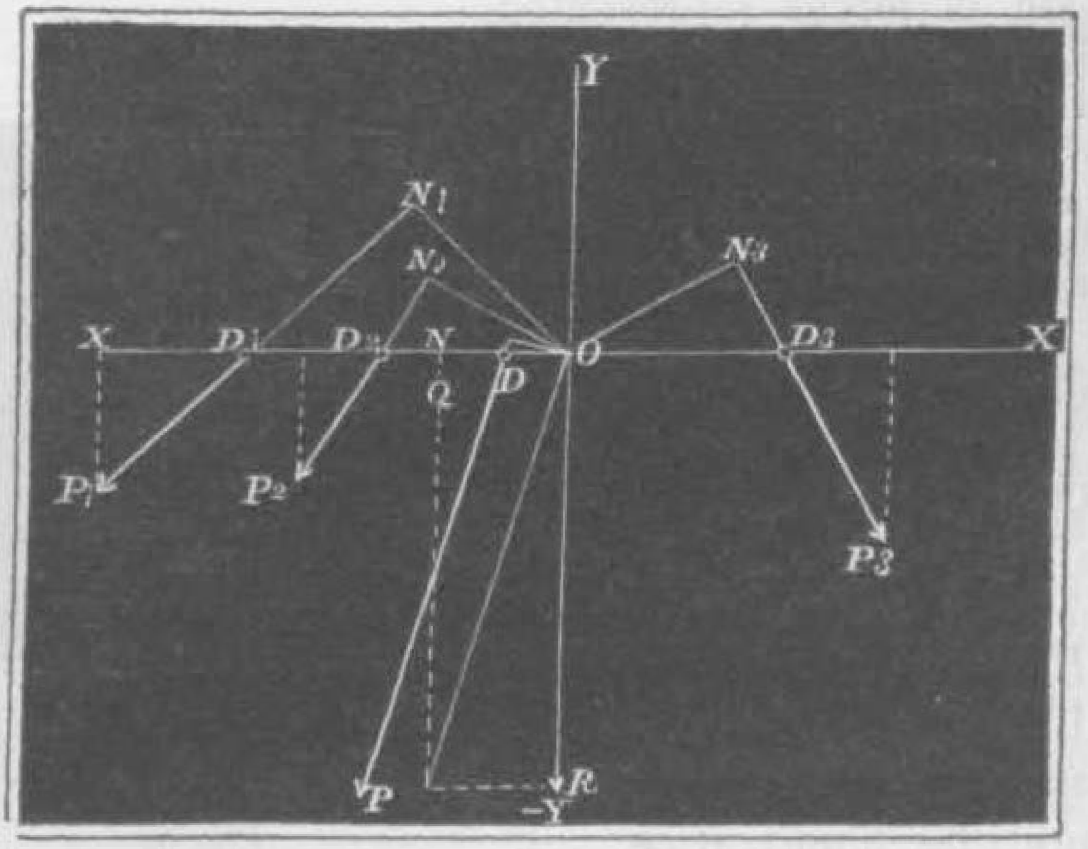
$Q \, . \, OK = P_1a_1 + P_2a_2$, provided Q is the resultant of P_1 and P_2, and OK the arm. If we combine these two equations we then obtain:

$Pa = P_1a_1 + P_2a_2 + P_3a_3$, and also for several forces:

$Pa = P_1a_1 + P_2a_2 + P_3a_3 + \ldots$, &c., *i. e.*, the (*statical*) *moment of the resultant is always equivalent to the algebraical sum of the* (statical) *moments of the components.*

§ 88. If P_1, P_2, P_3, Fig. 54, are the single forces of a system of forces; if, further, α_1, α_2, α_3, &c., are the angles P_1D_1X, P_2D_2X, P_3D_3X, &c., under which an arbitrarily chosen axis $X\overline{X}$ is intersected by the directions of force, and if a_1, a_2, a_3, designate the arms ON_1, ON_2, ON_3, of these forces with regard to the point of intersection O of both axes $X\overline{X}$ and $Y\overline{Y}$, we have from §§ 77 and 87:

Fig. 54.



1. The component parallel to the axis $X\overline{X}$:

$$Q = P_1 \cos. a_1 + P_2 \cos. a_2 + P_3 \cos. a_3 \ldots,$$

2. The component parallel to the axis $Y\overline{Y}$:

$$R = P_1 \sin. a_1 + P_2 \sin. a_2 + P_3 \sin. a_3 \ldots,$$

3. The resultant of the whole system:

$$P = \sqrt{Q^2 + R^2},$$

4. The angle which the resultant makes with the axis by

$$\text{tang. } \phi = \frac{R}{Q},$$

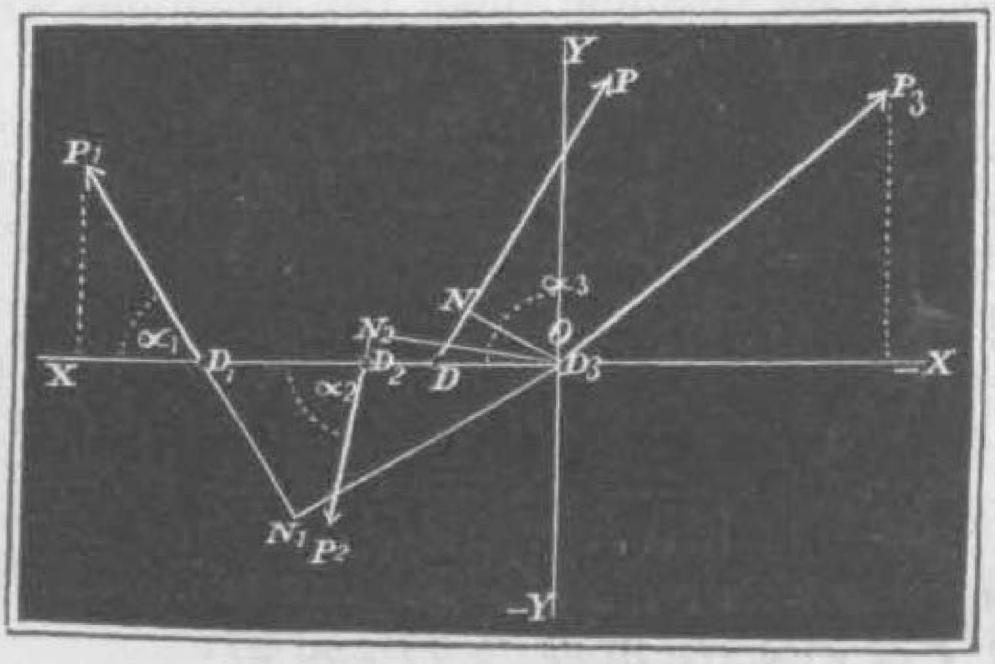
5. The arm of the resultant or the diameter of the circle to which the direction of the resultant is a tangent:

$$a = \frac{P_1 a_1 + P_2 a_2 + \ldots}{P}.$$

If this resultant be replaced by an equivalent opposite force $(-P)$, then the forces P_1, P_2, $P_3 \ldots (-P)$ are in equilibrium.

Example. The forces $P_1 = 40$ lbs., $P_2 = 30$ lbs., $P_3 = 70$ lbs., Fig. 55, intersect the axis $X\overline{X}$ at angles $\alpha_1 = 60°$, $\alpha_2 = -80°$, $\alpha_3 = 142°$, and the distances of the points of intersection D_1, D_2, D_3, of the directions of the forces with the axis: $D_1 D_2 = 4$ ft.,

Fig. 55.



and $D_2 D_3 = 5$ ft. Required the elements of the resultant. The sum of the component forces parallel to $\overline{XX}$ is:

$$Q = 40 \cos. 60^\circ + 30 \cos. (-80^\circ) + 70 \cos. 142^\circ$$
$$= 40 \cos. 60^\circ + 30 \cos. 80^\circ - 70 \cos. 38^\circ$$
$$= 20 + 5{,}209 - 55{,}161 = -29{,}952 \text{ lbs.}$$

The sum of the components parallel to $\overline{YY}$:

$$R = 40 \sin. 60^\circ + 30 \sin. (-80^\circ) + 70 \sin. 142^\circ$$
$$= 40 \sin. 60^\circ - 30 \sin. 80^\circ + 70 \sin. 38$$
$$= 34{,}641 - 29{,}544 + 48{,}096 = 48{,}193 \text{ lbs.}$$

The resultant sought is therefore:

$$P = \sqrt{Q^2 + R^2} = \sqrt{29{,}952^2 + 48{,}193^2} = \sqrt{3219{,}68} = 56{,}742 \text{ lbs.}$$

The angle ϕ, which it makes with the axis, is further determined by:

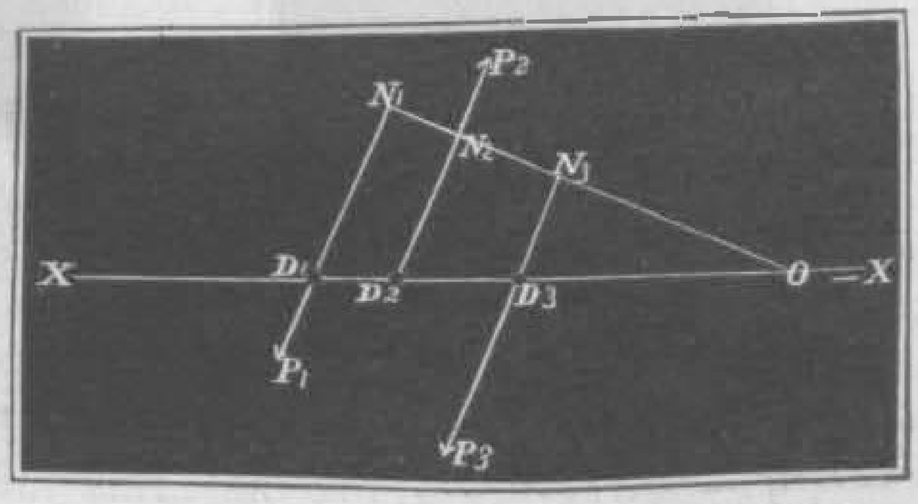
$$tang.\ \phi = \frac{R}{Q} = \frac{48{,}193}{-29{,}952} = -1{,}6090, \text{ it is therefore } \phi = 180^\circ - 58^\circ\ 8' = 121^\circ\ 52.'$$

The arm ON_1 of the force P_1 is $= OD_1 \sin. \alpha_1 = (4+5) \sin. 60^\circ = 9 \times 0.86603 = 7{,}794$ feet; the arm ON_2 of $P_2 = OD_2 \sin. \alpha_2 = 5 \sin. 80^\circ = 4{,}924$ feet; lastly, the arm ON_3 of $P_3 = O$, when the point of application O is transferred to D_3. The arm of the resultant is finally given by:

$$a = \frac{40 \times 7{,}794 - 30 \times 4{,}924}{56{,}742} = \frac{311{,}76 - 147{,}72}{56{,}742} = \frac{164{,}04}{56{,}742} = 2{,}891 \text{ feet.}$$

§ 89. *Parallel Forces.*—If the forces P_1, P_2, P_3, &c., Fig. 56, of a rigid system are parallel, the arms ON_1, ON_2, ON_3, are in the same straight line; if now we draw through the point of application O an arbitrary line $\overline{XX}$, the directions of the forces cut off the parts OD_1, OD_2, OD_3, &c., which are proportional to the arms ON_1, ON_2, ON_3, &c., because $\triangle\, OD_1N_1 \backsim \triangle\, OD_2N_2 \backsim \triangle\, OD_3N_3$. If the angle $D_1ON_1 = D_2ON_2$ be designated by α, &c., the arms ON_1, ON_2, &c., by a_1, a_2, &c., the abscisses OD_1, OD_2, &c., by b_1, b_2, &c., we then have:

Fig. 56.



$$a_1 = b_1 \cos. \alpha,\ a_2 = b_2 \cos. \alpha, \text{ \&c.}$$

If, lastly, these values be substituted in the formula:

$$Pa = Pa_1 + P_2a_2 + \ldots,$$

we then obtain:

$$Pb \cos. \alpha = P_1b_1 \cos. \alpha + P_2b_2 \cos. \alpha + \ldots,$$

or if the common factor $\cos. \alpha$ be left out:

$$Pb = P_1b_1 + P_2b_2 + \ldots$$

In every system of parallel forces it is allowable to replace the arms by the distances OD_1, OD_2, cut off from any line $\overline{XX}$. Because the magnitude and direction of the resultant is the same, the forces may act at one or at different points; hence the resultant of a system of parallel forces has the same direction with the single forces, and is equivalent to their algebraical sum.

Therefore

1. $P = P_1 + P_2 + P_3 + \ldots$ and

2. $a = \dfrac{P_1a_1 + P_2a_2 + \ldots}{P_1 + P_2 + \ldots}$, or also:

$$3.\ b = \frac{P_1 b_1 + P_2 b_2 + .s.\ .}{P_1 + P_2 + \ldots s}$$

Example. The forces $P_1 = 12$ lbs, $P_2 = -32$ lbs., $P_3 = 25$ lbs., and their directions intersect a straight line at the points D_1, D_2, and D_3, Fig. 56, whose distances from each other are $D_1D_2 = 21$ inches, $D_2D_3 = 30$ inches; required the resultant. The magnitude of this force is $P = 12 - 32 + 25 = 5$ lbs., its distance D_1O from D_1, is therefore:

$$b = \frac{12 \times 0 - 32 \times 21 + 25 \times (21 + 30)}{5} = \frac{0 - 672 + 1275}{5} = 120{,}6 \text{ inches.}$$

§ 90. *Couples.*—Two parallel, equal and opposite forces, P_1 and $-P_1$, Fig. 57, have the resultant

$P = P_1 + (-P_1) = P_1 - P_1 = 0$, with the arm

$$a = \frac{P_1 a_1 + P_2 a_2}{0} = \infty .$$

Fig. 57.

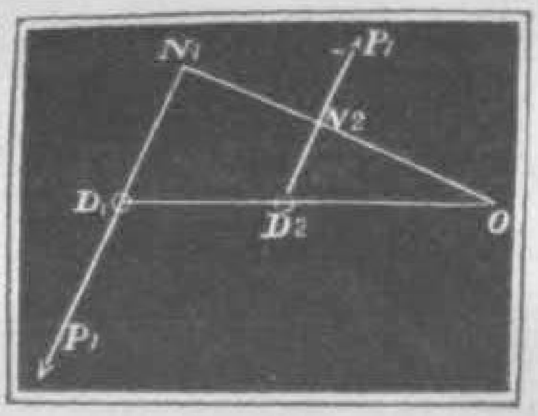
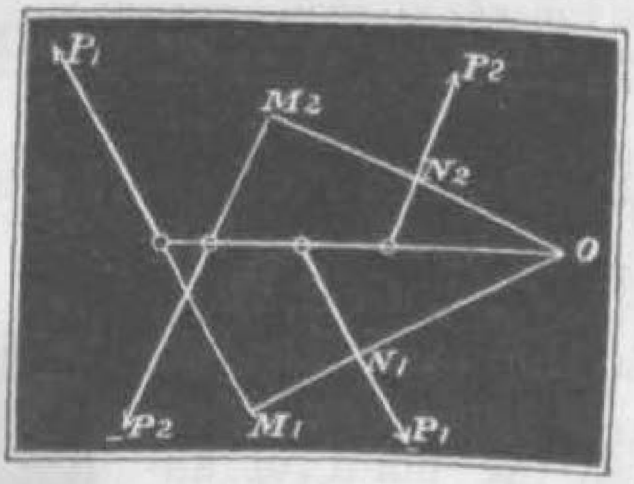


Fig. 58.



For restoring equilibrium to such a couple, according to this, a single finite force P acting at a finite distance, is not sufficient, but two such couples may easily hold each other in equilibrium. If P_1 and $-P_1$ and $-P_2$ and P_2, Fig. 58, are two such couples, and $OM_1 = a_1$, $ON_1 = OM_1 - M_1N_1 = a_1 - b_1$; if further, $OM_2 = a_2$ and $ON_2 = OM_2 - M_2N_2 = a_2 - b_2$ are the arms taken from a certain point O, we have for equilibrium:

$$P_1 a_1 - P_1 (a_1 - b_1) - P_2 a_2 + P_2 (a_2 - b_2) = 0, \text{ i.e.}$$

$$P_1 b_1 = P_2 b_2.$$

Two such couples are, therefore, in equilibrium if the product of one force, and its distance from the opposite force, are as great in the one couple as in the other.

A pair of equal opposite forces is called simply a *couple*, and the product of one of the forces and its normal distance from the other force, the *moment of the couple.* From the above, two couples acting in opposite directions are in equilibrium, if they have equal moments.

If we substitute in the formula (§ 87) for the arm a of the resultant:

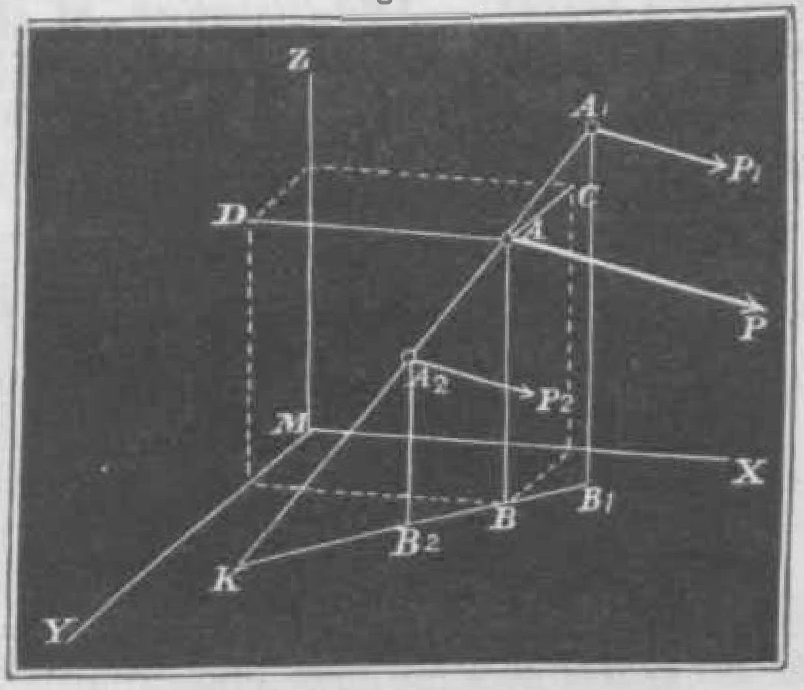
$$a = \frac{P_1 a_1 + P_2 a_2 + \ldots}{P}$$

$P = 0$ without the sum of the statical moments becoming null, we obtain likewise $a = \infty$, a proof that in this case also there is no resultant, but only a couple, possible.

Example. If a couple consists of the forces $P_1 = 25$ lbs., and $-P_1 = -25$ lbs., and another of the forces $-P_2 = -18$ lbs., and $P_2 = 18$ lbs.; if, lastly, the normal distance of the first pair $= 3$ feet for the condition of equilibrium, the normal distance of the second must amount to $= \frac{25 \times 3}{18} = 4\frac{1}{6}$ feet.

§ 91. *Centre of Parallel Forces.*—If the parallel forces lie in different planes, their union may be effected in the following manner. If the straight line $A_1 A_2$, Fig. 59, which unites the points of application of two parallel forces P_1 and P_2, be prolonged to the plane XY between the rectangular axes MX, MY, and if the point of intersection K be taken for the initial point, we shall in this manner obtain for the point of application A of the resultant $(P_1 + P_2)$ of these forces,

Fig. 59.



$$(P_1 + P_2) . KA = P_1 . KA_1 + P_2 . KA_2.$$

As now B, B_1, and B_2 are the projections of the points of application A, A_1, A_2, on the plane XY, we have:

$AB : A_1B_1 : A_2B_2 = KA : KA_1 : KA_2$, and therefore also

$(P_1 + P_2) AB = P_1 . A_1 B_1 + P_2 . A_2 B_2$.

If we designate by z_1, z_2, z_3, &c., the normal distances A_1B_1, A_2B_2, A_3B_3, &c., of the points of application from the principal plane XY, and by w_1 that of the point A from the same plane, we have for the two forces:

$(P_1 + P_2) w_1 = P_1 z_1 + P_2 z_2$; for three or more, and generally

$(P_1 + P_2 + P_3 + \ldots) w = P_1z_1 + P_2z_2 + P_3z_3 \ldots$ Consequently

$$1.\ w = \frac{P_1z_1 + P_2z_2 + \ldots}{P_1 + P_2 + \ldots}.$$

If we put likewise the distances AC and AD of the point of application of the resultant from the planes XZ and $YZ = v$ and u, we then obtain:

$$2.\ v = \frac{P_1y_1 + P_2y_2 + \ldots}{P_1 + P_2 \ldots} \text{ and}$$

$$3.\ u = \frac{P_1x_1 + P_2x_2 + \ldots}{P_1 + P_2 + \ldots}$$

The three distances u, v, w from the principal planes, as for example, from the floor and the two side walls of a room, fully determine the point A, for it is the eighth terminating point of the parallelopiped, constructed from u, v, w, consequently, in such a system there is but one single point of application of the resultant.

As the three formulæ for u, v, w, do not contain the angles which the forces make with the principal planes, the point of application is independent of these forces, and also of their directions; the whole system admits, therefore, of being turned about this point without its ceasing to be the point of application, provided only that in this turning the parallelism of the forces be preserved.

In such a system of parallel forces the product of a force, and the distance of its point of application from a plane or line, is called the moment of the force with reference to this plane or line, and generally, the point of application of the resultant is called the *centre of parallel forces*. The distance of the centre of a system of parallel forces from any plane or line whatever, (the latter when the forces lie in the same plane,) is obtained, when the sum of the moments is divided by the sum of the forces.

						SUM.
Example.	If the forces are	P_n	5	—7	10	3 lbs.
	the distances	x_n	1	2	0	9 ft.
	" "	y_n	2	4	5	3 "
	" "	z_n	8	3	7	10 "
	The moments are	$P_n x_n$	5	—14	0	36 ft. lbs.
	" "	$P_n y_n$	10	—28	50	12 "
	" "	$P_n z_n$	40	—21	70	40 "

Now, if the sum of the forces $= 19 - 7 = 12$ lbs., the distances of the central point of this system from the three principal planes are consequently:

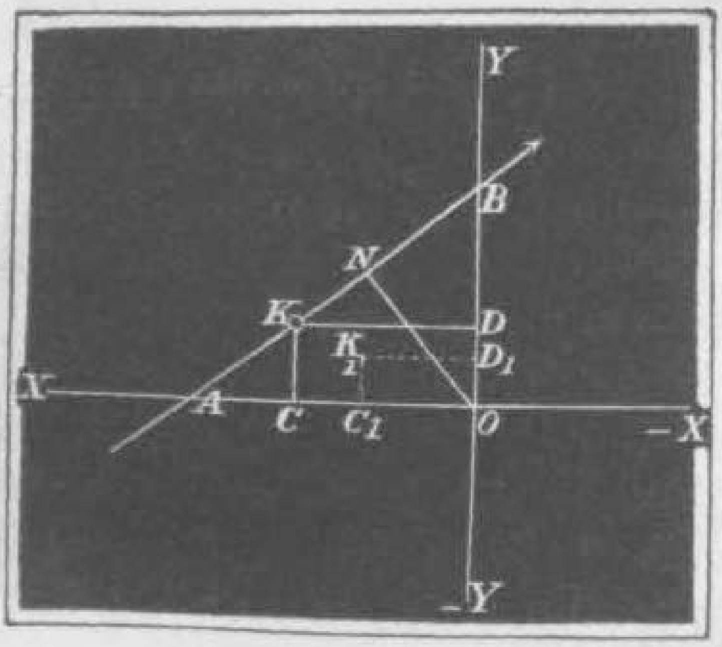
$$u = \frac{5+36-14}{12} = \frac{27}{12} = \frac{9}{4} = 2{,}25 \text{ feet};$$

$$v = \frac{10+50+12-28}{12} = \frac{44}{12} = \frac{11}{3} = 3{,}66\ldots \text{feet};$$

$$w = \frac{40+70+40-21}{12} = \frac{129}{12} = \frac{43}{3} = 10{,}75 \text{ feet.}$$

§ 92. *Forces in Space.*—If it be required to unite a system constituted of differently directed forces, a plane must be carried through the system, the different points of application transferred to this plane, and each force resolved into two component forces, the one coinciding with the plane, the other at right angles to it. If $\beta_1, \beta_2 \ldots$ are the angles under which the plane is intersected by the directions of the forces, then the normal forces are $P_1 \sin. \beta$, $P_2 \sin. \beta \ldots$, and those in the plane $P_1 \cos. \beta_1$, $P_2 \cos. \beta_2$, &c. The latter from § 88, and the former from the last § 91 may be combined to a resultant. In general, the directions of both resultants will nowhere intersect each other, and accordingly a composition of these is impossible, but if the resultant of parallel forces passes through a point K, Fig. 60, in the direction AB of the resultant of the forces in the plane (the plane of the paper) a composition is then possible. If we put the distances $OC = DK = u$, and $OD = CK = v$ for the point of application of the first resultant, on the other hand the arm ON of the second $= a$, and the angle BAO, at which it intersects the axis $\overline{XX} = \alpha$, the condition for the possibility of a composition is:

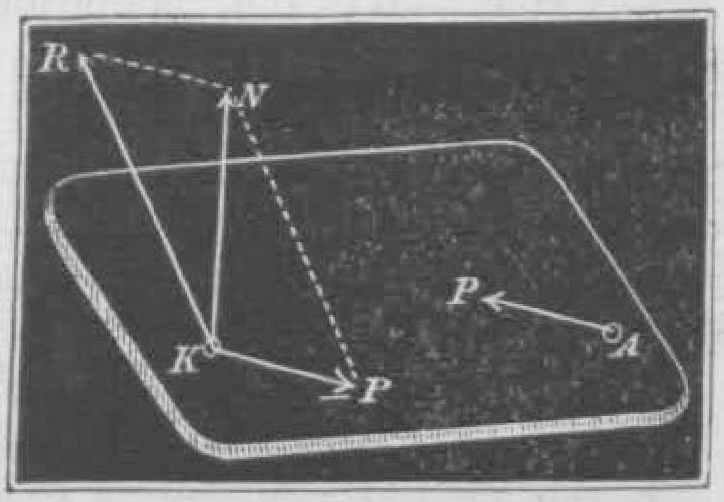
Fig. 60.



$$u \sin. \alpha + v \cos. \alpha = a.$$

If this equation is not satisfied, if, for example, the resultant of the normal forces passes through K_1, the reduction of the whole system of forces to a resultant is then impossible, but it readily admits of being reduced to a resultant R, Fig. 61, and a couple $P, -P$, if the resultant $\mathcal{N}$ of the parallel components is resolved into the forces $-P$ and R, of which the one is equal, and directed parallel and opposite to the resultant P of the forces in the plane.

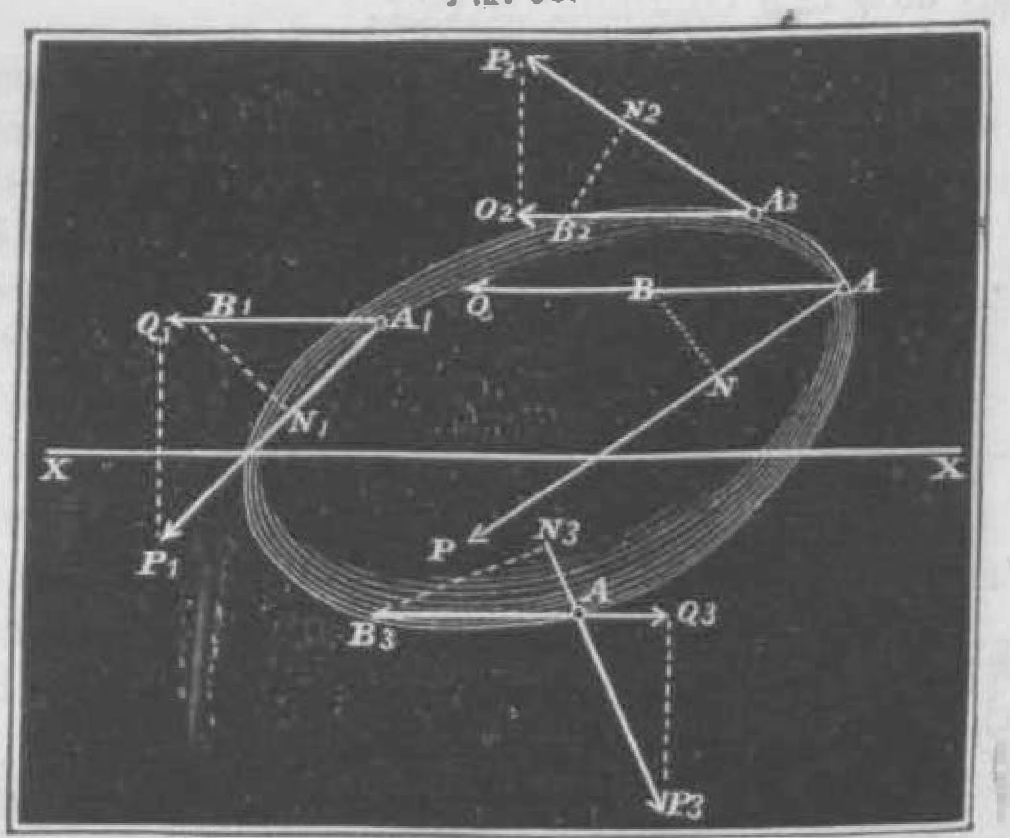
Fig. 61.



§ 93. *Principle of Virtual Velocities.*—If a system of forces P_1, P_2, P_3, acting in a plane, Fig. 62, is progressive, *i. e.* moves forward so that all the points of application $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$, . . . pass through equal parallel spaces $\mathcal{A}_1 B_1$, $\mathcal{A}_2 B_2$, $\mathcal{A}_3 B_3$, the effect of the resultant (in the sense of § 80) is equivalent to those of the components, and in a state of equilibrium therefore $= 0$. If the projections $\mathcal{A}_1 \mathcal{N}_1$, $\mathcal{A}_2 \mathcal{N}_2$, &c., coinciding with the directions of the forces of the common spaces $\mathcal{A}_1 B_1, = \mathcal{A}_2 B_2$, &c., $= s_1, s_2$, then the mechanical effect of the resultant is:

$$Ps = P_1 s_1 + P_2 s_2 + \ldots$$

Fig. 62.



This law follows from one of the formulæ of § 88, according to which the component of the resultant running parallel with the axis XX is equal to the sum $Q_1 + Q_2 + \ldots$ &c., of the similarly running components of the forces P_1, P_2; now from the similarity of the triangles $\mathcal{A}_1 B_1 \mathcal{N}_1$ and $\mathcal{A}_1 P_1 Q_1$, there follows the proportion

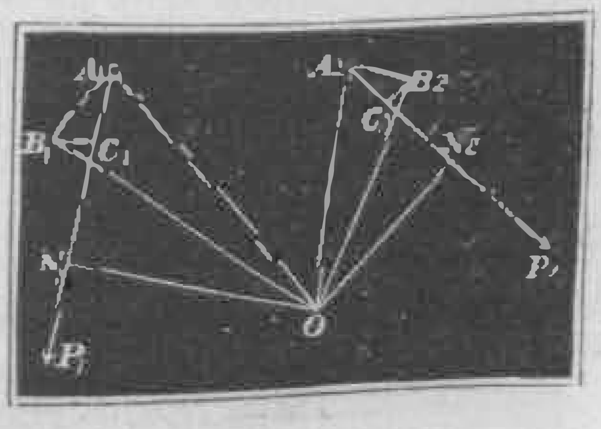
$$\frac{Q_1}{P_1} = \frac{\mathcal{A}_1 \mathcal{N}_1}{\mathcal{A}_1 B_1} = \frac{s_1}{\mathcal{A}B}, \text{ and from this:}$$

$$Q_1 = \frac{P_1 s_1}{\mathcal{A}B},\ Q_2 = \frac{P_2 s_2}{\mathcal{A}B}, \text{ \&c.,}$$

we may, therefore, in place of

$$Q = Q_1 + Q_2 + \ldots \text{ put}$$
$$Ps = P_1 s_1 + P_2 s_2 + \ldots$$

Fig. 63.



§ 94. If the system of forces P_1, P_2, &c., Fig. 63, be made to revolve a very little about the point O, the law of the principle of virtual velocities enunciated above in § 80 and § 93 holds equally good, as may be proved in the following manner. From § 86 the moment $P \cdot ON$ of the resultant is equivalent to the sum of the moments of the components, so that:

$$Pa = P_1 a_1 + P_2 a_2 + \ldots n$$

The space $A_1 B_1$ corresponding to a revolution through the small angle $A_1 OB_1 = \phi^0$ or the arc $\phi = \frac{\phi^0}{180} \cdot \pi$, is perpendicular to the diameter OA_1, therefore, the triangle $A_1 B_1 C_1$, which is formed if a perpendicular line $B_1 C_1$ be let fall on the direction of the force, is similar to the triangle $OA_1 N_1$ determined by the arm $ON_1 = a_1$, and accordingly

$$\frac{ON_1}{OA_1} = \frac{A_1 C_1}{A_1 B_1}$$

If the virtual velocity $A_1 C_1 = \sigma_1$ and the arc $A_1 B_1 = OA_1 \cdot \phi$, we then obtain:

$$a_1 = \frac{OA_1 . \sigma_1}{OA_1 . \phi} = \frac{\sigma_1}{\phi}, \text{ also } a_2 = \frac{\sigma_2}{\phi}, \text{ \&c.}$$

If these values be substituted in the above equation for a_1, a_2, we then have

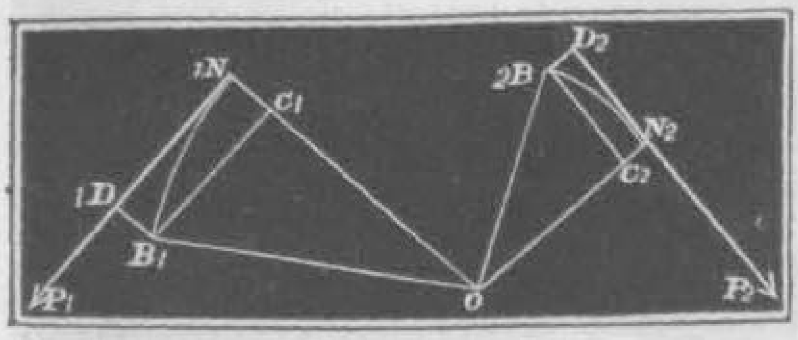
$$\frac{P\sigma}{\phi} = \frac{P_1 \sigma_1}{\phi} + \frac{P_2 \sigma_2}{\phi} + \ldots \text{ \&c.},$$

or, as ϕ is a common divisor,

$$P\sigma = P_1 \sigma_1 + P_2 \sigma_2 + \ldots \text{ \&c., the same as in § 80.}$$

So that, for small revolutions the mechanical effect $(P\sigma)$ of the resultant is equivalent to the sum of the mechanical effects of the components.

Fig. 64.



§ 95. The principle of virtual velocities holds likewise for arbitrarily great revolutions, if instead of the virtual velocities of the points of application, the projections $N_1 D_1$, $N_2 D_2$, &c., Fig. 64, of the spaces commencing at the points N_1, N_2 be introduced, and their values

$$B_1 C_1 = OB_1 \; sin. \; N_1 OB_1 = a_1 \; sin. \; \phi,$$
$$B_2 C_2 = OB_2 \; sin. \; N_2 OB_2 = a_2 \; sin. \; \phi, \text{ \&c.},$$

be substituted for σ_1, σ_2, we then obtain

$$Pa \; sin. \; \phi = P_1 a_1 \; sin. \; \phi + P_2 a_2 \; sin. \; \phi \ldots n +, \text{ or, dividing by } sin. \; \phi,$$

$$Pa = P_1 a_1 + P_2 a_2 + \ldots,$$

the known equation for statical moments.

This principle is correct also for finite revolutions, if the directions of the forces revolve simultaneously with the system, or if, while the point of application incessantly changes, the arm $ON_1 = OB_1$ remains invariable, then from

$$Pa = P_1 a_1 + P_2 a_2,$$

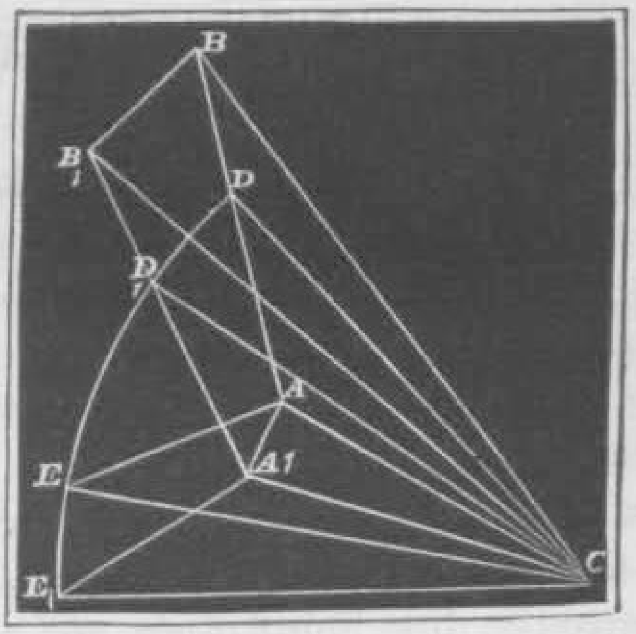
and multiplying by ϕ, we have

$$Pa\phi = P_1 a_1 \phi + P_2 a_2 \phi + \ldots, \text{ i. e.}$$
$$P\sigma = P_1 \sigma_1 + P_2 \sigma_2 + \ldots,$$

if σ, σ_1, σ_2, &c., designate the circular arcs, $N_1 B_1$, $N_2 B_2$, &c., of the points N, N_1, &c.

§ 96. Every small motion or displacement of a body in a plane may be regarded as a small revolution about a movable centre, and may be proved in the following manner. Let two points A and B, Fig. 65, of this body (this surface or line) be advanced by a small motion to A_1 and B_1, let also $A_1B_1 = AB$. If at these points we draw perpendiculars to the small spaces described AA_1 and BB_1, they will intersect at a point C, from which as a centre AA_1 and BB_1 may be considered the circular arcs described. Now from the equalities $AB = A_1B_1$, $AC = A_1C$, and $BC = B_1C$, the triangles ABC and A_1B_1C are equal, therefore, also the $\angle B_1CA_1 = \angle BCA$ and the $\angle ACA_1 = \angle BCB_1$. If we make $A_1D_1 = AD$, we obtain from the equality of the $\angle$s D_1A_1C and DAC, and from that of the sides CA_1 and CA in CA_1D_1 and CAD, again two congruent triangles in which $CD_1 = CD$, and $\angle A_1CD_1 = \angle ACD$. Consequently any arbitrary point D in AB, by its small advancement, describes a circular arc DD_{12} If lastly E be any point without the line AB, and rigidly connected with it, the small space EE_1 may be regarded as the arc of a circle from C as a centre, for if we make the $\angle E_1A_1B_1 = EAB$ and the distance $A_1E_1 = AE$, we again obtain two congruent triangles E_1A_1C and EAC with equal sides CE_1 and CE, and equal $\angle$s A_1CE_1 ACE, and the same may be shown for every other point rigidly connected with AB. We may consequently regard every small motion of a surface rigidly connected with AB, or of a rigid body, as a small revolution about a centre, which is given when the point of intersection C is determined, in which the perpendiculars to the paths AA_1 and BB_1 of the two points of the body intersect each other.

Fig. 65.



§ 97. From § 94, for a small revolution of a system of forces, the mechanical effect of the resultant is equivalent to the algebraical sum of its components; from § 95, every small displacement may be regarded as a small revolution: hence the law of the principle of virtual velocities above enunciated is, therefore, applicable to every small motion of a rigid body or system of forces.

If equilibrium obtain in a system of forces, *i. e.* if the resultant be null, the sum of the mechanical effects must be also null for a small arbitrary motion. If inversely for a small motion of the body, the sum of the effects be null, equilibrium does not from this necessarily follow; the sum for all possible small displacements must be $= 0$, if equilibrium is to take place. Since the formula expressing the law of virtual velocities only fulfils one condition of equilibrium, it is requisite for equilibrium that this law be satisfied, at least for as many motions as can be made from these conditions for example, in a system of forces in a plane, for the three motions independent of each other.

CHAPTER II.

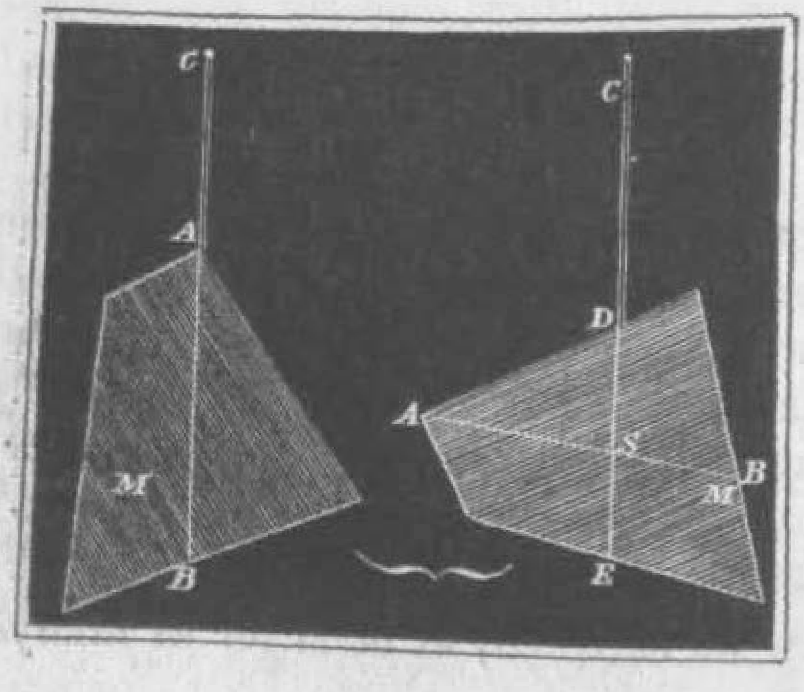
CENTRE OF GRAVITY.

§ 98. *Centre of Gravity.*—The weights of the parts of a heavy body form a system of parallel forces, whose resultant is the weight of the whole, and whose centre may be determined from the three formulæ of § 91. This middle point of a body or system of bodies is called the centre of gravity, and also the centre of the mass of the body or system of bodies. If a body be turned about its centre of gravity, this point does not cease to be the central point of gravity, for if the three planes, to which the points of application of the separate weights are referred, revolve at the same time with the body, the position of the directions of force to these planes alone changes by this revolution, the distances of the points of application from these planes remain invariable. The centre of gravity is, therefore, that point of a body in which its weight acts vertically downwards, and which must be, therefore, supported, and fixed, in order that in every position the body may remain at rest.

§ 99. Every vertical straight line in which this point lies is called the *line of gravity;* and every plane passing through the centre of gravity, a *plane of gravity.* The centre of gravity is determined by the intersection of two lines of gravity, or that of a line of gravity and a plane of gravity, or by the intersection of the planes of gravity.

Since the point of application may be displaced at will in the direction of force, without changing the action of the force, so a body is in any position in equilibrium if a point in the vertical line passing through the centre of gravity is fixed.

Fig. 66.



If a body M, Fig. 66, be suspended by a thread CA, in its prolongation AB we have a line of gravity, and if it be similarly suspended by a second line, we get a second line of gravity DE. The intersection S of both lines is the centre of gravity of the body. If the body be suspended upon an axis, or be brought upon a sharp edge (knife edge) into a state of equilibrium, we shall obtain in the vertical plane passing through the axis, or through the knife edge, a plane of gravity, &c. Experimental determinations of the centre of gravity, as just pointed out, are rarely applicable; we have generally to make use of geometrical rules, which will presently be given for the determination of this point with accuracy.

In many bodies, for example, in rings, the centre of gravity falls without the mass of the body. If such a body is to be fixed in its centre of gravity, it is necessary to connect a second body with the first, in such a manner that the centres of gravity of both may coincide.

§ 100. ***Determination of the Centre of Gravity.***—If x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3, &c., be the distances of the parts of a heavy body from the three planes xz, yz, xy, and the weights of these parts be P_1, P_2, P_3, &c., we then have the distances of the centre of gravity from these three planes,

$$x = \frac{P_1x_1 + P_2x_2 + P_3x_3 + \ldots}{P_1 + P_2 + P_3 + \ldots},$$

$$y = \frac{P_1y_1 + P_2y_2 + P_3y_3 + \ldots}{P_1 + P_2 + P_3 + \ldots},$$

$$z = \frac{P_1z_1 + P_2z_2 + P_3z_3 + \ldots}{P_1 + P_2 + P_3 + \ldots}.$$

If the volumes of these parts be V_1, V_2, V_3, &c., and their densities γ_1, γ_2, γ_3, &c., we may put therefore

$$x = \frac{V_1\gamma_1x_1 + V_2\gamma_2x_2 + \ldots}{V_1\gamma_1 + V_2\gamma_2 + \ldots}.$$

If the body be homogeneous, *i. e.* all parts of the same density γ, then:

$$x = \frac{(V_1x_1 + V_2x_2 + \ldots)\gamma}{(V_1 + V_2 + \ldots)\gamma},$$

or since the common factor γ above and below is cancelled:

$$1.\quad x = \frac{V_1x_1 + V_2x_2 + \ldots}{V_1 + V_2 + \ldots},$$

$$2.\quad y = \frac{V_1y_1 + V_2y_2 + \ldots}{V_1 + V_2 + \ldots},$$

$$3.\quad z = \frac{V_1z_1 + V_2z_2 + \ldots}{V_1 + V_2 + \ldots}.$$

We may also, instead of the weights, substitute the volumes of the separate parts, and thereby make the determination of the centre of gravity a problem of pure geometry.

When bodies are a little extended in one or in two dimensions, as thin plates, fine wires, &c., they may be regarded as surfaces or lines, and their centres of gravity likewise determined with the help of the three last formulæ, if for the volumes V_1, V_2, the arms or lengths be substituted.

§ 101. In regular figures the centre of gravity coincides with the centre of figure, as in dice, cubes, spheres, equilateral triangles, circles, &c. Symmetric figures have their centre of gravity in the plane or axis of symmetry. The plane of symmetry $ABCD$ divides a body $ADFE$, Fig. 67, into two congruent halves; the portions on both

Fig. 67.

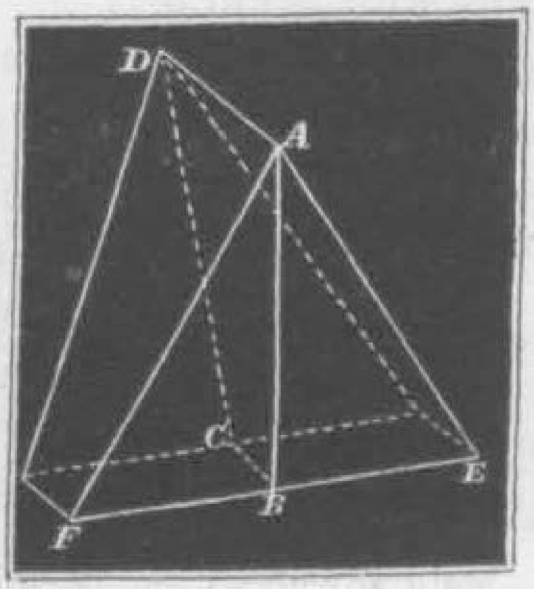
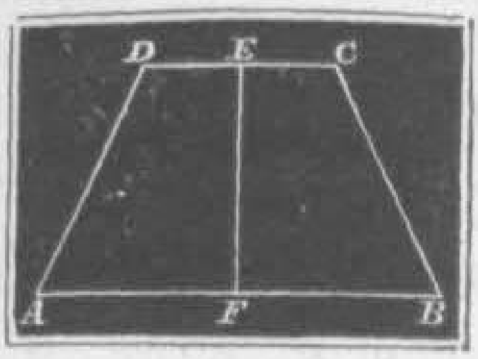


Fig. 68.



sides of this plane are equal; the moments also on the one side are equal to those on the other, and, consequently, the centre of gravity falls within this plane. Because the axis of symmetry EF cuts the plane surface $ABCD$, Fig. 68, into two congruent parts, here the portions on the one side are equal to those on the other; the moments also on both sides are equal, and the centre of gravity of the whole lies in this line. Lastly, the axis of symmetry KL of a body $ABGH$, Fig. 69, is its line of gravity, because it arises from the intersection

Fig. 69.

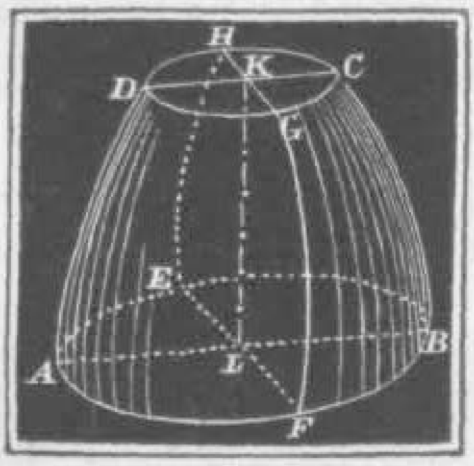
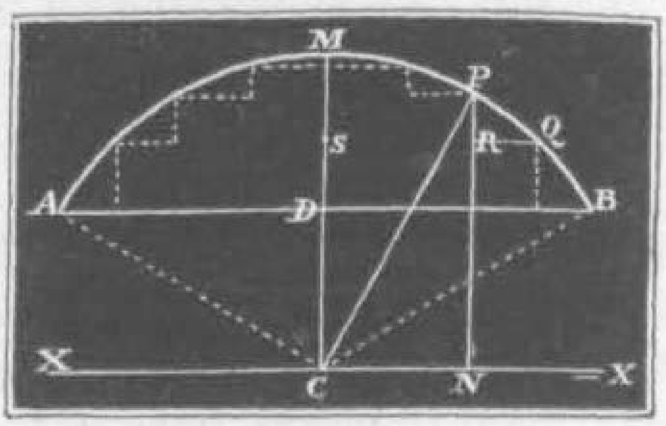


Fig. 70.



of two planes of symmetry, $ABCD$ and $EFGH$. For this reason, the centre of gravity of a cylinder, of a cone, and of a surface of revolution, or of a rotating body formed on the potter's wheel, lies in the axis of these bodies.

§ 102. *Centre of Gravity of Lines.*—The centre of gravity of a straight line lies in its middle.

The centre of gravity of a circular arc $AB = b$, Fig. 70, lies in the diameter CM, and passes through the middle M of the arc, for this diameter is the axis of symmetry of this arc. But in order to find the distance $CS = x$ of the centre of gravity S from the middle point, or *centre of the circle*, the arc must be divided into many elementary parts, and statical moments of these, with reference to an axis $X\overline{X}$

passing through the centre C and parallel to the chord $AB=s$, be determined.

If PQ be a part of the arc, and PN be its distance from $X\overline{X}$, then the statical moment of this portion of the arc $= PQ.PN$. If now the radius $PC=MC=r$ be drawn, and QR parallel to AB, we obtain the two similar Δ^s PQR and CPN, for which:

$$PQ : QR = CP : PN,$$

from which the statical moment of the elementary arc $PQ.PN = QR.CP = QR.r$ is determined.

Now, for the statical moments of all the remaining arcs, the radius r is a common factor, and the sum of all the projections QR of the elementary arcs is equal to the chord corresponding to the projection of the whole arc; it follows, therefore, that the moment of the whole arc is also = the chord (s) times the radius r. If this moment be put equal to the arc (b) times the distance x, and therefore $bx = sr$, we then obtaino

$$\frac{x}{r} = \frac{s}{b}, \text{ and } x = \frac{sr}{b}.$$

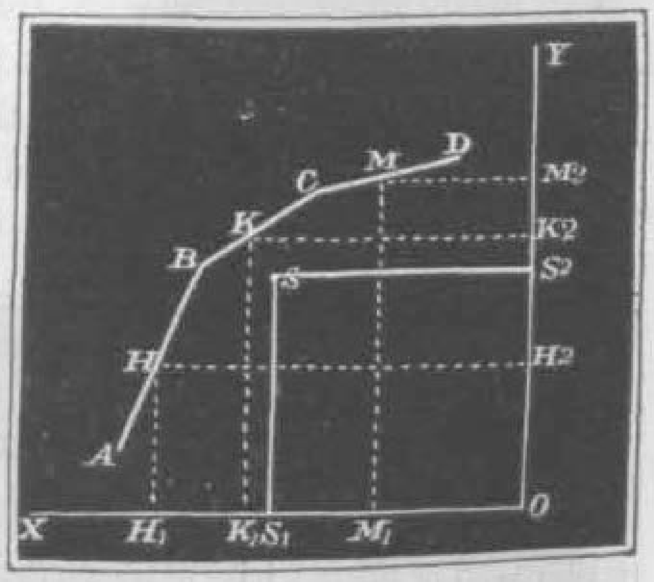
So that the distance of the centre of gravity, from the middle point is to the radius, in the ratio of the arc to the chord.

If the angle at the centre ACB of the arc b be $= \beta^0$, the arc corresponding to the diameter 1 is $\beta = \frac{\beta^0}{180^0}.\pi$. We have then $b = \beta r$, and $s = 2\, r\, sin. \frac{\beta}{2}$; whence it follows that, $x = \frac{2\, sin. \frac{1}{2}\beta . r}{\beta}$.

For the semicircle $\beta = \pi$ and $sin. \frac{\beta}{2} = 1$; therefore, $x = \frac{2}{\pi} r = 0{,}6366 \ldots r = \frac{7}{11} r$ nearly.

§ 103. To find the centre of gravity of a polygon or a connection of lines $ABCD$, Fig. 71, we must seek the distances of the middle points H, K, M, of the lines $AB=L_1$, $BC=L_2$, $CD=L_3$, &c., from two axes OX and OY, viz: $HH_1 = y_1$, $HH_2 = x_1$, $KK_1 = y_2$, $KK_2 = x_2$, &c.; the distances of the centre of gravity sought from these axes are then:

Fig. 71.



$$SS_2 = x = \frac{L_1 x_1 + L_2 x_2 + \ldots}{L_1 + L_2 + \ldots},$$

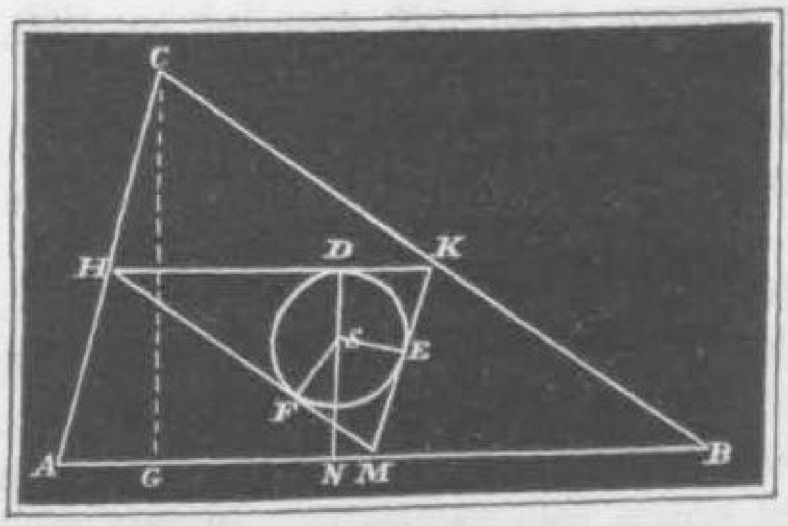
$$SS_1 = y = \frac{L_1 y_1 + L_2 y_2 + \ldots}{L_1 + L_2 + \ldots},$$

For example, the distance of the centre of gravity S of a wire bent into the form of a Δ ABC, Fig. 72, from the base is:

$$NS = x = \frac{\frac{1}{2}ah + \frac{1}{2}bh}{a+b+c} = \frac{a+b}{a+b+c} \cdot \frac{h}{2},$$

if the sides opposite to the angles A, B, C be designated by a, b, c, and the height CG by h.

Fig. 72.



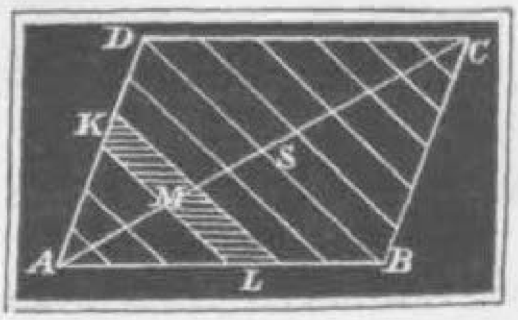
If the middle points H, K, M, of the sides of the triangle be connected with each other, and in the triangle so obtained a circle be described, its centre will coincide with the centre of gravity S, for the distance SD from one side HK is

$$= DN - SN = \frac{h}{2} - \frac{a+b}{a+b+c} \cdot \frac{h}{2} = \frac{c\,h}{2(a+b+c)} = \frac{\Delta\, ABC}{a+b+c} =$$

the distances SE and SF from the other sides.

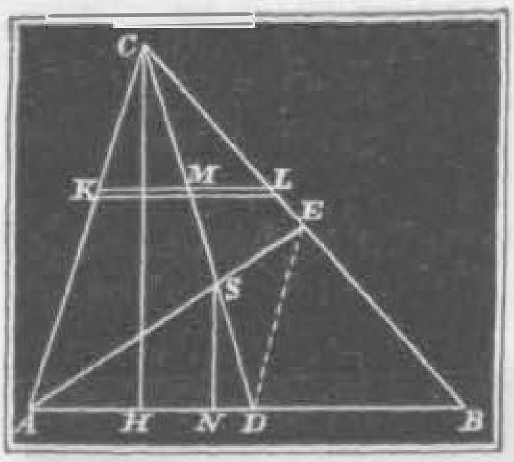
§ 104. *Centre of Gravity of Plane Figures.*—The centre of gravity of a parallelogram $ABCD$, Fig. 73, lies in the point of intersection of its diagonals, for all strips, such as KL, which are formed by drawing lines parallel to one of its diagonals BD, are bisected by the other diagonals AC; each of the diagonals, therefore, is a line of gravity.

Fig. 73.



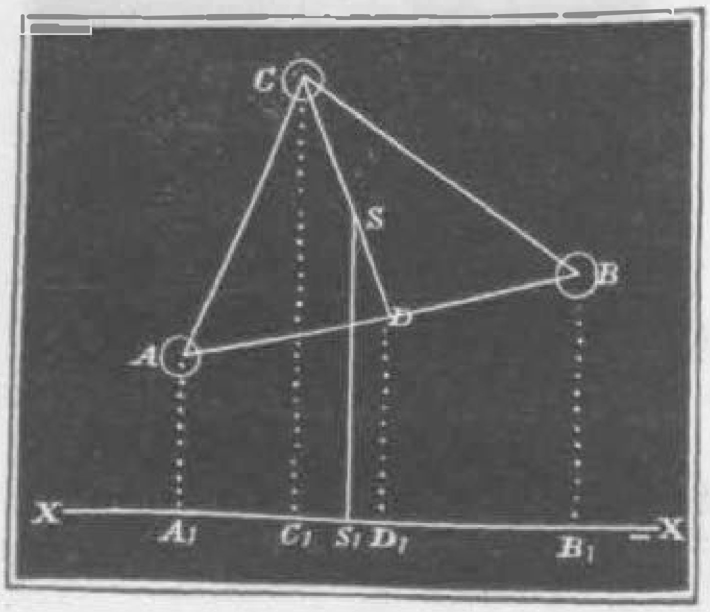
In a plane $\Delta\, ABC$, Fig. 74, every line CD from one angle to the middle D of the opposite side AB, is a line of gravity, for the same bisects all the elements KL of the Δ which are given when lines parallel to AB are drawn. If from a second angle A a second line of gravity be drawn to the middle E of the opposite side BC, the point of intersection of the two will give the centre of gravity of the whole Δ.

Fig. 74.



Because $BD = \frac{1}{2} BA$ and $BE = \frac{1}{2} BC$, DE is parallel to AC and $= \frac{1}{2} AC$, and $\Delta\, DES$ similar to the $\Delta\, CAS$, and lastly, $CS = 2\, SD$. If further we add SD, it follows that $CS + SD$, or $CD = 3\, DS$, and, therefore, inversely, $DS = \frac{1}{3} CD$. The centre of gravity S lies at $\frac{1}{3}$ of the line CD from the middle point D of the base, and at $\frac{2}{3}$ of the same from the angle C. If CH and SN be drawn perpendicular to the base, we have also $SN = \frac{1}{3}\, CH$; the centre of gravity S is at $\frac{1}{3}$ of the height from the base of the Δ.

Fig. 75.



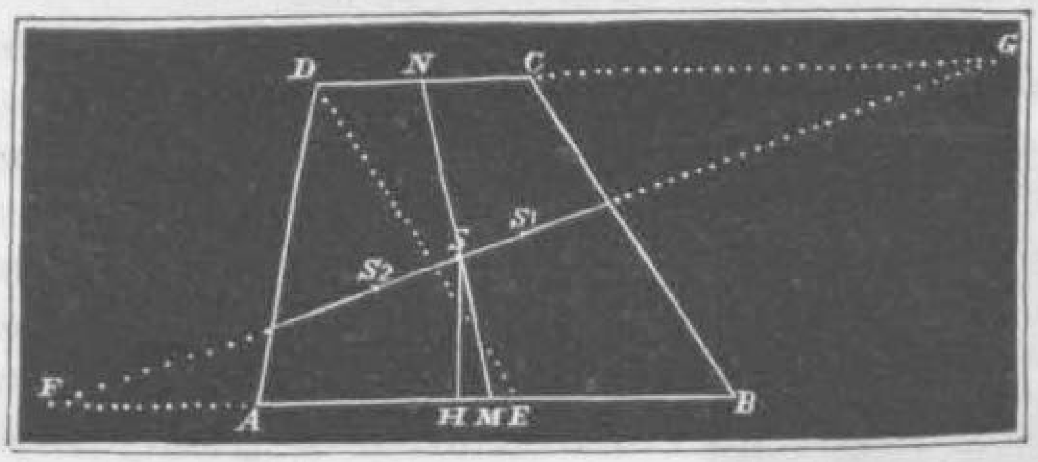
The distance SS_1 of the centre of gravity of a $\Delta\, ABC$, Fig. 75, from an axis $X\bar{X}$ is $= DD_1 + \frac{1}{3}$

$(CC_1 - DD_1)$, but $DD_1 = \frac{1}{2}(AA_1 + BB_1)$, consequently, $x = SS_1$ $= \frac{1}{3} CC_1 + \frac{2}{3} \cdot \frac{1}{2}(AA_1 + BB_1)$:
$= \frac{AA_1 + BB_1 + CC_1}{3}$, *i. e.*, the arithmetical mean of the distances of the three angular points.

Since the distance of the centre of gravity is determined in the same manner by three equal weights at the angular points of a $\triangle$, so the centre of gravity of a plane triangle coincides with the centre of gravity of these three equal weights.

§ 105. The determination of the centre of gravity S of a trapezium $ABCD$, Fig. 76, may be made in the following manner. The straight

Fig. 76.



line MN, which connects the middle points of the two bases AB and CD with each other, is a line of gravity of the trapezium; for lines drawn parallel to the bases decompose the trapezium into elementary parts, whose middle points or centres of gravity lie in MN. Now to determine completely the centre of gravity S, we have only, therefore, to find its distance SH from a base AB.

Let B represent the one, and b the other of the parallel sides AB and CD of the trapezium, h the height or the normal distance of these sides. Let DE be now drawn parallel to the side BC, we shall then obtain a parallelogram $BCDE$ of the area bh, and whose centre of gravity is S_1, and distance from $AB = \frac{h}{2}$, and a $\triangle$ ADE of the area $\frac{(B-b)h}{2}$ and centre of gravity S_2, and whose distance from $AB = \frac{h}{3}$.

The statical moment of the trapezium, about the line AB, is therefore

$$= bh \cdot \frac{h}{2} + \frac{(B-b)h}{2} \cdot \frac{h}{3} = (B+2b)\frac{h^2}{6},$$

but the area of the trapezium is $= (B+b)\frac{h}{2}$; it follows, therefore, that the normal distance of the centre of gravity S from the base is

$$HS = \frac{\frac{1}{6}(B+2b)h^{22}}{\frac{1}{2}(B+b)h} = \frac{B+2b}{B+b} \cdot \frac{h}{3}.$$

To find the centre of gravity by construction, let the two bases be prolonged, the prolongations CG made $= B$ and $AF = b$, and the two extreme points obtained, F and G, connected by a straight line: the point of intersection S with the middle line MN will be the centre of gravity sought; for, from $HS = \frac{B+2b}{B+b} \cdot \frac{h}{3}$, it follows that

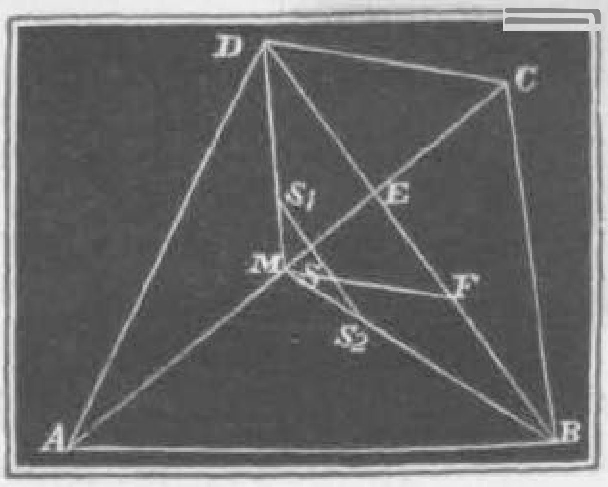
$$MS = \frac{B+2b}{B+b} \cdot \frac{MN}{3} \text{ and } NS = \frac{2B+b}{B+b} \cdot \frac{MN}{3}; \text{ and}$$

$$\frac{MS}{NS} = \frac{B+2b}{2B+b} = \frac{\frac{1}{2}B+b}{B+\frac{1}{2}b} = \frac{MA+AF}{CG+NC} = \frac{MF}{NG},$$

which actually arises from the similarity of the triangles MSF and NSG.

§ 106. To find the centre of gravity of any other four-sided figure $ABCD$, Fig. 77, we may decompose it by the diagonal AC into two triangles, and from the foregoing, determine their centres of gravity S_1 and S_2, and thereby a line of gravity S_1S_2. If now the four-sided figures be decomposed into two other triangles by the diagonal BD, and their centres of gravity determined, we obtain another line of gravity, whose intersection with the first will give the centre of gravity of the whole figure.

Fig. 77.

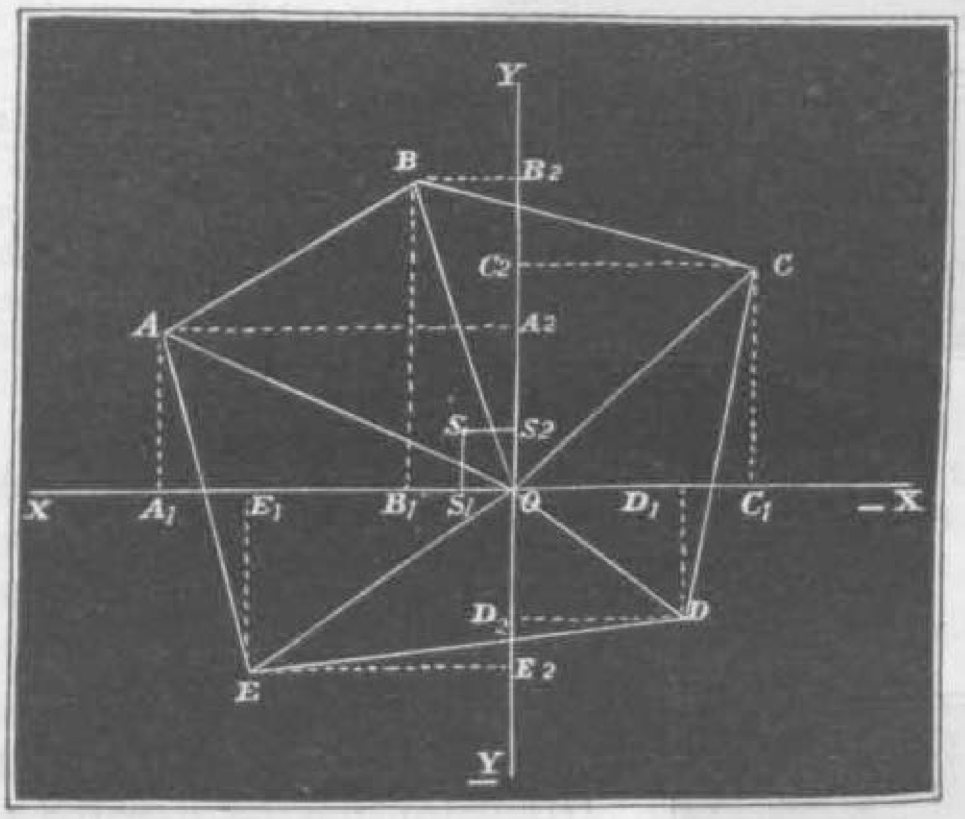


We may effect this more simply if we bisect the diagonal AC in M, apply the greater part BE of the second diagonal to the less, so that $DF=BE$, join FM and divide it into three equal parts; the centre of gravity lies in the first point S from M, as may be proved in the following manner. $MS_1 = \frac{1}{3}$ MD and $MS_2 = \frac{1}{3} MB$, consequently S_1S_2 are parallel to BD, but SS_1 times $\triangle ACD = SS_2$ times $\triangle ACB$, or $SS_1 \,.\, DE = SS_2 \,.\, BE$; therefore, $SS_1 : SS_2 = BE : DE$. Now, $BE=DF$ and $DE=BF$, consequently $SS_1 : SS_2 = DF : BF$. The straight line MF intersects, therefore, the line of gravity S_1S_2 in the centre of gravity of the figure.

§ 107. If it be required to find the centre of gravity S of a polygon $ABCDE$, Fig 78, we must decompose the polygon into triangles, and determine their statical moments with reference to two rectangular axes $X\overline{X}$ and $Y\overline{Y}$.

If the co-ordinates $OA_1=x_1$, $OA_2=y_1$, $OB_1=x_2$, $OB_2=y_2$, &c., of the extremities are given, the statical moments of the triangles ABO, BCO, COD, &c., may be determined simply in the following manner. The area of $\triangle ABO$, from the remark below, $= D_1 = \frac{1}{2}(x_1y_2 - x_2y_1)$; of the following $\triangle BCO = D_2 = \frac{1}{2}(x_2y_3 - x_3y_2)$, &c., the distance of the centre of gravity of $\triangle ABO$ from $Y\overline{Y}$ according to § 104 $= u_1 = \frac{x_1+x_2+0}{3} = \frac{x_1+x_2}{3}$, from $X\overline{X} = v_1 = \frac{y_1+y_2}{3}$; of the centre of gravity of $\triangle BCO = u_2 = \frac{x_2 + x_3}{2}$ and $v_2 = \frac{y_2 + y_3}{3}$, &c. If these distances are multiplied by the areas of the triangles, the moments of these last

Fig. 78.



are obtained; and if the values so obtained are substituted in the formulæn

$$u = \frac{D_1 u_1 + D_2 u_2 + \dots}{D_1 + D_2 + \dots}$$

$$v = \frac{D_1 v_1 + D_2 v_2 + \dots}{D_1 + D_2 + \dots},$$

we have the distances u and v of the centre of gravity from the axes YY and XX.

Example. A pentagon $ABCDE$, Fig. 78, is given by the following co-ordinates of its extremities A, B, C, &c.: to find the co-ordinates of its centre of gravity:

Co-ordinates given.		Twice the area of triangles.	Triple co-ordinates of centre of gravity.		Six times the statical moments.	
x	y		$3\,u_n$	$3\,v^n$	$6D_n u_n$	$6D_n v_n$
24	11	24 . 21 — 7 . 11 = 427	31	32	13237	13664
7	21	7 . 15 + 21 . 16 = 441	— 9	36	—3969	15876
—16	15	16 . 9 + 12 . 15 = 324	—28	6	—9072	1944
—12	— 9	12 . 12 + 18 . 9 = 306	+ 6	—21	1836	—6426
18	—12	18 . 11 + 24 . 12 = 486	+42	— 1	20412	— 486
		Sum: 1984			22444	24572

The distance of the centre of gravity from the axis $\overline{YY}$ is:

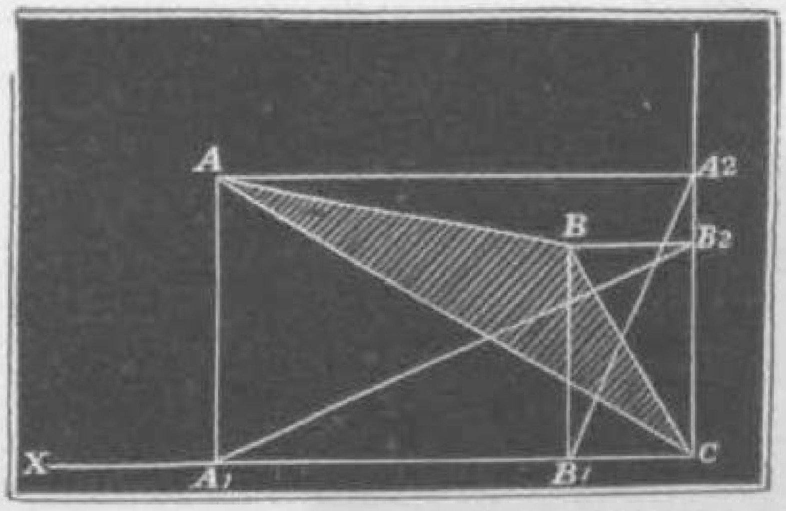
$$SS_2 = u = \frac{1}{3} \cdot \frac{22444}{1984} = 3{,}771,$$

and from the axis $\overline{XX}$:

$$SS_1 = v = \frac{1}{3} \cdot \frac{24572}{1984} = 4{,}128.$$

Remark. If $CA_1 = x_1$, $CB_1 = x_2$, $CA_2 = y_1$, and $CB_2 = y_2$, the co-ordinates of the two angles of a triangle ABC, Fig. 79, whose third angle C coincides with the point of application of the system of co-ordinates, we have the area of the same:

Fig. 79.



$D =$ trapezium $ABB_1A_1 +$ triangle $CBB_1 -$ triangle CAA_1

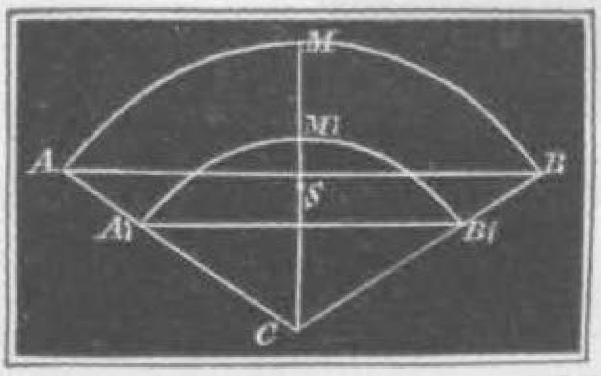
$$= \left(\frac{y_1 + y_2}{2}\right)(x_1 - x_2) + \frac{x_2 y_2}{2} - \frac{x_1 y_1}{2}$$

$$= \frac{x_1 y_2 - x_2 y_1}{2}.$$

The area of this triangle is the difference of two other triangles, CB_2A_2 and CA_2B_1, and the one co-ordinate of a point is the base of the one, and the other co-ordinate the height of the other triangle, and inversely.

§ 108. The centre of gravity of the sector of a circle ACB, Fig. 80, coincides with that of a circular arc A_1B_1 which has the same angle with the sector, and whose radius CA_1 is two-thirds of the radius CA of the sector; for the sector may be divided by an infinity of radii into very small triangles, whose centres of gravity are distant two-thirds of the radius from the centre C, and these form by their continuity the arc $A_1M_1B_1$. The centre of gravity S of the sector lies in the radius CM, bisecting the surface, and at the distance $CS = x = \frac{\text{chordi.}}{\text{arc}} \cdot \frac{2}{3} CA = \frac{4}{3} \cdot \frac{sin.\, \frac{1}{2}\beta}{\beta} \cdot r$; r representing the radius CA of the sector, and β the arc which measures the angle at the centre ACB.

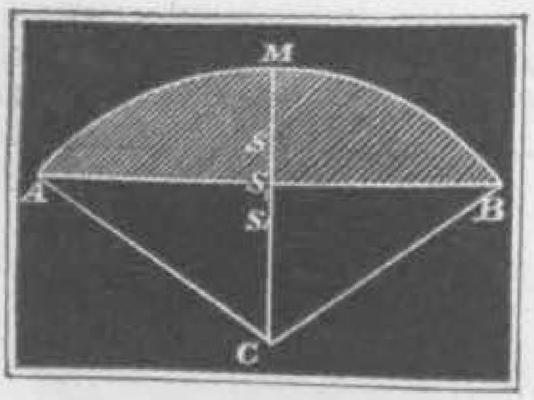
Fig. 80.



For the semi-circle $\beta = \pi$, $sin.\, \frac{1}{2}\beta = sin.\, 90° = 1$, therefore $x = \frac{4}{3\pi} r = 0{,}4244\, r$ or about $\frac{14}{33} r$. For a quadrant $x = \frac{4}{3} \cdot \frac{\sqrt{\frac{1}{2}}}{\frac{1}{2}\pi} r = \frac{4\sqrt{2}}{3\pi} r = 0{,}6002\, r$ and for a sixth part $x = \frac{4}{3} \cdot \frac{\frac{1}{2}}{\frac{1}{3}\pi} r = \frac{2}{\pi} r = 0{,}6366\, r$.

§ 109. The centre of gravity of a segment of a circle ABM, Fig. 81, is given, if we put the moment of the sector $ACBM$ equal to the sum of the moments of the segment and the moment of the triangle ACB. If r be the radius CA, s the chord AB, and A the area of the segment ABM, the moment of the sector $=$ the sector $\times$ $CS_1 = \frac{r \cdot \text{arc}}{2} \cdot \frac{\text{chord}}{\text{arc}} \cdot \frac{2}{3} r = \frac{1}{3} s r^2$, further the moment of the triangle $=$ triangle $\times$ $CS_2 = \frac{s}{2}\sqrt{r^2 - \frac{s^2}{4}}$.

Fig. 81.

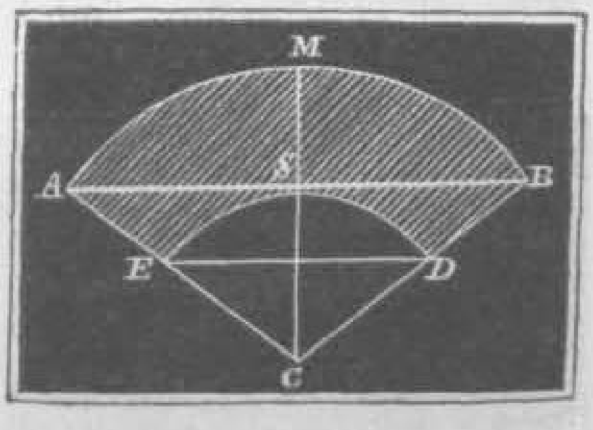


$\frac{2}{3}\sqrt{r^2 - \frac{s^2}{4}} = \frac{sr^2}{3} - \frac{s^3}{12}$, and from this the moment of the segment: $A \cdot CS = Ax = \frac{1}{3}sr^2 - \left(\frac{sr^2}{3} - \frac{s^3}{12}\right) = \frac{s^3}{12}$; consequently the distance sought is $x = \frac{s^3}{12eA}$.

For the semi-circle $s = 2r$ and $A = \frac{1}{2}\pi r^2$, hence $x = \frac{8r^3}{12.\frac{\pi r^2}{2}} = \frac{4r}{3\pi}$, as found above.

In like manner we may find the centre of gravity S of a portion of a ring $ABDE$, Fig. 82, which is the difference of two sectors ACB and DCE. If the radii be $CA = r$ and $CD = r_1$, and the chords $AB = s$ and $DE = s_1$, the statical moments of the sectors are: $\frac{s\,r^2}{3}$ and $\frac{s_1\,r_1^{\,2}}{3}$, therefore the statical moments of the portion of ring: $= \frac{sr^2 - s_1 r_1^{\,2}}{3}$, or (since $\frac{s_1}{s} = \frac{r_1}{r}$) is $= \frac{r^3 - r_1^{\,3}}{3} \cdot \frac{s}{r}$. But the area $= \frac{\beta\, r^2}{2} - \frac{\beta r_1^{\,2}}{2} = \beta\left(\frac{r^2 - r_1^{\,2}}{2}\right)$, provided that β represents the arc corresponding to the angle at the centre ACB; the centre of gravity, therefore, of the portion follows from the distance $CS = x = \frac{\text{moment}}{\text{area}} = \frac{r^3 - r_1^{\,3}}{r^2 - r_1^{\,2}} \cdot \frac{2}{3} \cdot \frac{s}{r\beta} = \frac{2}{3}\left(\frac{r^3 - r_1^{\,3}}{r^2 - r_1^{\,2}}\right) \cdot \frac{\text{chords}}{\text{arc}} = \frac{4}{3}\,\frac{sin.\,\frac{1}{2}\,\beta s}{\beta} \cdot \frac{r^3 - r_1^{\,3}}{r^2 - r_1^{\,2}}$.

Fig. 82.

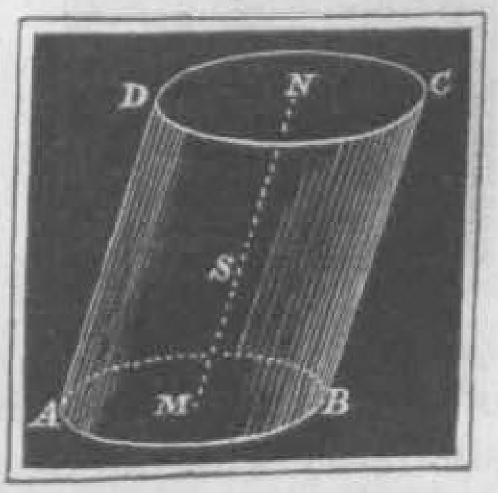


Example. The radii of the surfaces of a dome are: $r = 5$ ft., $r_1 = 3\frac{1}{2}$ ft., and the angle at the centre, $\beta^0 = 130^0$, then is the distance of the centre of gravity of these surfaces from their central point:

$$x = \frac{4}{3}\,\frac{sin.\,65^0}{arc.\,130^0} \cdot \frac{5^3 - 3{,}5^3}{5^2 - 3{,}5^2} = \frac{4 \,.\, 0{,}9063}{3 \,.\, 2{,}2689} \cdot \frac{125 - 42{,}875}{25 - 12{,}25} = \frac{3{,}62520 \times 82{,}125}{6{,}8067 \times 12{,}75}$$
$$= 3{,}430 \text{ feet.}$$

§ 110. *Centre of Gravity of Curved Surfaces.*—The centre of gravity of a curved surface (envelope) of a cylinder $ABCD$, Fig. 83, lies in the middle S of the axis MN of this body, for all the annular elements of the cylindrical envelope which are obtained by sections drawn through the body parallel to the base, are equal, and their centres of gravity lie in the axis; these centres of gravity form a uniform line of gravity. For the same reason the

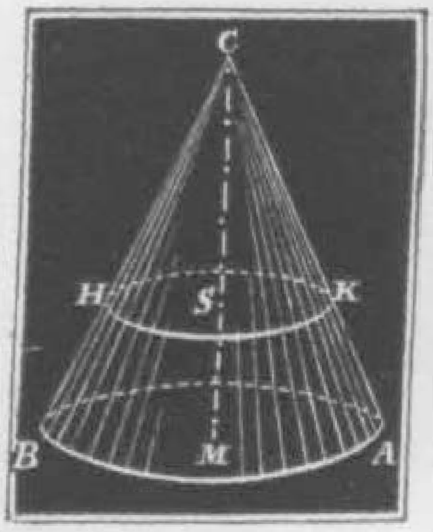
Fig. 83.



centre of gravity of the surfaces of a prism lies in the middle point of the straight lines connecting the centres of gravity of both the bases.

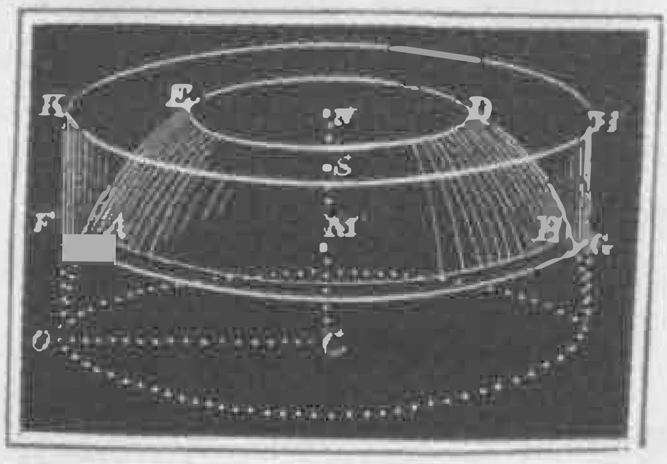
The centre of gravity of the envelope of a right cone ABC, Fig. 84, lies in the axis of the cone, and is one-third of this line from the base, or two-thirds from the vertex; for this curved surface may be decomposed into an infinite number of small triangles by straight lines, which are called the sides of the cone whose ceutres of gravity form a circle HK, which is distant two-thirds of the axis from the vertex, and whose centre of gravity or centre S lies in the axis CM.

Fig. 84.



The centre of gravity of a spherical zone $ABDE$, Fig. 85, and likewise that of a spherical cup lies in the centre S of its height MN; for from the rules of geometry the zone has the same surface as a cylindrical envelope $FGHK$, whose height is equal to that of MN, and whose radius is equal to that of the radius CO of the spherical zone; and this equality also exists in the annular elements, which are obtained by carrying an infinite number of planes parallel to the circular bases through the same; according to this the centre of gravity of the zone coincides with that of the cylindrical envelope.

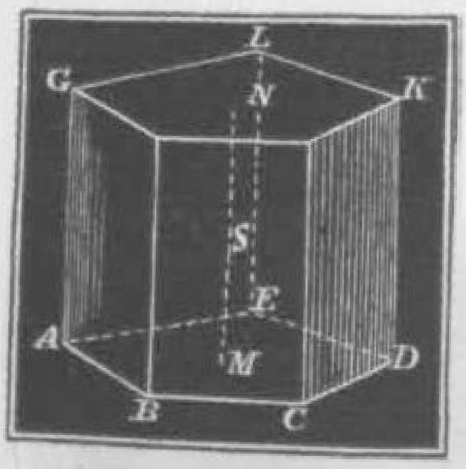
Fig. 85.



Remark. The centre of gravity of the surface of an oblique cone or oblique pyramid lies at about one-third of the height from the base, but not in the straight line passing from the vertex to the centre of gravity of the base, because slices parallel to the base decompose the surface into rings, which vary in breadth at different parts of their surface.

§ 111. *Centre of Gravity of Bodies.*—The centre of gravity of a prism AK, Fig. 86, is the centre S of the straight line which connects the centres of gravity M and N of both bases AD and GK, for the prism may be decomposed by sections parallel to the base into exactly congruent slices, whose centres of gravity lie in MN, and by their superposition make the line MN a uniform line of gravity.

Fig. 86.



For the same reason the centre of gravity of a cylinder lies in the middle of its axis.

The centre of gravity of a pyramid ADF, Fig. 87, lies in the straight line MF from the vertex F to the centre of gravity M of the base, for all slices as $NOPQR$, have from their similarity with the base, their centres of gravity in this line.

If the pyramid be triangular as $ABCD$, Fig. 88, each of the four angular points may be considered as vertices, and the opposite surfaces as bases; the centre of gravity S is determined by the intersec-

Fig. 87.

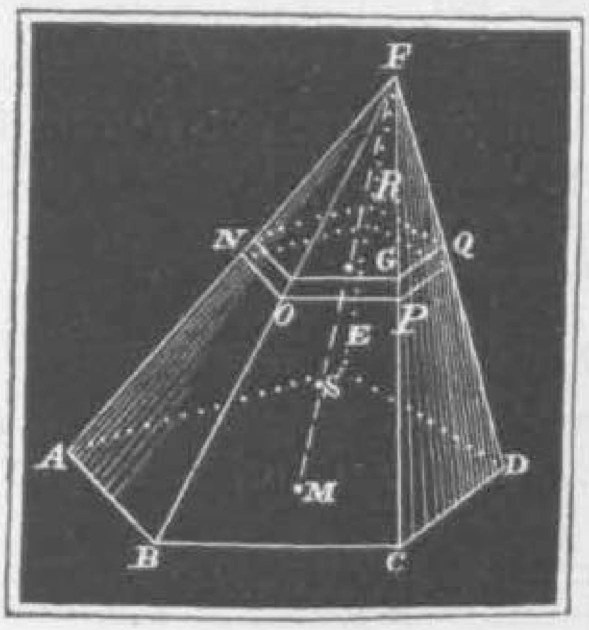
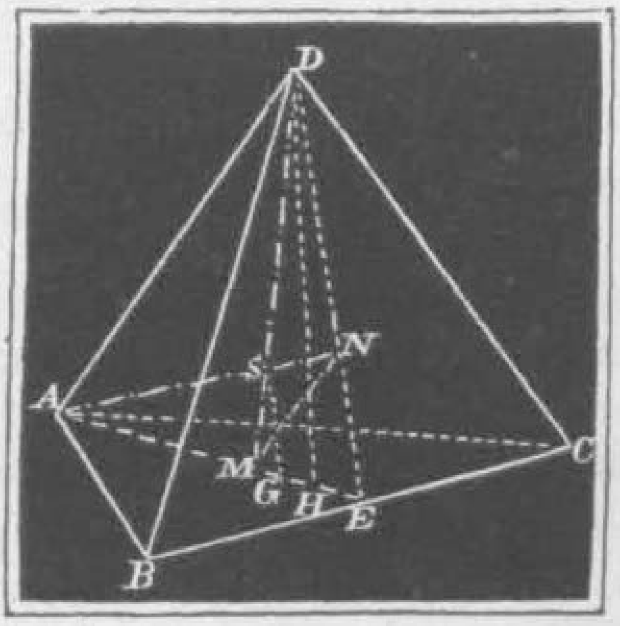


Fig. 88.



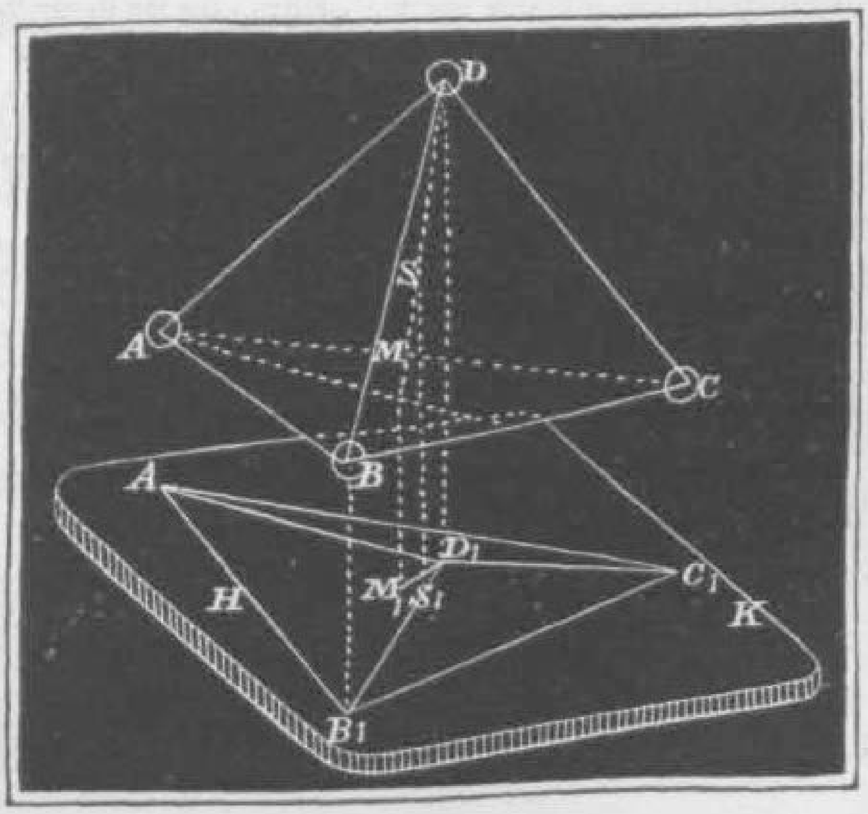
tion of two straight lines drawn from D and A to the centres of gravity M and N of the opposite surfaces ABC and BCD.

If the straight lines EA and ED be given, we then have from § 104 $EM = \frac{1}{3}\, EA$ and $EN = \frac{1}{3}\, ED$; therefore MN is parallel to AD and $= \frac{1}{3}\, AD$, and the $\triangle\ MNS$ similar to $\triangle\ DAS$. Again from this similarity we have $MS = \frac{1}{3}\, DS$, or $DS = 3\, MS$, also $MD = SD + MS = 4\, MS$, and inversely $MS = \frac{1}{4}\, MD$. Hence the centre of gravity is found to be one-fourth of the line joining the centre of gravity M of the base with the vertex D.

Further, if the heights DH and SG be given, and HM be drawn, we then obtain the two similar $\triangle^s$ DHM and SGM, in which from the foregoing $SG = \frac{1}{4}\, DH$. We may, therefore, say that the distance of the centre of gravity S of a triangular pyramid from the base is equal to one-fourth, and that from the vertex three-fourths of the height of the pyramid.

As every pyramid, and also every cone, is made up of an infinite number of three sided pyramids of the same height, the centre of gravity of every pyramid and cone is a fourth of the height from the base and three-fourths from the vertex. We may, therefore, find the centre of gravity of a pyramid or cone, if a plane be drawn parallel to the base at a distance one-fourth from the base, and the centre of gravity of the section or its intersection with the line joining the vertex and the centre of gravity of the base be determined.

Fig. 89.



§ 112. If the distances AA_1, BB_1, of the four angles of a triangular pyramid $ABCD$, Fig. 89, from a plane HK be known, the distance of the centre of gravity S from this plane is found from the mean value

$$SS_1 = \frac{AA_1 + BB_1 + CC_1 + DD_1}{4}.$$

The distance of the centre of gravity M of the base ABC from this plane is (§ 104):

$$MM_1 = \frac{AA_1 + BB_1 + CC_1}{3},$$

and that of the pyramid S is:

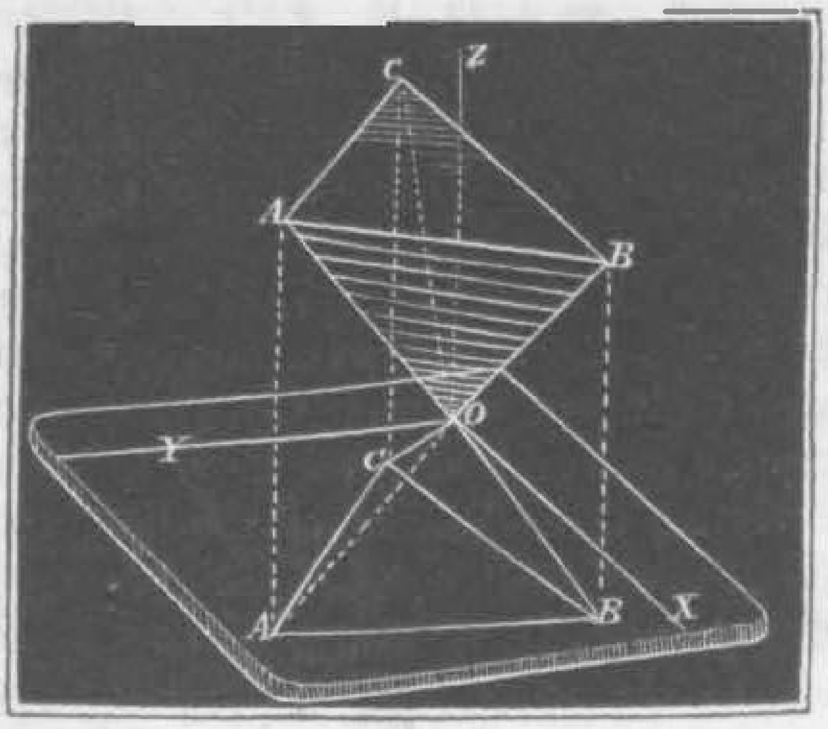
$$SS_1 = MM_1 + \tfrac{1}{4}(DD_1 - MM_1),$$

where DD_1 is the distance of the vertex: hence it follows by combining the two last equations, that:

$$SS_1 = \tfrac{3}{4} MM_1 + \tfrac{1}{4} DD_1 = \frac{AA_1 + BB_1 + CC_1 + DD_1}{4}.$$

The distance of the centre of gravity of four equal weights applied to the angles of a triangular pyramid, is equivalent to the arithmetical mean $\frac{AA_1 + BB_1 + CC_1 + DD_1}{4}$, consequently the centre of gravity of the pyramid corresponds with that of the system of weights.

Fig. 90.



Remark. The determination of the volume of a triangular pyramid, from the co-ordiuates of its angles, is simple. If we draw planes XY, XZ, YZ, through the vextex O of such a pyramid $ABCO$, Fig. 90, and represent the distances of the angles ABC from these planes by z_1, z_2, z_3, y_1, y_2, y_3, and x_1, x_2, x_3, the volume of the pyramid will be

$$V = \tfrac{1}{6}(x_1y_2z_3 + x_2y_3z_1 + x_3y_1z_2 - x_1y_3z_2 - x_2y_1z_3 - x_3y_2z_1),$$

which will be given, if the pyramid be considered as an aggregate of four oblique prisms.

The distances of the centre of gravity of these pyramids from the three planes are:

$$x = \frac{x_1 + x_2 + x_3}{4},\ y = \frac{y_1 + y_2 + y_3}{4},\ \text{and}\ z = \frac{z_1 + z_2 + z_3}{4}.$$

§ 113. Since every polyhedron as $ABCDO$, Fig. 91, may be decomposed into triangular pyramids $ABCO$, $BCDO$, we may also find its centre of gravity S if we calculate the volumes, and the statical moments of the single pyramids.

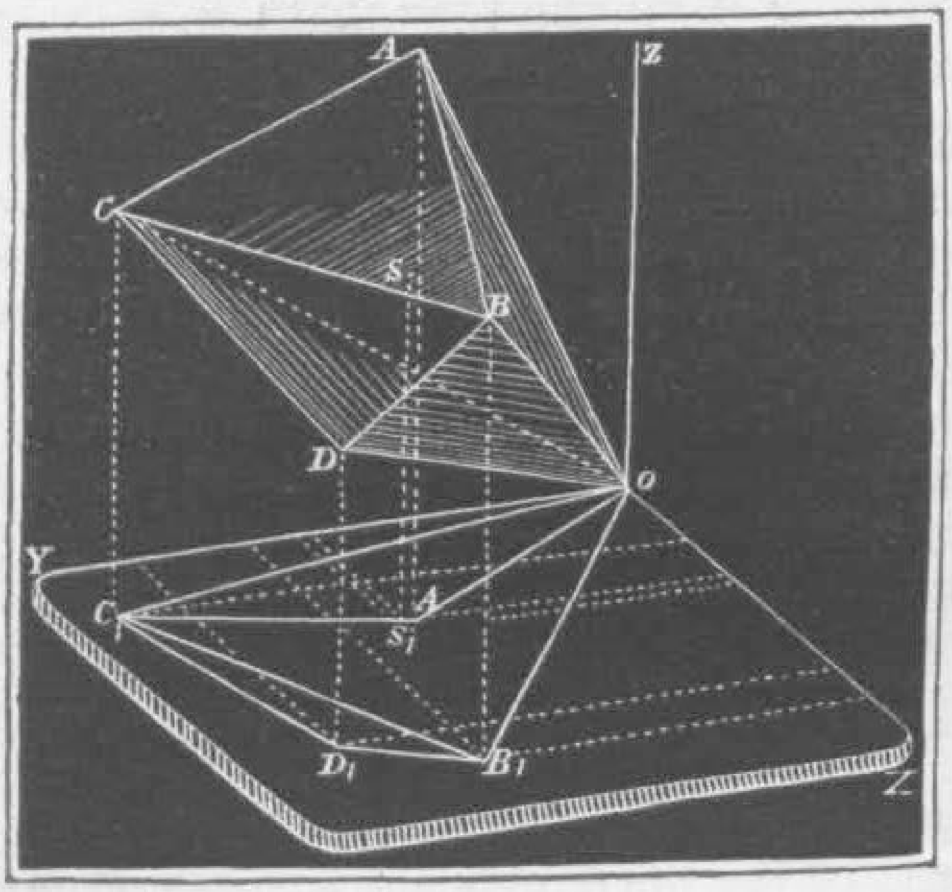
If the distances of the angles A, B, C, &c., from the co-ordinate planes passing through the common vertex O of all the pyramids, are x_1, x_2, x_3, &c., y_1, y_2, y_3, &c., z_1, z_2, z_3, &c., the volumes of the single pyramids are:

$$V_1 = \pm \tfrac{1}{6}(x_1y_2z_3 + x_2y_3z_1 + x_3y_1z_2 - x_1y_3z_2 - x_2y_1z_3 - x_3y_2z_1),$$

$$V_2 = \pm \tfrac{1}{6}(x_2y_3z_4 + x_3y_4z_2 + x_4y_2z_3 - x_2y_4z_3 - x_3y_2z_4 - x_4y_3z_2),$$

and the distances of their centres of gravity:

Fig. 91.



$$u_1 = \frac{x_1 + x_2 + x_3}{4}, v_1 = \frac{y_1 + y_2 + y_3}{4}, w_1 = \frac{z_1 + z_2 + z_3}{4},$$

$$u_2 = \frac{x_2 + x_3 + x_4}{4}, v_2 = \frac{y_2 + y_3 + y_4}{4}, w_2 = \frac{z_2 + z_3 + z_4}{4}, \text{\&c.}$$

From these values the distances of the centre of gravity of the whole body may be finally calculated by the formula:

$$u = \frac{V_1 u_1 + V_2 u_2 + \ldots}{V_1 + V_2 + \ldots}, v = \frac{V_1 v_1 + V_2 v_2 + \ldots}{V_1 + V_2 + \ldots},$$

$$w = \frac{V_1 w_1 + V_2 w_2 + \ldots}{V_1 + V_2 + \ldots}.$$

Example. A body bounded by six triangles ADO, Fig. 91, is determined by the following values for the co-ordinates of angles; whence the co-ordinates of its centre of gravity may be found.

Given co-ordinates			Six times the area of the triangular pyramids.	Four times the co-ordinates of centre of gravity.			Twenty-four times the statical moments.		
x	y	z		$4u_n$	$4v_n$	$4w_n$	$24 V_n u_n$	$24 V_n v_n$	$24 V_n w_n$
20 45	23 29	41 30	$6\,V_1 = \left\{\begin{matrix}20.29.28\\41.45.40\\23.30.12\end{matrix}\right\} - \left\{\begin{matrix}20.40.30\\23.28.45\\41.12.29\end{matrix}\right\} = 31072$	77	92	99	2392544	2858624	3076128
12 38	40 35	28 20	$6\,V_2 = \left\{\begin{matrix}45.35.28\\29.20.12\\30.38.40\end{matrix}\right\} - \left\{\begin{matrix}45.40.20\\29.28.38\\30.12.35\end{matrix}\right\} = 17204$	95	104	78	1634380	1789216	1341912
			Sum: 48276				4026924	4647840	4418[illegible]40

From the results of this calculation, the distances of the centres of gravity from the three planes YZ, XZ, and XY follow.

$$u = \frac{1}{4} \cdot \frac{4026924}{48276} = 20{,}853,$$

$$v = \frac{1}{4} \cdot \frac{4647840}{48276} = 24{,}069,$$

$$w = \frac{1}{4} \cdot \frac{4416040}{48276} = 22{,}879.$$

§ 114. The centre of gravity of a truncated pyramid $ADQN$ (Fig. 88), lies in the line MG, which connects the centres of gravity of the two parallel bases; in order to determine the distance of this point from one of the bases, we must determine the volumes and moments of the entire pyramid ADF, and the supplementary pyramid NQF. If the areas of the bases AD and $NQ = G$ and g, and the normal distance of both $= h$, the height of the supplementary pyramid will be given from the formulæ:

$\frac{G}{g} = \frac{(h+x)^2}{x^2}$, or $\frac{h}{x} + 1 = \sqrt{\frac{G}{g}}$, and $x = \frac{h\sqrt{g}}{\sqrt{G}-\sqrt{g}}$, as also $h + x = \frac{h\sqrt{G}}{\sqrt{G}-\sqrt{g}}$.

The moment of the whole pyramid with reference to the base G is now

$\frac{G\,(h+x)}{3} \cdot \frac{h+x}{4} = \frac{1}{12} \cdot \frac{h^2\,G^2}{(\sqrt{G}-\sqrt{g})^2}$, that of the supplementary pyramid $= \frac{gx}{3}\left(h+\frac{x}{4}\right) = \frac{1}{3}\,\frac{h^2\sqrt{g^3}}{\sqrt{G}-\sqrt{g}} + \frac{1}{12} \cdot \frac{h^2 g^2}{(\sqrt{G}-\sqrt{g})^2}$; hence it follows that the moment of the truncated pyramid:

$$= \frac{h^2}{12\,(\sqrt{G}-\sqrt{g})^2} \cdot \left(G^2 - 4\,(\sqrt{Gg^3} - g^2) - g^2\right)$$

$$= \frac{h^2(G^2 - 4g\sqrt{Gg} + 3g^2)}{12\,(G - 2\sqrt{Gg} + g)} = \frac{h^{22}}{12} \cdot (G + 2\sqrt{Gg} + 3g).$$

Now the solid contents of the truncated pyramid are:

$$V = \frac{h}{3}\,(G + \sqrt{Gg} + g);$$

hence it follows finally that the distance of its centre of gravity S from the base is

$$MS = y = \frac{h}{4} \cdot \frac{G + 2\sqrt{Gg} + 3g}{G + \sqrt{Gg} + g}.$$

The radii of the bases of a truncated cone are R and r, and therefore $G = \pi R^2$ and $g = \pi r^2$, we have then for this

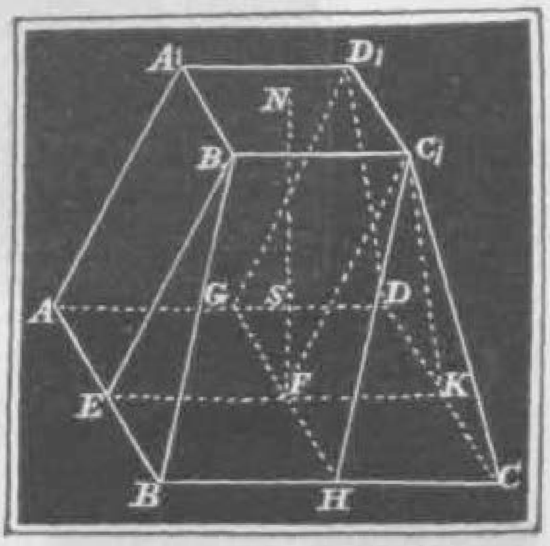
$$y = \frac{h}{4} \cdot \frac{R^2 + 2Rr + 3r^2}{R^2 + Rr + r^2}.$$

Example. The centre of gravity of a truncated cone of the height $h = 20$ inches, and radius $R = 12$ and $r = 8$ inches, always lies in the line joining the centres of the two circular bases, and is distant from the greater by:

$$y = \frac{20}{4} \cdot \frac{12^2 + 2.12.8 + 3.8^2}{12^2 + 12.8 + 8^2} = \frac{5.528}{304} = \frac{2640}{304} = 8{,}684 \text{ inches.}$$

§ 115. A pontoon is a body enclosed by two dissimilar rectangular bases and four trapeziums ACC_1A_1, Fig. 92, and may be decomposed into a parallelopiped AFC_1A_1, two triangular prisms EHC_1B_1, GKC_1D_1, and a quadrangular pyramid HKC_1; we may, therefore, with the help of these constituents, find the centre of gravity of the body.

Fig. 92.



It is easy to see that the line from the one bases to the other is the line of gravity of this body; there remains only to determine the distance of the centre of gravity from either base. If we represent the length BC and breadth AB of one base by l and b, and that of A_1B_1 and B_1C_1 of the other base by l_1 and b_1, and the height of the body by h, then the volume of the parallelopiped $= b_1l_1h$, and its moment $b_1l_1h \cdot \frac{h}{2} = \frac{1}{2} b_1l_1h^2$, further the volumes of the two triangular prismss $= ([b-b_1]\,l_1 + [l-l_1]\,b_1)\,\frac{h}{2}$ and their moments $= ([b-b_1]\,l_1 + [l-l_1]\,b_1)\,\frac{h}{2} \cdot \frac{h}{3}$, lastly the volume of the pyramid $= (b-b_1)\,.\,(l-l_1)\,\frac{h}{3}$ and its moment $= (b-b_1)\,.\,(l-l_1)\,\frac{h}{3} \cdot \frac{h}{4}$. The volume of the whole body is, therefore:

$$V = (6b_1l_1 + 3bl_1 + 3lb_1 - 6b_1l_1 + 2bl + 2b_1l_1 - 2bl_1 - 2b_1l) \cdot \frac{h}{6}$$

$$= (2\,bl + 2\,b_1l_1 + bl_1 + lb_1) \cdot \frac{h}{6}, \text{ and its moment}$$

$$Vy = (6b_1l_1 + 2bl_1 + 2lb_1 - 4b_1l_1 + bl + b_1l_1 - bl_1 - lb_1) \cdot \frac{h^2}{12}$$

$$= (3b_1l_1 + bl + bl_1 + b_1l)\,\frac{h^2}{12}.$$

Hence it follows that the distance of the centre of gravity from the base bl is:

$$y = \frac{bl + 3b_1l_1 + bl_1 + b_1l}{2bl + 2b_1l_1 + bl_1 + b_1l} \cdot \frac{h}{2}.$$

Remark. This formula is also applicable to bodies with elliptical bases. The axes of the one base are a and b, and of the other a_1 and b_1; the volume of such a body, therefore, is: $V = \frac{\pi h}{24}\,(2\,ab + 2\,a_1b_1 + ab_1 + a_1b)$, and the distance of the centre of gravity:

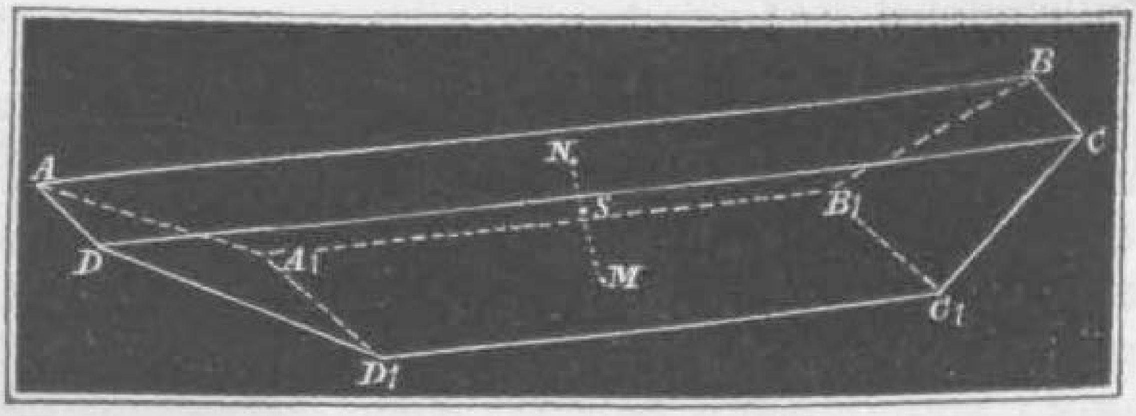
$$y = \frac{ab + 3\,a_1b_1 + ab_1 + a_1b}{2\,ab + 2\,a_1b_1 + ab_1 + a_1b} \cdot \frac{h}{2}.$$

Example. A dam, ACC_1A_1, Fig. 93, is of the height 20 feet, 250 feet long at the bottom and 40 feet wide, at the top 400 feet long and 15 wide; to find the distance of its centre of gravity from the base. Here $b = 40$, $l = 250$, $b_1 = 15$, $l_1 = 400$, $h = 20$, therefore the vertical distance sought is:

$$MS = y = \frac{40.250 + 3.15.400 + 40.400 + 15.250}{2.40.250 + 2.15.400 + 40.400 + 15.250} \cdot \frac{20}{2}$$

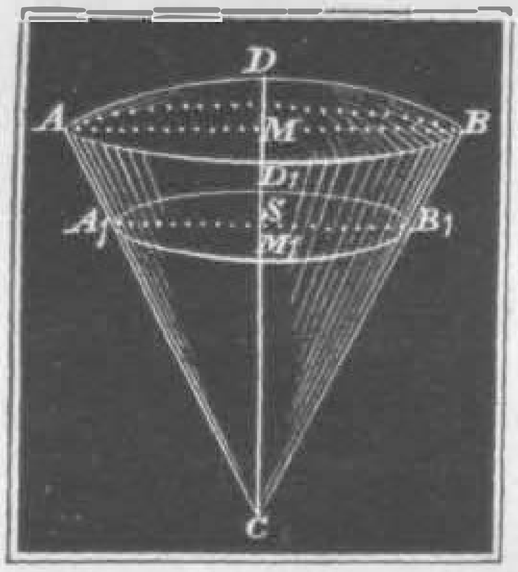
$$= \frac{4775}{5175} \cdot 10 = \frac{1910}{207} = 9{,}227 \text{ feet.}$$

Fig. 93.



§ 116. If the sector of a circle ACD, Fig. 94, revolves about its radius CD, there is generated the spherical sector ACB, whose centre of gravity we wish to determine. We may represent the body as containing infinitely many and infinitely thin pyramids, whose common vertex is the centre C, and whose base forms the spherical surface ADB. The centres of gravity of all these pyramids are at $\frac{3}{4}$ of the radius of the sphere from the centre Ci; they therefore form a second spherical surface $A_1D_1B_1$ of the radius $CA_1 = \frac{3}{4}\, CA$. But the centre of gravity S of this curved surface is the centre of gravity of the spherical sectors; because the weights of the elementary pyramids are uniformly distributed over this surface, and therefore it is uniformly heavy.

Fig. 94.



If we now put the radius $CA = CD = r$ and the height DM of the outer surfacei$= h$, we get for the inner $CD_1 = \frac{3}{4}\, r$, and $M_1D_1 = \frac{3}{4}\, h$; consequentlyi(§ 110) $D_1S = \frac{1}{2}\, M_1D_1 = \frac{3}{8}\, h$, and the distance of the centre of gravity of the sector from the centrei

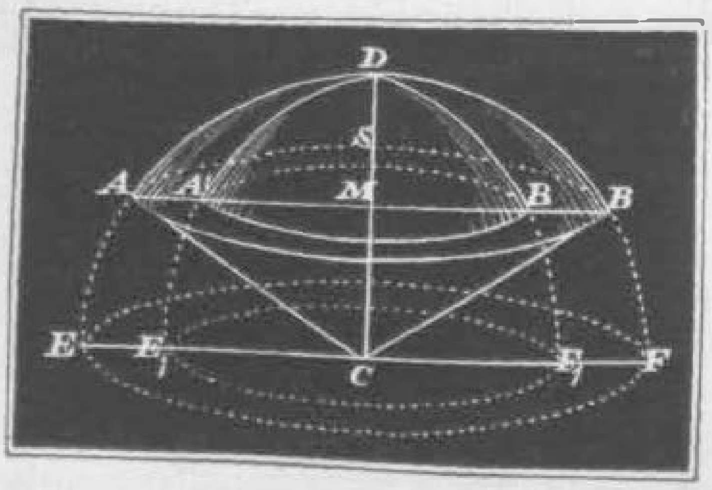
$$CS = CD_1 - D_1S = \frac{3}{4}\, r - \frac{3}{8}\, h = \frac{3}{4}\left(r - \frac{h}{2}\right).$$

For the semicircle, for example, $h = r$, therefore the distance of its centre of gravity S from the centre C is:

$$CS = \frac{3}{4} \cdot \frac{r}{2} = \frac{3}{8}\, r.$$

§ 117. The centre of gravity S of the segment of a sphere ABD, Fig. 95, is obtained when its moment is put equal to the difference of the moments of the sector $ADBC$ and that of the cone ABC. Again, if we put the radius of the cone $CD = r$ and the height $DM = h$, the moment of the sector $= \frac{2}{3}\pi r^2 h \cdot \frac{3}{8}\,(2r - h) = \frac{1}{4}\pi r^2 h\,(2r - h)$ and that of the cone $= \frac{1}{3}\pi h(2r - h) \cdot (r - h) \cdot \frac{3}{4}\,(r - h) = \frac{1}{4}\,\pi h\,(2r - h)\,(r - h)^2$; the moment of the segment of the sphere is therefore $= \frac{1}{4}\,\pi\, h\,(2r - h)\,(r^2 - [r - h]^2) = \frac{1}{4}\,\pi\, h^2\,(2r - h)^2$. The

Fig. 95.



volume of the segment $= \frac{1}{3}\pi h^2 (3r-h)$; hence, the distance in question is:

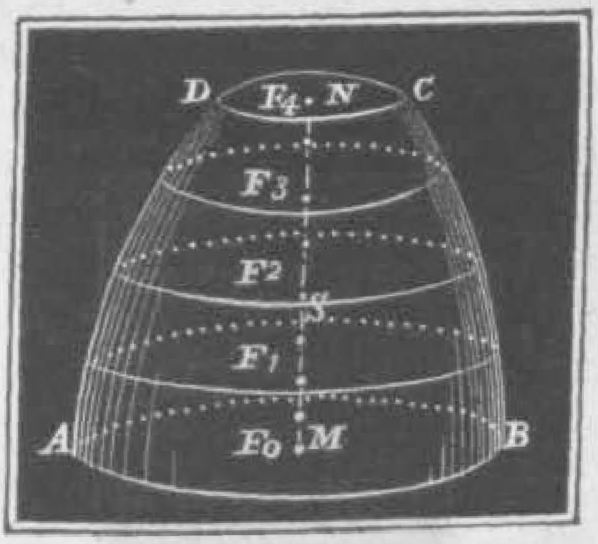
$$CS = \frac{\frac{1}{4}\pi h^2 (2r-h)^2}{\frac{1}{3}\pi h^2 (3r-h)} = \frac{3}{4} \cdot \frac{(2r-h)^2}{3r-h}.$$

If, again, we put $h = r$, the segment becomes a semicircle, and as above, $CS = \frac{3}{8} r$.

This formula holds good for the segment of a spheroid A_1DB_1, which is generated by the revolution of an elliptical arc DA_1 about its major semi-axis $CD = r$; for both segments may be divided into thin slices by planes parallel to the base AB, so that the ratio of any two is constant and $= \frac{\overline{MA_1}^2}{\overline{MA}_1^2} = \frac{\overline{CE_1}^2}{\overline{CE}^2} = \frac{b^2}{r^2}$, if b represent the semi-axis minor of the ellipse. The volume, as well as the moment of the segment of the sphere must be multiplied by $\frac{b^2}{r^2}$, in order to give the volume and moment of the segment of the spheroid, and thereby the quotient $CS = \frac{\text{moment}}{\text{volume}}$ will remain unchanged.

§ 118. To find the centre of gravity of an irregular body $ABCD$, Fig. 96, we must decompose it into thin slices, by planes equi-distant from each other, determine the solid contents of each slice, their moments with reference to the first parallel plane AB serving for the base, and finally connect them together by Simpson's rule.

Fig. 96.



The contents of these slices are F_0, F_1, F_2, F_3, F_4, and the whole height or distance of the outermost parallel plane is $= h$; the volume of the body, therefore, according to Simpson's rule (approximately) is:

$$V = (F_0 + 4F_1 + 2F_2 + 4F_3 + F_4)\frac{h}{12}.$$

If we multiply in this formula each of these volumes by their distance, we obtain the moment:

$$Vy = (0 \cdot F_0 + 1 \cdot 4F_1 + 2 \cdot 2F_2 + 3 \cdot 4F_3 + 4F_4)\frac{h}{4} \cdot \frac{h}{12};$$

lastly, by dividing one expression by the other, we get the distance required:

$$MS = y = \frac{(0 \cdot F_0 + 1 \cdot 4F_1 + 2 \cdot 2F_2 + 3 \cdot 4F_3 + 4F_4)}{F_0 + 4F_1 + 2F_2 + 4F_3 + F_4}\frac{h}{4}.$$

If the number of elementary slices $= 6$, we have:

$$y = \frac{0.F_0 + 1.4F_1 + 2.2F_2 + 3.4F_3 + 4.2F_4 + 5.4F_5 + 6.F_6}{F_0 + 4F_1 + 2F_2 + 4F_3 + 2F_4 + 4F_5 + F_6} \cdot \frac{h}{6}.$$

It is easy to understand how this formula may be altered when the number of slices is different from the above. This rule requires only that the number of the slices should be even, and, therefore, that of the surfaces uneven.

In most cases of application, the determination of one distancenis enough, because, besides this, a line of gravity is known. The bodies commonly met with in practice are solids of rotation, generated in a lathe whose axis of rotation is the line of gravity.

This formula, lastly, is applicable to the determination of the centre of gravity of a surface, in which case the sections F_0, F_1, F_2, become lines.

Example.—1. For the parabolic conoid ABC. Fig. 97, which is generated by the revolution of a parabola ABM about its axis AM, we obtain by making the section DNE, the following:

The height $AM = h$, the radius $BM = r$, $AN = NM = \frac{h}{2}$, and hence the radius $DN = r\sqrt{\frac{1}{2}}$. The area of the section through A is $F_0 = 0$, of that through $N = F_1 = \pi \overline{DN^2} = \frac{\pi r^2}{2}$, and of that through $M = F_2 = \pi r^2$. Hence the volume of this body is:

$$V = \frac{h}{6}(0 + 4F_1 + F_2) = \frac{h}{6}(2\pi r^2 + \pi r^2) = \tfrac{1}{2}\pi r^2 h = \tfrac{1}{2}F_2 h;$$

on the other hand, the moment is $= \frac{h^2}{12}(1 \cdot 2\pi r^2 + 2 \cdot \pi r^2) = \frac{1}{3}\pi r^2 h^2 = \frac{1}{3}F_2 h^2$; lastly, the distance of the centre of gravity S from the vertex, is:

$$AS = \frac{\frac{1}{3}F_2 h^2}{\frac{1}{2}F_2 h} = \tfrac{2}{3}h.$$

Fig. 97.

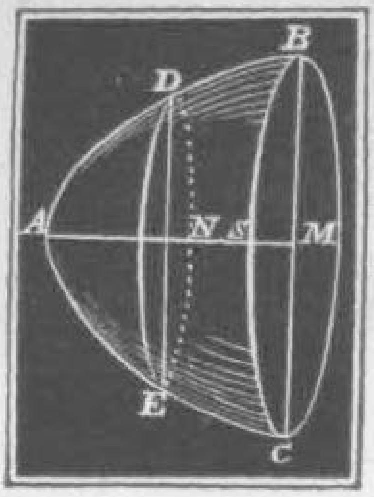
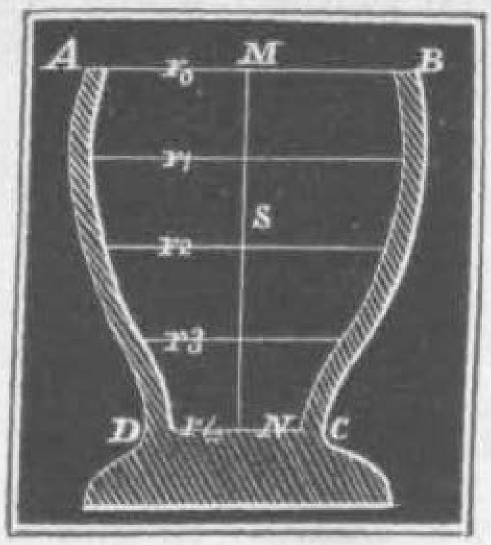


Fig. 98.



Example 2. A vessel $ABCD$, Fig. 98, has its mean half breadths, $r_0 = 1$ inch, $r_1 = 1{,}1$ inch, $r_2 = 0{,}9$ inch, $r_3 = 0{,}7$ inch, $r_4 = 0{,}4$ inch, with a height $MN = 2{,}5$ inch. The sections are $F_0 = 1 . \pi$, $F_1 = 1{,}21 . \pi$, $F_2 = 0{,}81t . \pi$, $F_3 = 0{,}49t . \pi$, $F_4 = 0{,}16t . \pi$; hence, the distance of the centre of gravity from the horizontal plane AB, is:

$$MS = \frac{0 . 1\pi + 1 . 4 . 1{,}21 . \pi + 2 . 2 . 0{,}81\pi + 3 . 4 . 0{,}49\pi + 4 \; 0{,}16 . \pi}{1\pi + 4 . 1{,}21\pi + 2 . 0{,}81\pi + 4 . 0{,}49\pi + 0{,}16\pi} \cdot \frac{2{,}5}{4}$$

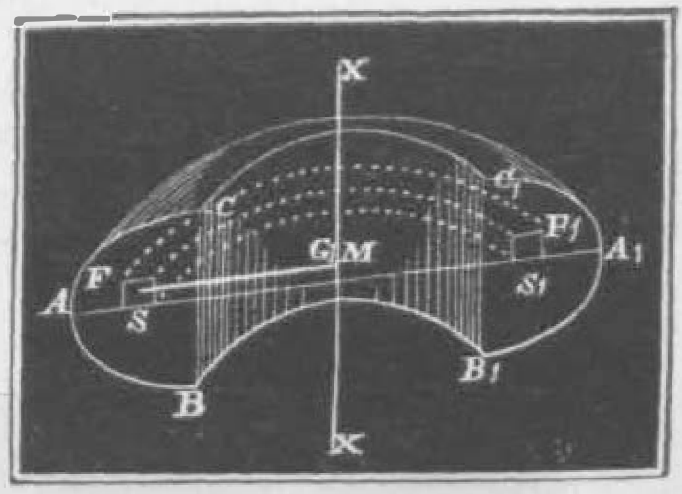
$$= \frac{14.60}{9{,}58} \cdot \frac{2{,}5}{4} = \frac{3650}{38{,}32} = 0{,}9502 \text{ inches.}$$

The capacity, therefore, is $= 9{,}58\pi \cdot \frac{2.5}{12} = 6{,}270$ cubic inches.

§ 119. An interesting and sometimes very useful application of the laws of the centre of gravity is the *properties of Guldinus*, or the barocentric method. According to these, the volume of a body of revolution (or of a surface of revolution) is equal to the product of the generating surface (or generating line), and the space described by its centre of gravity during the generation of the body or surface of revolution. The correctness of this proposition may be made evident in the following manner.

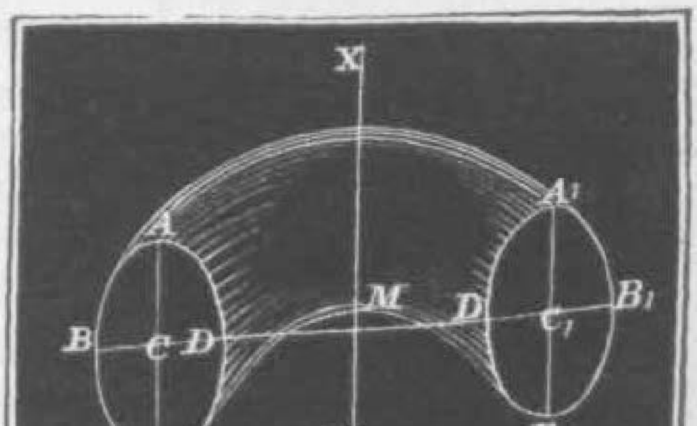
Guldinus' Properties.—If the plane area ABC, Fig. 99, revolve about an axis $X\overline{X}$, each element $F_1 F_2$, &c., of the same will describe an annulus; if the distances $F_1 G_1$, $F_2 G_2$, &c., of these elements from the axis of revolution $X\overline{X}_1$ be $= r_1, r_2$, &c., and the angle of revolution $AMA_1 = \alpha^0$, therefore the arc corresponding to the radius $1 = \alpha$, the circular paths of the elements will be $= r_1\alpha$, $r_2\alpha$, &c. The spaces described by the elements F_1, F_2, &c., may be considered as curved prisms having the bases F_1, F_2, &c., and the heights $r_1\alpha$, $r_2\alpha$, &c., and the volumes $F_1 r_1 \alpha$, $F_2 r_2 \alpha$, &c., and therefore the volume of the whole body $ABCB_1A_1C_1$: $V = F_1 r_1 \alpha + F_2 r_2 \alpha \ldots = (F_1 r_1 + F_2 r_2 + \ldots) \cdot \alpha$. If $MS = x$ be the distance of the centre of gravity S of the generating surface from the axis of revolution, we have also $(F_1 + F_2 + \ldots) x = F_1 r_1 + F_2 r_2 + \ldots$, consequently the volume of the whole body $V = (F_1 + F_2 + \ldots) x\alpha$. But $F_1 + F_2 + \ldots$ are the contents of the whole surface F, and $x\alpha$ the circular arc $w\ SS_1$, described by the centre of gravity S; consequently, $V = Fw$, as above enunciated. This formula holds good also for the revolution of a line, because it may be considered as a surface made up of infinitely small breadths; F is namely $= Lw$: *i.e.* the surface of revolution is a product of the generating line (L) and the path (w) of its centre of gravity.

Fig. 99.



Example.—1. In a half ring of an elliptical section $ABED$, Fig. 100, let the semi-axis of the section be $CA = a$ and $CB = b$, and let the distance CM of the centre C from the axis $X\overline{X} = r$; then the elliptical generating surface $F = \pi ab$, and the path of the centre of gravity (C) $w = \pi r$; hence the volume of this half-ring $V = \pi^2 abr$, and that of the whole ring $= 2\pi^2 abr$. If the dimensions be, $a = 5$ inches, $b = 3$ inches, $r = 6$ inches, the volume of one-fourth of the ring $= \frac{1}{2} \cdot \pi^2 . 5 . 3 . 6 = 9{,}8696 . 5 . 9 = 444{,}132$ cubic inches.

Fig. 100.



Example.—2. For a ring with a semicircular section ABD, Fig. 101, if $CA = CB = a$, represent the radius of this section, and $MC = r$ that of the hollow space or neck, the volume is

$$V = \frac{\pi a^2}{2} . 2\pi \left(r + \frac{4a}{3\pi} \right) = \pi a^2 \left(\pi r + \frac{4}{3} a \right).$$

Example.—3. To find the surface and volume of a cupola ADB of the dome of a convent, Fig. 102, half the width $MA = MB = a$, and the height $MD = h$ are given. From both dimensions it follows that the radius $CA = CD$ of the generating circle $= r = \frac{a^2 + h^2}{2a}$, and the angle ACD subtended at the centre by $AD = \alpha^0$, if we put the $\sin. \alpha =$

$\frac{h}{r}$. The centre of gravity S of an arc $DAD_1 = 2\,AD$ is determined by the distance $CS = r.\ \frac{\text{chord } MD}{\text{arc } AD} = \frac{r\,sin.\alpha}{\alpha}$; further, $CM = r\,cos.\,\alpha$, consequently the distance MS of the centre of gravity S from the axis $MD = \frac{r\,sin.\,\alpha}{\alpha} - r\,cos.\,\alpha = r\left(\frac{sin.\,\alpha}{\alpha} - cos.\,\alpha\right)$, and the path described by the centre of gravity in the generation of the surface $ADB = 2\pi r.\left(\frac{sin.\,\alpha}{\alpha} - cos.\,\alpha\right)$. The generating line $DAD_1 = 2\,r\,\alpha$, and since it only is required to determine the half ADB, this line $= r\,\alpha$, and consequently we must put the whole surface $O = r\,\alpha\,.\,2\,\pi\,r\left(\frac{sin.\,\alpha}{\alpha} - cos.\,\alpha\right) = 2\,\pi\,r^2\,(sin.\,\alpha - \alpha\,cos.\,\alpha)$.

Fig. 101.

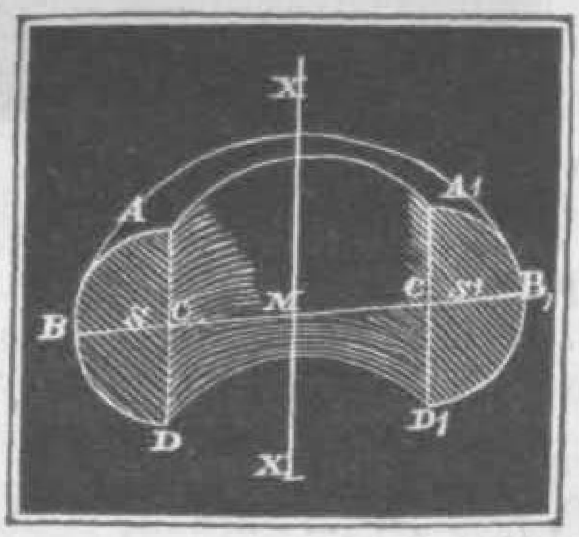
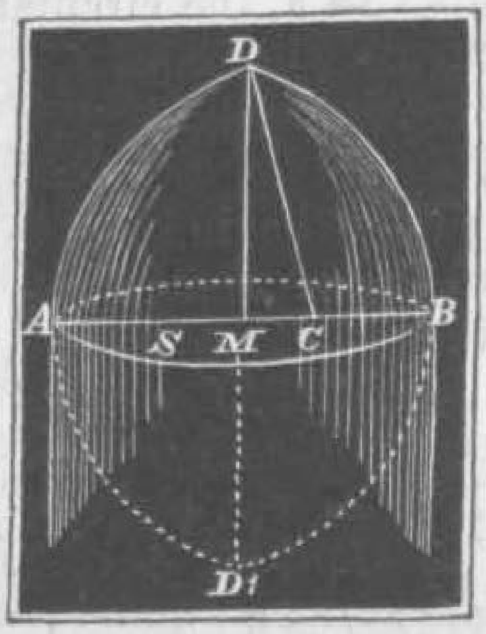


Fig. 102.



Very commonly $\alpha^0 = 60^0$; therefore, $\alpha = \frac{\pi}{3}$, $sin.\,\alpha = \frac{1}{2}\sqrt{3}$, and the $cos.\,\alpha = \frac{1}{2}$; hence it follows that $O = \pi\,r^2\left(\sqrt{3} - \frac{\pi}{3}\right) = 2{,}1515\,.\,r^2$.

For the segment $DAD_1 = A = r^2\,(\alpha - \frac{1}{2}\,sin.\,2\,\alpha)$ the distance of the centre of gravity from the centre C is $= \frac{(2\,.\,MD)^3}{12\,A} = \frac{2}{3}\,.\,\frac{r^3\,sin.\,\alpha^3}{A}$, hence the distance from the axis $MS = CS - CM = \frac{2}{3}\,.\,\frac{r^3\,sin.\,\alpha^3}{A} - r\,cos.\,\alpha$; finally, the path of this centre of gravity described in one revolution is:

$$w = \frac{2\,\pi\,r}{A}\left(\tfrac{2}{3}\,r^2\,sin.\,\alpha^3 - A\,cos.\,\alpha\right) = \frac{2\,\pi\,r^3}{A}\left(\tfrac{2}{3}\,sin.\,\alpha^3 - [\alpha - \tfrac{1}{2}\,sin.\,2\,\alpha]\,cos.\,\alpha\right).$$

The volume of the whole body generated by the segment DAD_1, is given if this path be multiplied by A, and the volume of the dome found by taking the half of this: therefore, $V = \pi\,r^3\,(\frac{2}{3}\,sin.\,\alpha^3 - [\alpha - \frac{1}{2}\,sin.\,2\,\alpha]\,cos.\,\alpha)$. For example, $\alpha^0 = 60^0 = \alpha = \frac{\pi}{3}$ $sin.\,\alpha = \frac{1}{2}\sqrt{3}$, and $cos.\,\alpha = \frac{1}{2}$; hence:

$$V = \pi\,r^3\left(\frac{3}{8}\sqrt{3} - \frac{\pi}{6}\right) = 0{,}3956\,.\,r^3.$$

Remark. Guldinus' properties find their application in those bodies which arise when the generating surface so moves that in every position it remains perpendicular to the path of its centre of gravity, because we may assume every small part of a curvilinear motion to be circular. From this we may find the solid contents of the threads of screws, and sometimes also calculate the masses of earth, heaped up or removed, as in the case of canals, roads, railroads, &c.

§ 120. Another application of the doctrine of the centre of gravity, nearly allied to the last rule, is the following.

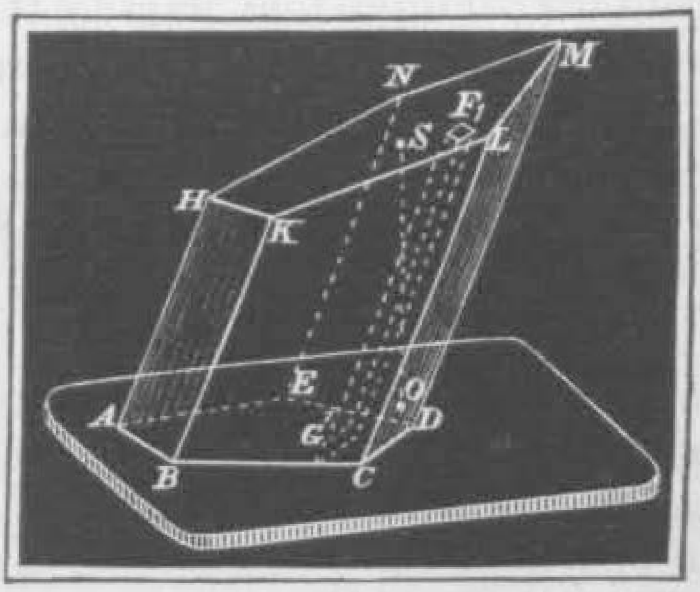
We may assume that every oblique prismatic body $ABCHKL$, Fig. 103, consists of an infinite number of thin prisms, similar to $F_1\,G_1$.

If G_1, G_2 are the bases, and h_1, h_2 the heights of these elementary prisms, we have for their solid contents $G_1 h_1$, $G_2 h_2$, &c., and the volume of the whole oblique prism $V = G_1 h_1 + G_2 h_2 + \ldots$ Now an element F_1 of the oblique section HKL is to the element G_1 of the base ABC as the whole oblique surface F to the base G; therefore,

$$G_1 = \frac{G}{F} F_1,\ G_2 = \frac{G}{F} F_2, \text{ \&c. and}$$

$$V = \frac{G}{F}(F_1 h_1 + F_2 h_2 + \ldots).$$

Fig. 103.



And because $F_1 h_1 + F_2 h_2 + \ldots$ is the statical moment Fh of the whole oblique section, it follows that:

$$V = \frac{G}{F} \cdot Fh = Gh, \text{ i. e.,}$$

the volume of an oblique prism is equal to the volume of a perfect prism, which stands upon the same base, and whose height is equal to the distance SO of the centre of gravity S of the oblique surface from the base.

In a right or oblique triangular prism, if h_1, h_2, h_3, be the edges of the sides, the distance of the centre of gravity of the oblique surface from the base $h = \frac{h_1 + h_2 + h_3}{3}$, hence the volume

$$V = G \frac{(h_1 + h_2 + h_3)}{3}.$$

CHAPTER III.

EQUILIBRIUM OF BODIES RIGIDLY CONNECTED AND SUPPORTED.

§ 121. *Kinds of Support.*—The rules developed in the first chapter of this section, on the equilibrium of a rigid system of forces, are applicable to that of rigid bodies acted upon by forces, if we consider the weight of the body as a force applied to its centre of gravity, and acting vertically downwards.

Bodies balanced by forces, are either *freely movable, i. e.* yield to the action of forces, or they are *fixed by one or more points*, or *supported* by other bodies.

If a point of a rigid body is fixed, any other point may take up a motion whose path lies in the surface of a sphere, described from the fixed point as a centre by the distance of the other point as radius. If, on the other hand, two points of a body are fixed in every possible motion, the paths described by the remaining points are circles, which are the intersections of two spherical surfaces described from the fixed points. These circles are parallel to each other, and perpendicular to the straight line joining the two fixed points. The points of this line remain immovable; and the body revolves about this line, which is called the axis of revolution.

The radius of the circle in which each point moves, is found by letting fall from the point a perpendicular upon the axis of revolution. The greater this is, the greater also is the circle in which the point revolves.

If three points of a body, not falling within the same line, be fixed, the body can in no sense take up motion, because the three spherical surfaces, in which a fourth point must move, intersect each other in a point only.

§ 122. *Kinds of Equilibrium.*—If a body, fixed at one point, be balanced by one force or by the resultant of several forces, the direction of this force must pass through the fixed point; for a point is fixed when every force passing through it is counteracted. If this force consist merely of the weight of the body, it is then necessary that its centre of gravity should lie in the vertical line passing through the fixed point. If the centre of gravity coincide with the fixed, or the so called point of suspension, we then have indifferent equilibrium, because the body is balanced, in whatever direction it may revolve about the fixed point. If a body AB, Fig. 104, be fixed or sustained at a point C lying above its centre of gravity S, it then finds itself in a

Fig. 104.

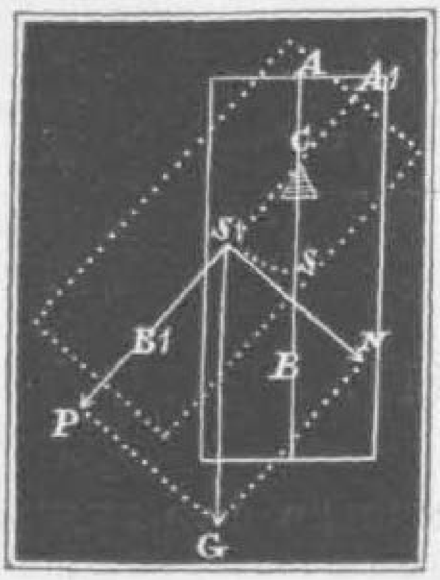
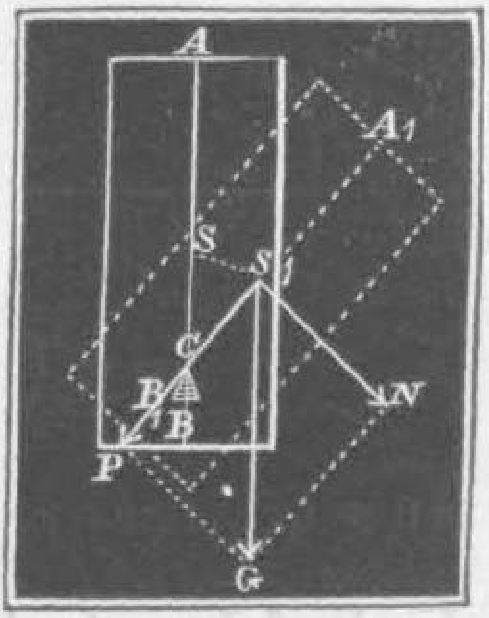


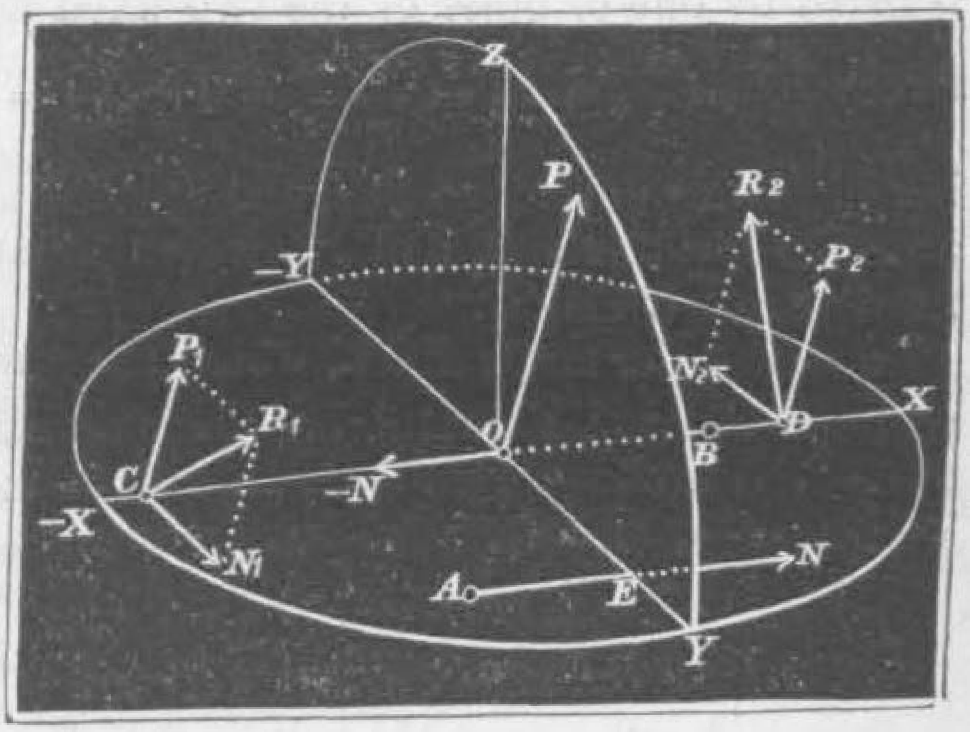
Fig. 105.



condition of stable equilibrium, because, if this body be brought into any other position, the component N of the weight G tends to bring it back into its first position, whilst the fixed point C counteracts the other component P. On the other hand, if a body AB, Fig. 105, be fixed at a point C below its centre of gravity S, the body is then in a state of unstable equilibrium; for if the centre of gravity be drawn out of the vertical passing through C, the component N of the weight of the body G not only does not bring it back into its former position, but draws it more and more out of that position, until the centre of gravity at last comes below the fixed point.

The same reasoning will also apply to the case of a body fixed by two points, or by an axis ; it will be either in indifferent, stable, or unstable equilibrium, according as the centre of gravity lies vertically above or vertically below the axis.

Fig. 106.



§ 123. *Pressure on the Axis.*—If a body, acted upon by forces in space, be fixed by two points or by a line, relations then take place, which we will now investigate. We may reduce, according to § 92, every system of forces to two, viz., one running parallel to the fixed axis, and the other acting in the plane normal to this line. Let $AN = N$, Fig. 106, be the first, parallel to the axis $X\overline{X}$, passing through the fixed points C and D; and $OP = P$, the second force acting in the plane $YZ\overline{Y}$ at right angles to the axis $X\overline{X}$. If we introduce other forces, as $B\overline{N} = -N$, $CN_1 = N_1$, and $DN_2 = -N_2$, we change nothing in the condition of equilibrium or of motion, because these forces are entirely taken up by the axis. Now the forces N and $-N$ form together a first couple, and the forces N_1 and $-N_1$, acting in the plane XY and perpendicular to XX, a second couple ; we may, therefore, so manage, that these shall perfectly replace each other. If EO is the normal distance between the force N and the axis $XX = y$, and CD that of the fixed points $= x$; from § 90, we have the moments of both couples $= Ny$ and N_1x, and these are equivalent to each other, if $Ny = N_1x$. We may also assume inversely that the force N is entirely taken up by the axis XX, whilst the axis has to sustain in its proper direction the pressure N, and the forces $N_1 = \frac{y}{x} N$, and $-N_1 = -\frac{y}{x} N$ applied perpendicularly to it at the points C and D.

That the body may be in a state of equilibrium, it is necessary that the direction also of the resultant acting in the normal plane YZ (at O) pass through the axis. This force P may be replaced by two parallel forces P_1 and P_2 applied at the points C and D, which may be determined, if we put $P_1 . CD = P . DO$ and $P_2 . CD = P . CO$; the axis $X\overline{X}$ will have, therefore, besides the forces $BN = -N$, $CN_1 = N_1$ and $DN_2 = -N_1$, also to react against the forces $P_1 = \frac{x_2}{x} . P$ and $P_2 = \frac{x_1}{x} . P$, which may be calculated from the distances $CD = x$, $OC = x_1$, and $OD = x_2$.

§ 124. From the results of the investigations of the foregoing paragraph we may easily calculate the forces sustained by the axis and the fixed points C and D. First, the axis has a pressure to sustain equivalent to the force N in its own direction, which may be entirely resisted by one or other of the two fixed points. Secondly, from the forces $N_1 = \frac{y}{x} N$, $P_1 = \frac{x_2}{x} P$ and $-N_1 = -\frac{y}{x} N$ and $P_2 = \frac{x_1}{x} P$, acting in planes normal to $X\overline{X}$, and applied at the points C and D, there arise the resultants R_1 and R_2, which must be also sustained by the fixed points C and D.

If we put the angle POY, which the direction of the force P makes with the plane XY containing the axis $X\overline{X}$ and the direction of the force $N = \alpha$, the angle $N_1 CP_1$ is also $= \alpha$; on the other hand, $N_2DP_2 = 180^\circ - \alpha$, and the resultant pressures are therefore given by:

$$R_1 = \sqrt{N_1^2 + P_1^2 + 2N_1 P_1 \cos. \alpha}, \text{ and}$$
$$R_2 = \sqrt{N_1^2 + P_2^2 - 2N_1 P_2 \cos. \alpha}.$$

Example. A set of forces of a body fixed by its axis $X\overline{X}$, is resolved into a normal force $P = 36$ lbs., and a parallel force $N = 20$ lbs.; the distance of the last from the axis is $y = 1\frac{1}{2}$ feet, and the distance $CD = x = 4$ feet. To find the forces sustained by the axis, or by the fixed points in it, with the condition that the direction of P deviate by an angle $\alpha = 65^\circ$ from the plane XY, and its point of application O be distant by $CO = x_1 = 1$ foot from the fixed point C? The force $N = 20$ lbs. imparts to the axis along its direction a thrust $N = 20$ lbs.; besides, it generates also the forces $N_1 = \frac{y}{x} N = \frac{1,5}{4} . 20 = 7,5$ lb. and $-N_1 = -7,5$ lb., against which the fixed points C and D react. From the force P arise the forces $P_1 = \frac{x_2}{x} P = \frac{4-1}{4} . 36 = 27$ lbs. and $P_2 = \frac{x_1}{x} P = \frac{1}{4} . 36 = 9$ lbs. and by substitution of these values we have the resultant forces:

$$R_1 = \sqrt{7,5^2 + 27^2 + 2 . 7,5 . 27 . \cos. 65^\circ} = \sqrt{56,25 + 729 + 171,160}$$
$$= \sqrt{956,410} = 30,926 \text{ lb., and}$$
$$R_2 = \sqrt{7,5^2 + 9^2 - 2 . 7,5 . 9 . \cos. 65^\circ} = \sqrt{56,25 + 81 - 57,054}$$
$$= \sqrt{80,196} = 8,955 \text{ lbs.}$$

§ 125. *Equilibrium of Forces about an Axis.*—The force P is the resultant of all those component forces whose directions lie in one or more planes normal to the axis. But now in these cases, from § 86, the statical moment Pa of the resultant is equivalent to the sum $P_1a_1 + P_2a_2 + \ldots$ of the statical moments of the components, and for the condition of equilibrium of the fixed body the arm a of the resultant $= 0$, because this passes through the axis; hence the sum is also:

$$P_1a_1 + P_2a_2 + \ldots = 0;$$

i. e. a body fixed by its axis is in a state of equilibrium, and remains also without revolving, if the sum of the moments about this axis $= 0$, or if the sum of the moments of the forces acting in one direction of revolution, is equivalent to the sum of the moments of those acting in the opposite direction. By the help of this last formula we may find either a *force* or an *arm* for an element of a system of forces in equilibrium.

Example. The forces of rotation $P_1 = 50$ lbs., and $P_2 = -35$ lbs., act upon a body capable of turning about an axis at the arms $a_1 = 1\frac{1}{4}$ foot, and $a_2 = 2\frac{1}{2}$ feet; required, the force P_3 which shall act at the arm $a_3 = 4$ feet, in order to restore the balance, *i. e.* to prevent rotation about the axis? It is:

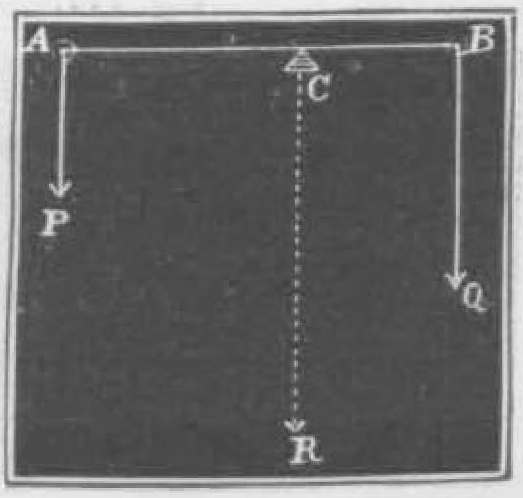
$$50e\ 1,25 - 35e\ 2,5 + 4\ P_3 = 0, \text{ hence}$$
$$P_3 = \frac{87,5 - 62,5}{4} = 6,25 \text{ lb.}$$

§ 126. *The Lever.*—A body capable of turning about a fixed axis, and acted upon by forces, is called a lever. If we imagine it to be devoid of weight, it is then called a *mathematical*, but otherwise, a *material* or *physical lever*.

It is generally assumed that the forces of a lever act in a plane at right angles to the axis, and that the axis is replaced by a fixed point, called the fulcrum. The perpendiculars let fall from this point on the direction of the forces, are called arms. If the directions of the forces of a lever are parallel, the arms form a single straight line, and the lever is called a straight lever. If the arms make an angle with each other, it is then called a bent lever. The straight lever acted upon by two forces, is either one-armed or two-armed, according as the points of application lie on the same or on opposite sides of the fulcrum. There is a distinction made of levers of the first, second and third order; the two-armed lever is termed a lever of the first order; the one-armed, of the second or third order, according as the weight acting vertically downwards, or the power acting vertically upwards, lies nearest to the fulcrum.

§ 127. The theory of the equilibrium of the lever has been already fully laid down; we have now, therefore, only to treat of each specially.

Fig. 107.



In the two-armed lever, *ACB*, Fig. 107, if the arm *CA* of the power *P* be designated by *a*, and the arm *CB* of the weight *Q* by *b*, from the general theory: $Pa = Qb$, *i. e.* the moment of the force is equal to the moment of the weight; or also, $P : Q = b\,e\,a$, *i. e.* the power is to the weight inversely as the arms. The pressure on the fulcrum is $R = P + Q$.

In the one-armed levers *ABC*, Fig 108, and *BAC*, Fig. 109, the same relation takes place between the power *P* and the weight *Q*, but here the direction

Fig. 108.

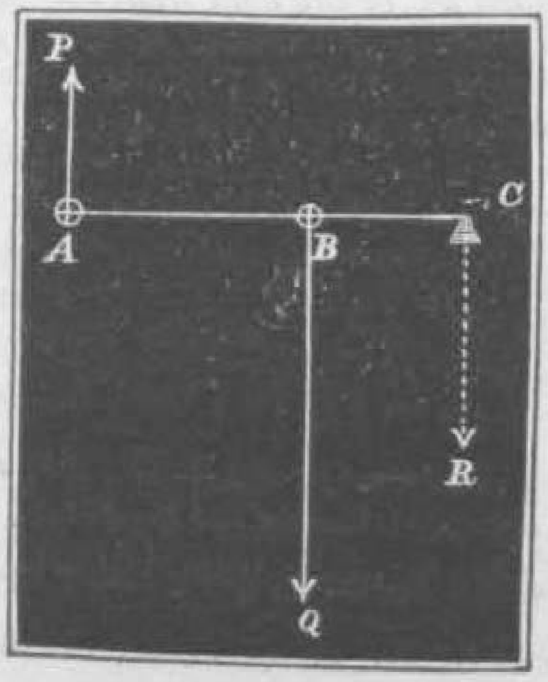
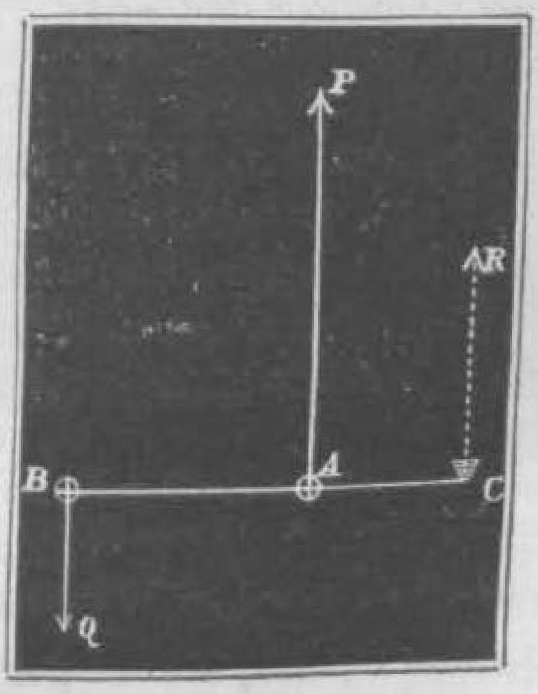


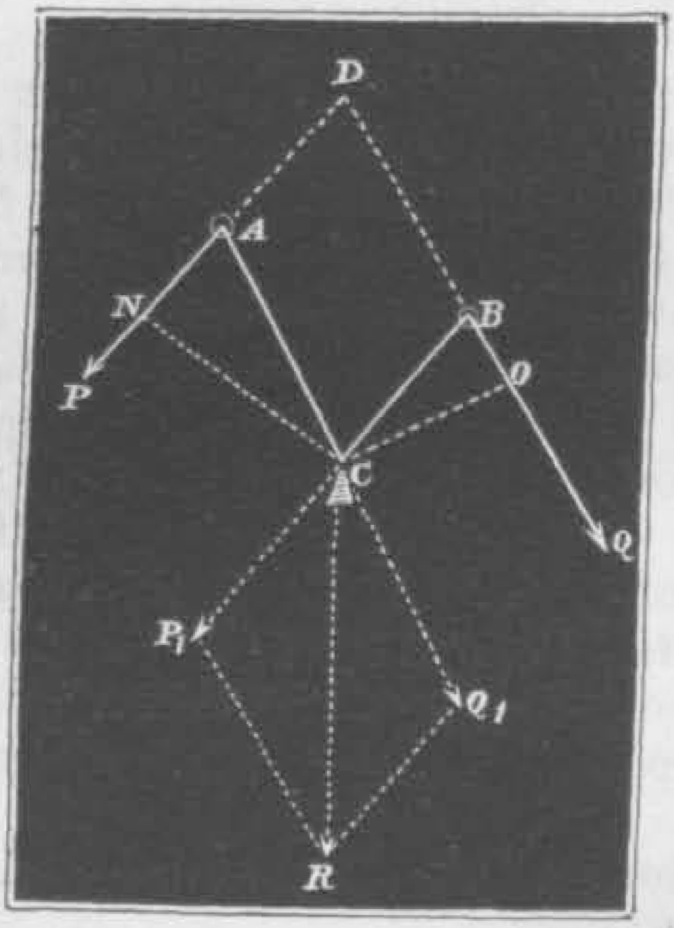
Fig. 109.



of the power is opposite to that of the weight, and therefore the pressure on the fulcrum is their difference, and in the first case $R = Q - P$, and in the second, $R = P - Q$.

Also in the bent lever ACB, with the arms $CN = a$ and $CO = b$, Fig. 110, $P : Q = b : a$, here the pressure on the fulcrum is equivalent to the diagonal R of the parallelogram CP_1RQ_1, which may be constructed from the power P, the weight Q and the angle $P_1CQ_1 = PDQ = \alpha$, which their directions make with each other.

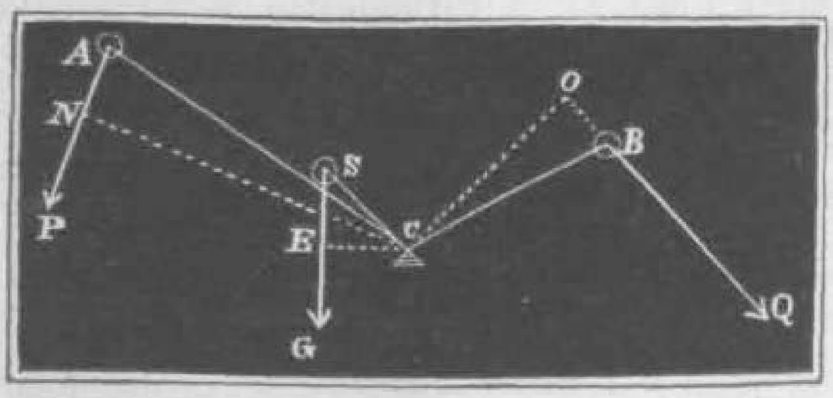
Fig. 110.



Let G be the weight of the lever, and $CE = e$, Fig. 111, the distance of the fulcrum C from the vertical line SG, passing through its centre of gravity; we shall then have to put $Pa \pm Ge = Qb$, and the plus or minus sign before G, according as the centre of gravity lies on the side of the power P, or on that of the weight Q.

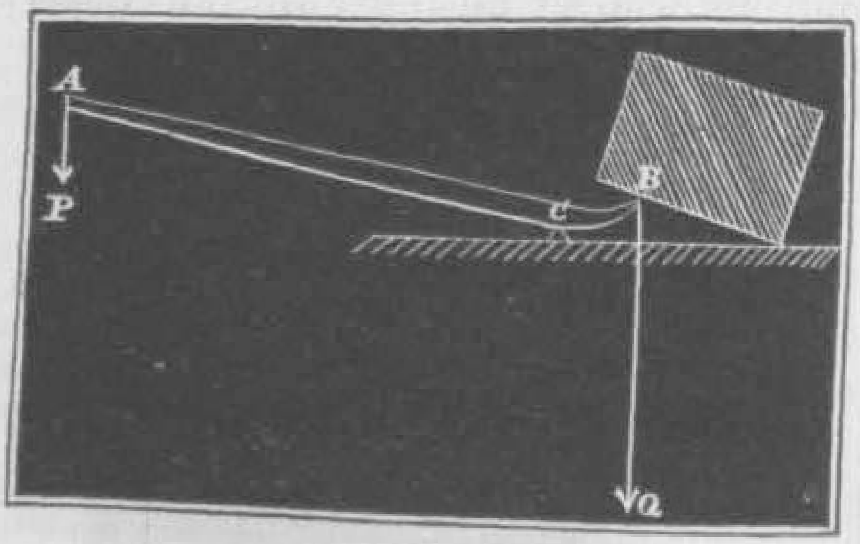
Remark. The theory of the lever finds its application in many tools and machines, viz. in the different kinds of balances, crow-bars, the brakes of pumps, wheelbarrows, &c. The second part will treat fully of these.

Fig. 111.



Example.—1. If we press down the end A of a crow-bar ACB, Fig. 112, with a force $P = 60$ lbs., and with the arm CA of the power equal to 12 times that of the arm CB of the weight, then will this, or rather the force exerted at B, be $= Q = 12$ times that of $P = 12 \,.\, 60 = 720$ lbs.—2. If a load Q, hanging from a pole, Fig. 113, be carried by two men, which pole the one lays hold of at A and the other at B, we may readily find out what weight each has to sustain. Let the load $Q = 120$ lbs., the weight of the pole $G = 12$ lbs., the distance AB of both points of application $= 6$ ft., the distance BC of the load from one of these points $= 2\frac{1}{2}$ feet, the distance of the centre of gravity S of the pole from this same point $BS = 3\frac{1}{2}$ feet. If we take B for the fulcrum, the power P_1 has to balance at A the weights Q and G, therefore $P_1 \,.\, BA = Q \,.\, BC + G \,.\, BS$, i. e. $6\,P_1 = 2{,}5 \,.\, 120 + 3{,}5 \,.\, 12 = 300 + 42 = 342$; hence, $P_1 = \frac{342}{6} = 57$ lbs. On the other hand, if

Fig. 112.



A be considered as the fulcrum, we must put $P_2 \,.\, AB = Q \,.\, AC + G \,.\, AS$, and in numbers, $6\,P_2 = 3{,}5 \,.\, 120 + 2{,}5 \,.\, 12 = 420 + 30 = 450$; hence, the power P_2 of the second man is $P_2 = \frac{450}{6} = 75$ lbs.; also, the sum of the forces $P_1 + P_2$ acting upwards, $= 57 + 75 = 132$ lbs., is exactly equal to the sum of the forces acting downwards,

$Q+G=120+12=132$ lbs.—3. In a bent lever, ACB, Fig. 114, of 150 lbs. wt. the vertically pulling force $Q=650$ lbs., and the arm $CB=4$ ft., but the arm CA of the power $P=$

Fig. 113.

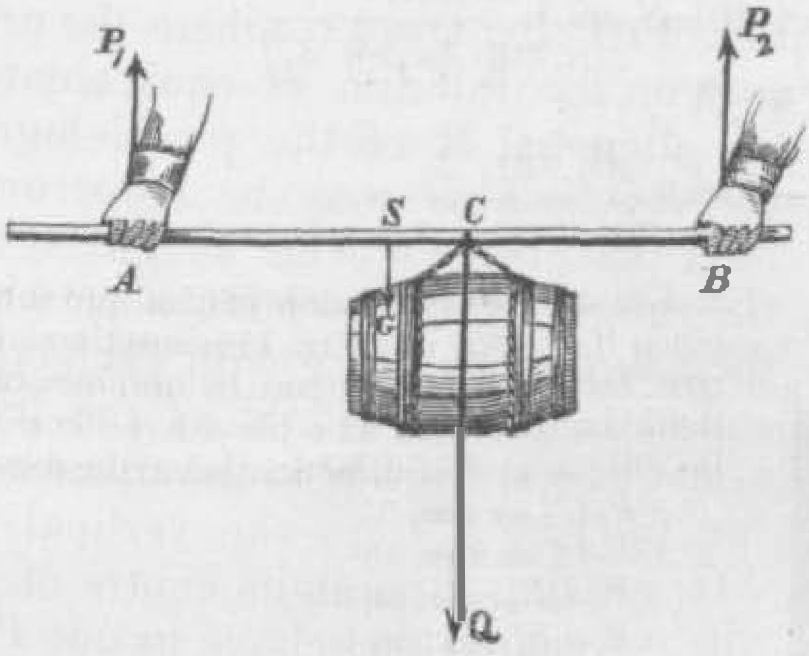
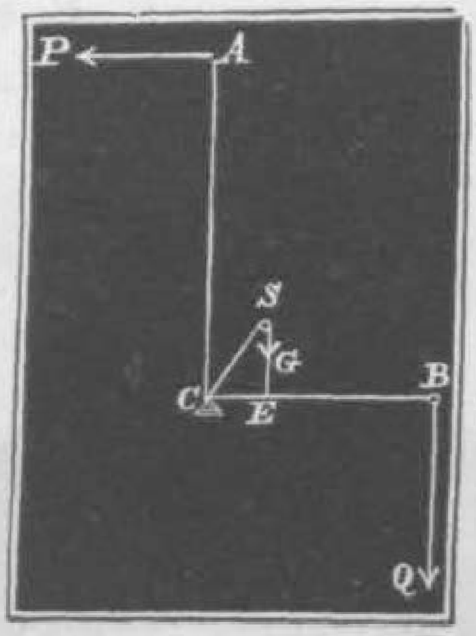


Fig. 114.

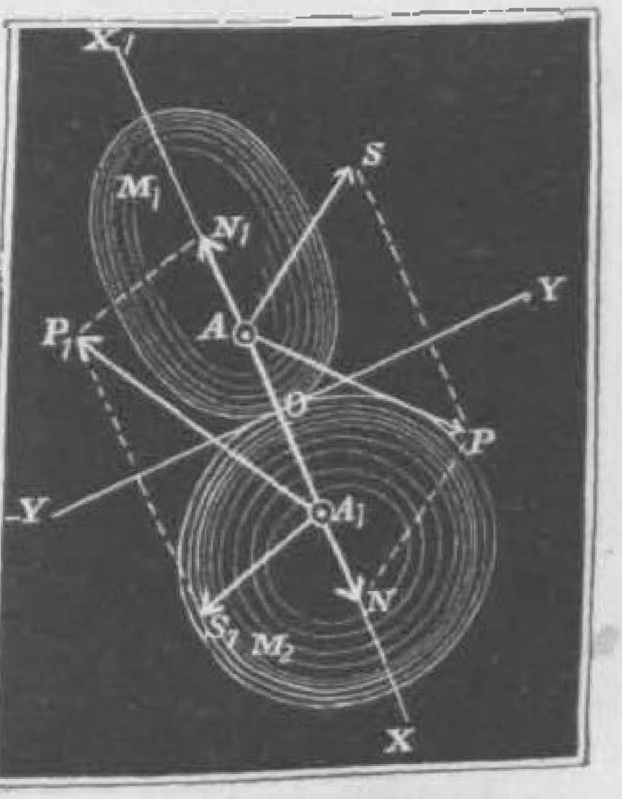


6 ft, and that of the weight $CE=1$ foot, what is the amount of the power P, and the pressure on the pivot R required to restore the balance? $CA \cdot P = CB \cdot Q + CE \cdot G$, i. e., $6\,P = 4 \cdot 650 + 1 \cdot 150 = 2750$; consequently, the power $P = \frac{2750}{6} = 458\frac{1}{3}$ lbs.; the pressure on the pivot consists of the vertical force $Q+G=650+150=800$ lbs., and the horizontal power $P=458\frac{1}{3}$ lbs., and is therefore:

$$= R = \sqrt{(Q+G)^2 + P^2}$$
$$= \sqrt{(800)^2 + (458\tfrac{1}{3})^2}$$
$$= \sqrt{850070} = 922 \text{ lbs.}$$

§ 128. *Pressure of Bodies on one another.*—The experimental law announced in § 62, that *action and reaction are equal to each other*, is the basis of the whole mechanics of machines. It is necessary in this place to make the meaning of this still clearer. When two bodies M_1 and M_2, Fig. 115, act upon each other with the forces P and P_1, whose directions deviate from the normal common $\overline{XX}$ to the two surfaces at their point of contact, a decomposition of the forces is always possible; the one component N or N_1, which is in the direction of the normal, passes over from the one body to the other, the other component S or S_1 remains in the body, and must be counteracted by another force or resistance, in order to maintain the bodies in equilibrium. From the principle set forth, perfect equilibrium is found to subsist between the normal components N and N_1.

Fig. 115.



If the direction of the force P deviates by the angle $NAP=\alpha$ from the normal AX and by the angle $SAP=\beta$ from the direction of the second component S, we have (§ 75)

$$\mathcal{N} = \frac{P \sin. \beta}{\sin. (\alpha + \beta)}, \quad S = \frac{P \sin. \alpha}{\sin. (\alpha + \beta)}.$$

If we represent $\mathcal{N}_1 \mathcal{A}_1 P_1$ by α_1 and $S_1 \mathcal{A}_1 P_1$ by β_1, we also have

$$\mathcal{N}_1 = \frac{P_1 \sin. \beta_1}{\sin. (\alpha_1 + \beta_1)} \text{ and } S_1 = \frac{P_1 \sin. \alpha_1}{\sin. (\alpha_1 + \beta_1)};$$

lastly, from the equality $\mathcal{N} = \mathcal{N}_1$,

$$\frac{P \sin. \beta}{\sin. (\alpha + \beta)} = \frac{P_1 \sin. \beta_1}{\sin. (\alpha_1 + \beta_1)}.$$

Example.—What resolution of the forces takes place if a body M_2, Fig. 116, sustained by a support DE, be pressed upon by another, capable of revolving about an axis C with a force $P = 250$ lbs., the angles of direction being the following:

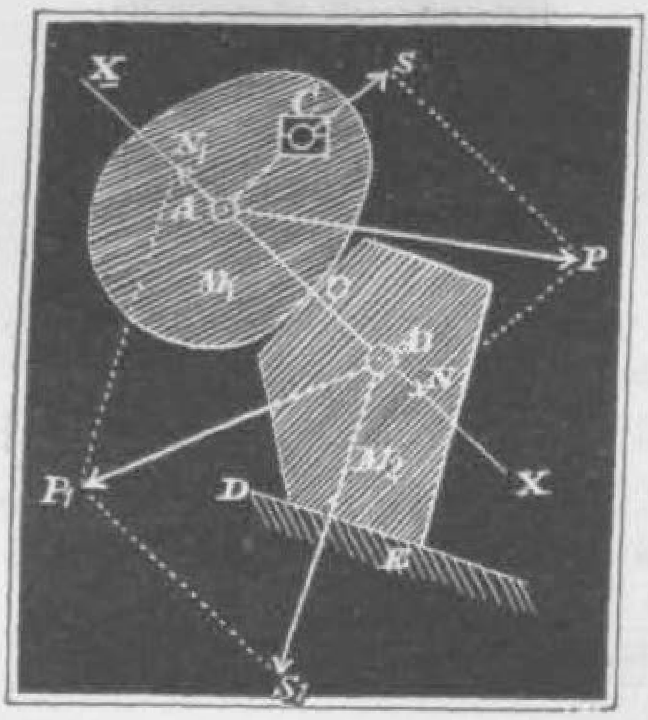
$P\mathcal{A}N = \alpha = 35^0$,

$P\mathcal{A}Se = \beta = 48^0$,

$P_1 \mathcal{A}_1 N_1 = \alpha_1 = 65^0$,

$P_1 \mathcal{A}_1 S_1 = \beta_1 = 50^0$.

Fig. 116.



From the first formula the normal pressure between the two bodies is determined by

$$N = N_1 = \frac{P \sin. \beta}{\sin. (\alpha + \beta)} = \frac{250 \sin. 48^0}{\sin. 83^0} =$$

187,18 lbs.; from the second the pressure on the axis, or on the point C, is

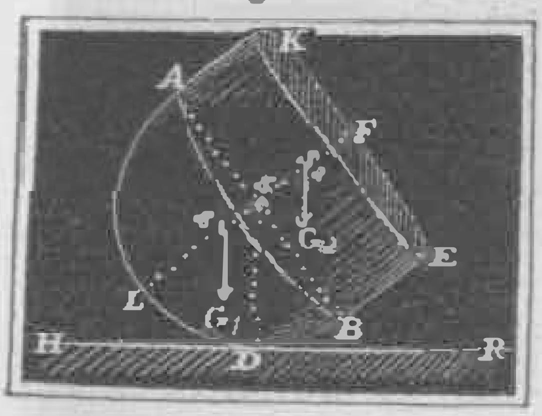
$$S = \frac{P \sin. \alpha}{\sin. (\alpha + \beta)} = \frac{250 \sin. 35^0}{\sin. 83^0} =$$

144,47 lbs.; and by combining the third and fourth equation, there follows finally for the component opposed to DE:

$$S_1 = \frac{N_1 \sin. \alpha_1}{\sin. \beta_1} = \frac{187,18 \sin. 65^0}{\sin. 50^0} = 221,46 \text{ lbs.}$$

§ 129. *Stability.*—When a body pressing against a horizontal plane is acted upon by no other force than gravity, it has no tendency to move forward, because the weight acting vertically downwards is exactly sustained by this plane; nevertheless, a revolution of the body is possible. If the body $\mathcal{A}DBF$, Fig. 117, rests at a point D upon the horizontal plane HR, it will remain at rest, if its centre of gravity S be supported, *i.e.*, if it lie in the vertical line passing through D. If a body is supported at two points on the horizontal surface of another, it is requisite for its equilibrium that the vertical line of gravity should intersect the line connecting the two points. Lastly, if a body rests at three or more points on a horizontal plane, equilibrium subsists if the vertical line containing the centre of gravity passes through the triangle or polygon which is formed by the straight lines connecting the points of support.

Fig. 117.



In bodies which are supported, we must distinguish between stable and unstable equilibrium. The weight G of a body $\mathcal{A}B$, Fig. 118, draws its centre of gravity downwards; if no resistance be opposed to this force it will cause the body to turn until its centre of gravity has attained its lowest position, and equilibrium will then be restored. We may mention that the equilibrium is stable when the centre of gravity is in its lowest possible position, Fig. 119, and unstable when in its highest, Fig. 120, and indifferent, when the centre of gravity in every position of the body remains at the same height, Fig. 121.

Fig. 118.

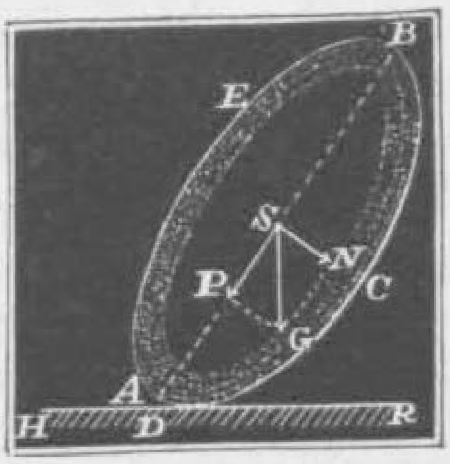


Fig. 119.

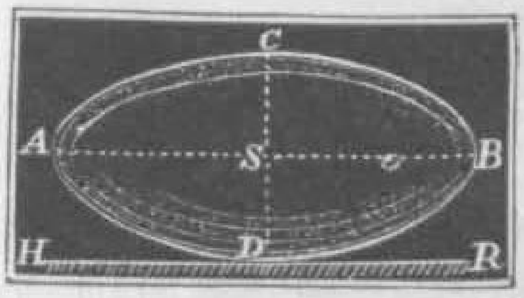


Fig. 120.

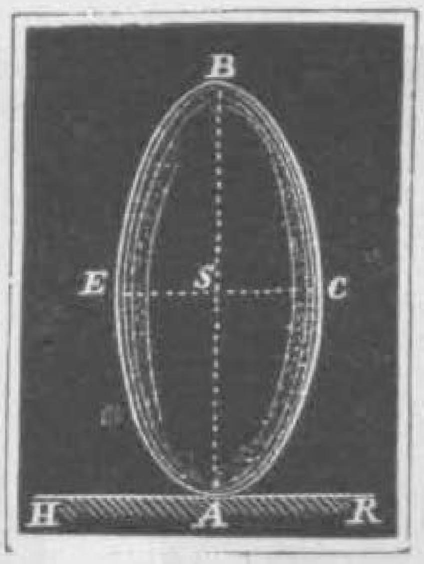
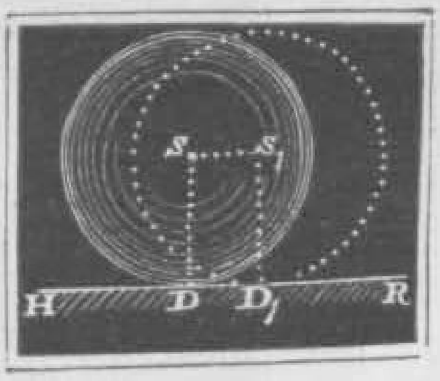


Fig. 121.

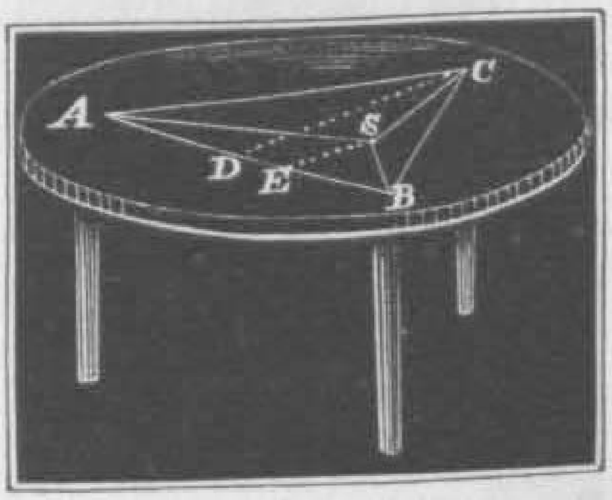


Example.—1. The homogeneous body $ADBF$, consisting of a hemisphere and a cylinder, Fig. 117, rests upon a horizontal plane HR. What height $SF = h$ must its cylindrical part have, that the body may be in equilibrium? The radius of a sphere is perpendicular to the corresponding plane of contact: now the horizontal plane is such a one; consequently the radius SD must be perpendicular to the horizontal plane, and the centre of gravity of the body lie in it. The axis FSL of the body passing through the centre of the sphere is its second line of gravity; the point S, the intersection of the two lines, is therefore the centre of gravity of the body.

Let us now put the radius of the sphere and cylinder $SA = SB = r$, and the height of the cylinder $SF = BE = h$, we then have for the volume of the hemisphere: $V_1 = \frac{2}{3}\pi r^3$, for the volume of the cylinder $V_2 = \pi r^2 h$; for the distance of the centre of gravity of the sphere $S_1 : SS_1 = \frac{3}{8} r$, and for that of the cylinder $S_2 : SS_2 = \frac{1}{2} h$. That the centre of gravity of the whole body may fall in S, the moment of the sphere $\frac{2}{3}\pi r^3 . \frac{3}{8} r$ must be put equal to the moment of the cylinder, $\pi r^2 h . \frac{1}{2} h$; from which we have:

$$h^2 = \tfrac{1}{2} r^2, \text{ i. e., } h = r\sqrt{\tfrac{1}{2}} = 0{,}7071 \text{ or } r.$$

Fig. 122.



2. The pressure which each of the three legs, A, B, C, Fig. 122, of any loaded table has to sustain, is determined in the following manner. Let S be the centre of gravity of the table with its load, and SE, CD, perpendiculars upon AB. If G be the weight of the whole table, and R the pressure on C, we may, considering AB as the axis, put the moment of $R =$ to the moment of G, i. e., $R . CD = G . SE$, and we then obtain $R = \frac{SE}{CD} . G = \frac{\Delta ABS}{\Delta ABC} . G$; likewise also the pressure on $B = Q = \frac{\Delta ACS}{\Delta ABC} . G$, and that on $A = P = \frac{\Delta BCS}{\Delta ABC} . G$.

§ 130. Let us now take the case of a body having a plane base resting on a horizontal plane. Such a body possesses stability, or is in stable equilibrium when its centre of gravity is supported, *i. e.* when the vertical line containing the centre of gravity of the body passes through its base, because, in this case, the tendency of the weight of the body to cause it to turn is prevented by its own rigidity. When the line of gravity passes through the edge of the base, the body is then in unstable equilibrium, and when the line passes outside the base, no equilibrium subsists. The body falls to one side and overturns. The triangular prism *ABCDE*, Fig. 123, is, according to the above, stable, because the vertical *SG* passes through a point *N* of the base. The parallelopiped *ABCG*, Fig. 124, is in unstable

Fig. 123.

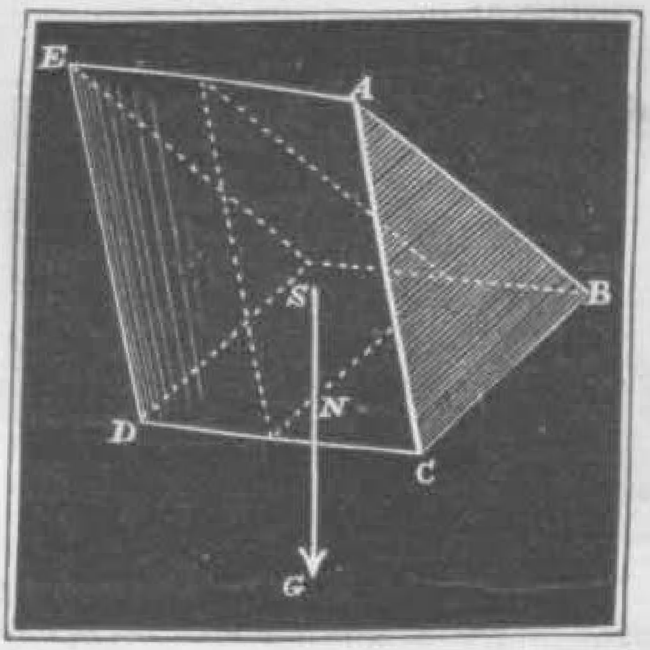
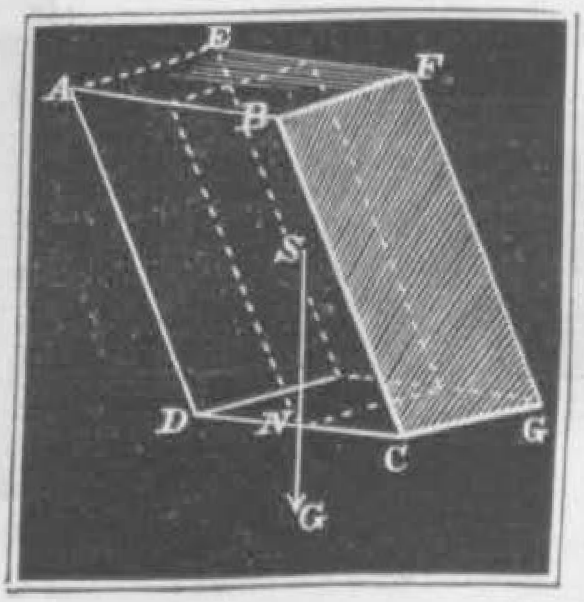
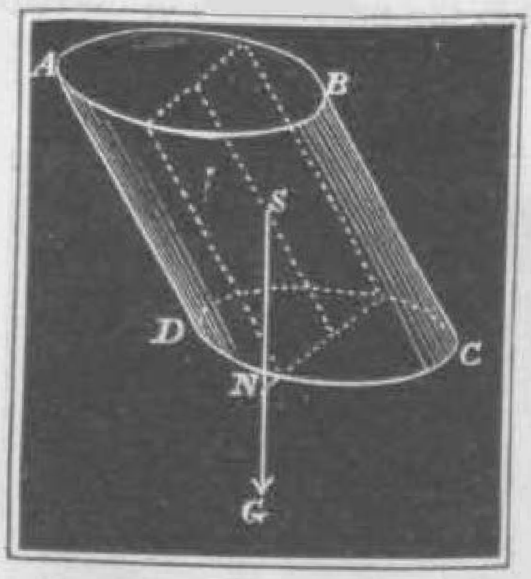


Fig. 124.



equilibrium, because *SG* intersects a side *CD* of the base. The cylinder *ABCD*, Fig. 125, is without stability because *SG* no where intersects the base *CD*.

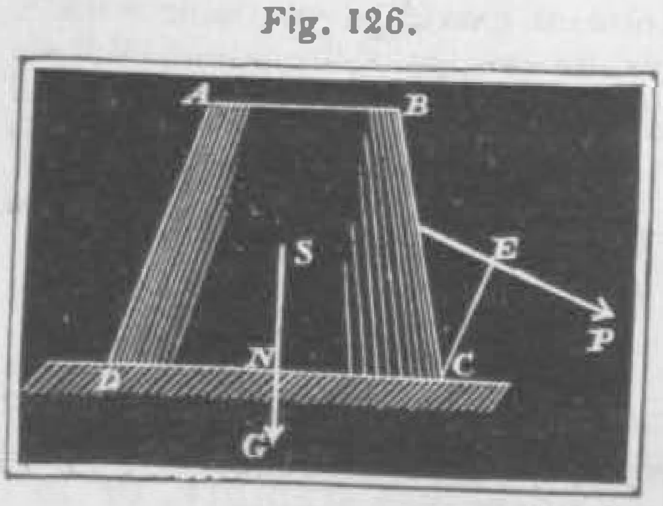
Fig. 125.



Stability is the power of a body to preserve its position by its weight alone, and to oppose resistance to any cause tending to overturn it. If we have to choose a measure of the stability of a body, we must distinguish whether this has reference to a displacement or to an actual overturning of the body. Let us now take into consideration the first only of these circumstances.

§ 131. *Formulæ of Stability.*—A force, *P*, not directed vertically, tends not only to overturn a body *ABCD*, Fig. 126, but also to push it forward; let us assume in the mean time that a resistance is opposed to the pushing or pulling forwards, as it may happen, and have regard only to its revolving about one of its edges *C*. If we let fall from this edge *C* a perpendicular *CE*n$=a$ upon the direction of the force, and *CN* $=x$ upon the vertical line *SG* passing through the centre of gravity, we have

Fig. 126.



only to consider a bent lever ECN, for which $Pa = Gx$, so that $P = \frac{x}{a} G$; if the external force P be greater than $\frac{x\,G}{a}$ the body revolves about the point C, and, therefore, loses its stability. Hence the stability depends upon the product (Gx) of the weight of the body, and the shortest distance between a side of the perimeter of the base and the vertical line passing through the centre of gravity; Gx may therefore be regarded as a measure of the stability, and for this reason is properly called the stability itself.

Hence we see that the stability increases simultaneously with the weight G and the distance x, and may conclude that under otherwise similar circumstances a body twice or thrice as heavy does not possess more stability than one of the single weight with twice or thrice the distance or arm x, &c.

§ 132.—1. In a parallelopiped $ABCF$, Fig. 127, of the length $AE = l$, breadth $AB = CD = b$, and height $AD = BC = h$, the weight $G = V\gamma = bhl\gamma$, and the stability $S = G \,.\, KN$s$= G.\ \frac{1}{2}\ CD$s$= \frac{Gb}{2} = \frac{1}{2} b^2hl\gamma$, provided γ represent the density of the mass of the parallelopiped.

2. In a body $ACFH$ consisting of two parallelopipeds, Fig. 128, the stabilities about the two edges of the base C and E are different

Fig. 127.

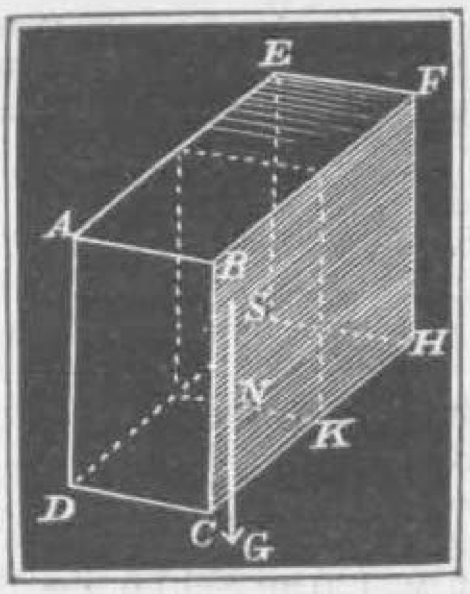
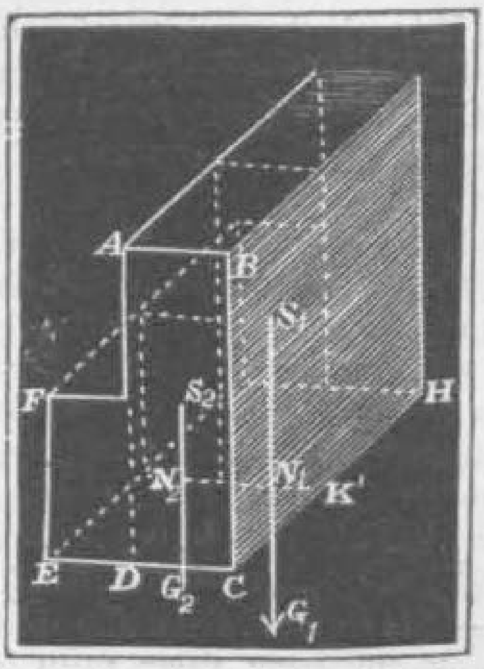


Fig. 128.



from one another. Let us take the heights BC and $EF = h$ and h_1, and the breadths CD and $DE = b$ and b_1, the weights of the parts G and $G_1 = bhl\gamma$ and $b_1h_1l\gamma$; then the arms about C will be $KN_1 = \frac{1}{2} b$ and $KN_2 = b_1 + \frac{1}{2} b_1$, and those about $E = b_1 + \frac{1}{2} b$ and $\frac{1}{2} b_1$. The stabilities accordingly are: first for the revolution about C,

$$S = \tfrac{1}{2} Gb + G_1 (b + \tfrac{1}{2} b_1), = (\tfrac{1}{2} b^2h + bb_1h_1 + \tfrac{1}{2} b_1{}^2h_1) l\gamma,$$

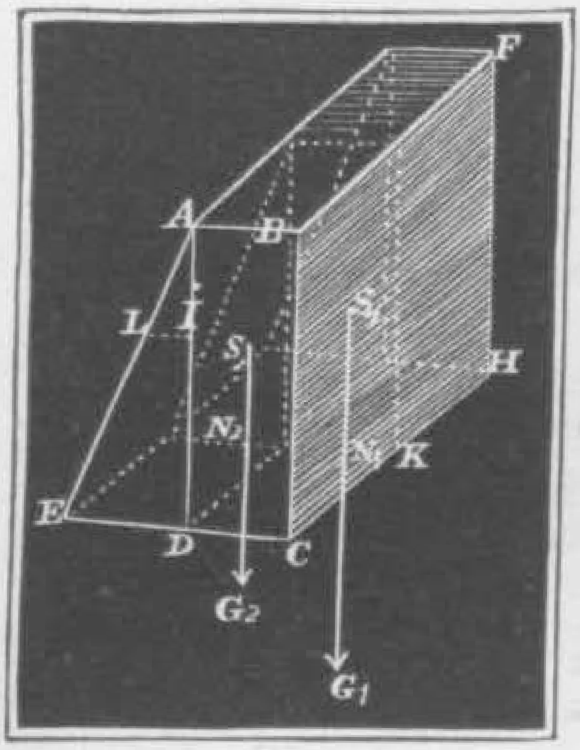
secondly for that about E,

$$S_1 = G (b_1 + \tfrac{1}{2} b) + \tfrac{1}{2} G_1b_1 = (\tfrac{1}{2} b_1{}^2h_1 + bb_1h + \tfrac{1}{2} b^2h)\, l\gamma.$$

The latter stability is about $S_1 - S = (h - h_1)\, bb_1l\gamma$ greater than the former; if we wish to increase the stability of a wall AC by offsets, these must be placed on that side of the wall towards which the force of revolution (wind, water, pressure of earth, &c.) acts.

3. The following is the stability of a wall $ABCEF$, Fig. 129, battering on one side. The upper breadth $AB = b$, the height $BC = h$ and the length $CH = l$, and the batters$= n$, *i.e.* upon AI = a height of 1 foot; $IL = n$ feet or inches of batter, therefore upon h feet $ED = nh$. The weight of the parallelopiped ACF is $G_1 = bhl\gamma$, that of the three sided prism $ADE = G_2 = \frac{1}{2} nh . hl\gamma$, the arms for a revolution about E are $= DE + \frac{1}{2} b = nh + \frac{1}{2} b$ and $\frac{2}{3} DE = \frac{2}{3} nh$, consequently for the stability we have

Fig. 129.



$$S = G_1 (nh + \tfrac{1}{2} b) + \tfrac{2}{3} G_2 nh$$
$$= (\tfrac{1}{2} b^2 + nhb + \tfrac{1}{3} n^2 h^2)\, hl\gamma.$$

A parallelopipedical wall of equal volume has the breadth $b + \frac{1}{2} nh$, hence the stability is:

$$S_1 = \tfrac{1}{2} (b + \tfrac{1}{2} nh)^2\, hl\gamma = (\tfrac{1}{2} b^2 + \tfrac{1}{2} nhb + \tfrac{1}{8} n^2 h^2)\, hl\gamma;$$

its stability is, therefore, about $S - S_1 = (b + \frac{5}{12} nh) . \frac{1}{2} nh^2 l\gamma$, less than that of the battered wall.

For a wall sloped upon the opposite side, the stability is $S_2 = (b^2 + nhb + \frac{1}{3} n^2 h^2) . \frac{1}{2} hl\gamma$, less also than S, and indeed about $S - S_2 = (b + \frac{1}{3} nh) . \frac{1}{2} nh^2 l\gamma$, as well as about $S_2 - S_1 = \frac{1}{24} n^2 h^3 l\gamma$ less than the stability of the parallelopipedical walls

Example. What is the stability for each foot in length of a stone wall of 10 feet in height, and $1\frac{1}{4}$ feet of upper breadth with batter of 1 in 5 on the back? The specific gravity of this wall (§ 58) is taken at 2,4, its density γ is, therefore, $= 62,5 . 2,4, =$ 130 lbs.; now $l = 1$, $h = 10$, $b = 1,25$, and $n = \frac{1}{5} = 0,2$; hence it follows, that the stability sought is:

$$S = (\tfrac{1}{2} . [1,25]^2 + 0,2 . 1,25e\ 10 + \tfrac{1}{3} . [0,2]^2 . 10^2)\ 10 . 1 . 130$$
$$= (0,78125 + 2,5 + 1,3333)\ 130 = 4,6146 . 130 = 603,4 \text{ ft. lbs.}$$

With the same quantity of material, and under otherwise similar circumstances, the stability of a parallelopipedical wall would be:

$$S_1 = (\tfrac{1}{2} . [1,25]^2 + \tfrac{1}{2} . 0,2e\ 1,25e\ 10 + \tfrac{1}{8} . 0,2^2 . 10^2) . 130$$
$$= (0,78125 + 1,25 + 0,5) . 130 = 2,531 . 130 = 329 \text{ ft. lbs.}$$

The same wall, with a sloping front, would have the stability:

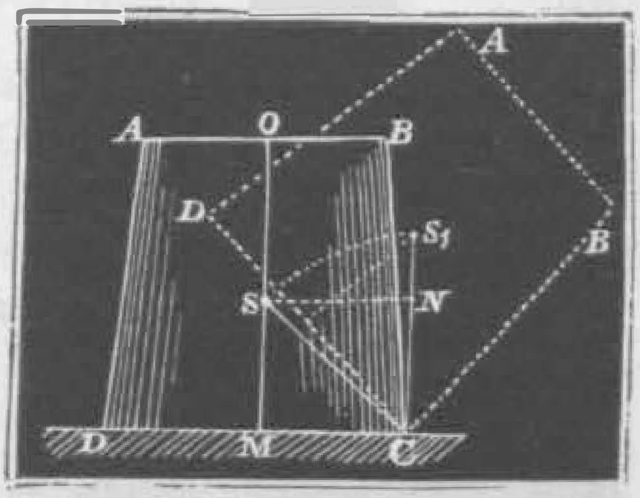
$$S_2 = (\tfrac{1}{2} . [1,25]^2 + \tfrac{1}{2} . 0,2e\ 1,25e\ 10 + \tfrac{1}{6} . [0,2]^2 . 10^2)e\ 130$$
$$= (0,78125 + 1,25 + 0,666 \ldots) . 130 = 2,6979e\ 130 = 350,7 \text{ ft. lbs.}$$

Remark.—It is evident from the foregoing that it allows of a saving of material to batter walls, to construct them with counterforts, to give them offsets, or to place them upon plinths, &c. The second part will give a further extension of this subject, when we come to treat of the pressure of earth, and of vaults, chain bridges, &c.

§ 133. *Dynamical stability.*—We may distinguish from the measure of stability treated of in the last paragraph, still another to a certain degree dynamical measure of stability, when we consider the effect which is to be expended in order to overturn a body. Now the mechanical effect of a force is equal to the product of the force and the space, but the force of a heavy body is its weight G, and the space equal to the vertical projection of that described by its centre of gravity, we may consequently take for the dynamical measure of the stability of a body the product Gs, if s be the height to which the centre of gravity of the body must ascend in order to bring the body from its stable condition into an unstable one.

Let C be the axis of revolution and S the centre of gravity of a body $ABCD$, Fig. 130, whose dynamical stability we wish to find. If we cause the body to revolve so that its centre of gravity comes to S_1, *i. e.* vertically over C, the body will be in unstable equilibrium, for if it only revolve a little further it will fall over. If we draw the horizontal line SN, this will cut off the height $NS_1 = s$ to which the centre of gravity has ascended, from which the stability Gs is given. If now $CS = CS_1 = z$, $CM = SN = x$, and the height $CN = MS = y$, it follows that the space $S_1N = s = z - y = \sqrt{x^2+y^2} - y$, and the stability in the last form of expression is

Fig. 130.



$$S = G(\sqrt{x^2 + y^2} - y).$$

If the body is a prism with a symmetrical trapezoidal transverse section, as Fig. 130 represents, and if the dimensions are the following: length $= l$, height $MO = h$, lower breadth $CD = b_1$, upper breadth $AB = b_2$, we then have $MS = y = \frac{b_1 + 2b_2}{b_1 + b_2} \cdot \frac{h}{3}$ (§ 105) and $CM = x = \frac{1}{2} b_1$, hence

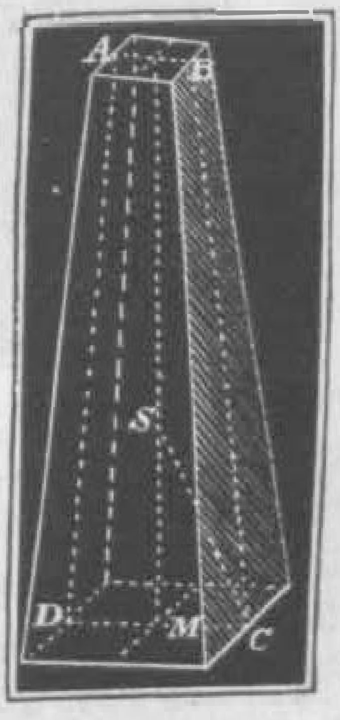
$$CS = \sqrt{\left(\frac{b_1}{2}\right)^2 + \left(\frac{b_1 + 2b_2}{b_1 + b_2} \cdot \frac{h}{3}\right)^2},$$

and the dynamical stability, or the mechanical effect, required to overturn it:

$$S = G\left[\sqrt{\left(\frac{b_1}{2}\right)^2 + \left(\frac{b_1 + 2b_2}{b_1 + b_2} \cdot \frac{h}{3}\right)^2} - \frac{b_1 + 2b_2}{b_1 + b_2} \cdot \frac{h}{3}\right].$$

Example.—What is the dynamical stability or the mechanical effect necessary for the overturning of an obelisk $ABCD$, Fig. 131, of granite, if its height $h = 30$ ft., its upper length and breadth $l_1 = 1\frac{1}{2}$, and $b_1 = 1$ ft., and lower length and breadth $l_2 = 4$ ft., $b_2 = 3\frac{1}{2}$ ft.? The volume of this body is (§ 115):

Fig. 131.



$$V = (2b_1 l_1 + 2b_2 l_2 + b_1 l_2 + b_2 l_1)\frac{h}{6}$$

$$= (2 \cdot \tfrac{3}{2} \cdot 1 + 2 \cdot 4 \cdot \tfrac{7}{2} + 1 \cdot 4 + \tfrac{3}{2} \cdot \tfrac{7}{2})\tfrac{30}{6}$$

$= 40{,}25 \cdot 5 = 201{,}25$ cubic feet. Now a cubic foot of granite weighs $= 3 \cdot 62{,}5 = 187{,}5$ lbs.; the whole weight of this body is: $G = 201{,}25 \cdot 187{,}5 = 37734{,}3$ lbs. The height of the centre of gravity above the base is:

$$y = \frac{b_2 l_2 + 3b_1 l_1 + b_2 l_1 + b_1 l_2}{2b_2 l_2 + 2b_1 l_1 + b_2 l_1 + b_1 l_2} \cdot \frac{h}{2}$$

$$= \frac{4 \cdot \frac{7}{2} + 3 \cdot \frac{3}{2} \cdot 1 + 1 \cdot 4 + \frac{3}{2} \cdot \frac{7}{2}}{40{,}25} \cdot \frac{30}{2} = \frac{27{,}75 \cdot 15}{40{,}25} = 10{,}342 \text{ ft.}$$

Provided it be a revolution about the longer edge of the base, the horizontal distance of the centre of gravity from this edge will be: $x = \frac{1}{2} b_2 = \frac{1}{2} \cdot \frac{7}{2} = \frac{7}{4}$ ft.; hence, the distance of the centre of gravity from the axis will be:

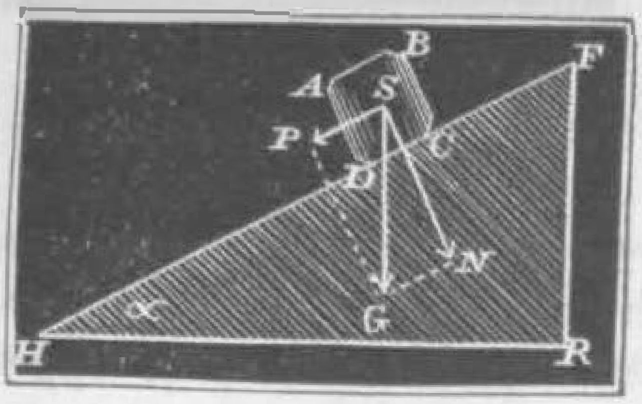
$CS = z = \sqrt{x^2 + y^2} = \sqrt{(1{,}75)^2 + (10{,}342)^2} = \sqrt{110{,}002} = 10{,}489$: and the height to which the centre of gravity must be raised to bring about an overthrow will be:

$s = z - y = 10{,}489 - 10{,}342 = 0{,}147$ ft; lastly, the corresponding mechanical effect or stability will be: $Gs = 37734.3 \,.\, 0{,}147 = 5547$ ft. lbs.

Remark. The factor $s = \sqrt{x^2 + y^2} - y$ gives for $y = 0$, $s = x$, for $y = x$, $s = x\,(\sqrt{2} - 1) = 0{,}414\ x$, for $y = nx$, $s = (\sqrt{n^2 + 1} - n)\ x$, approximately $= (n + \frac{1}{2n} - n)\, x = \frac{x}{2n}$, also for $y = 10\ x$, $s = \frac{x}{20}$, and for $y = \infty$, $s = \frac{x}{\infty} = 0$; the dynamical stability is therefore so much the greater, the lower the centre of gravity lies, and it approximates more and more to null, the higher the centre of gravity lies above the base. Sledges, carriages, ships, floating docks, &c., must on this account be so constructed and loaded, that the centre of gravity may lie as low as possible, and besides, be situated over the middle of the base.

§ 134. *Theory of the Inclined Plane.*—A body AC, Fig. 132, resting on an inclined plane, that is, on one inclined to the horizon, may take up two motions; it may slide down the inclined plane, and it may also revolve about one of the edges of its base and overturn. If the body is left to itself, its weight G is resolved into a force N normal, and to a force P parallel to the base, the first is resisted by the reaction of the plane, and the last urges the body down the plane. Let the angle of inclination FHR of the inclined plane to the horizon $= \alpha$, we have therefore the angle $GSN = \alpha$, and hence the normal pressure:

Fig. 132.



$$N = G \cos. \alpha,$$

and the force parallel to the plane:

$$P = G \sin. \alpha.$$

If the vertical line SG passes through the base CD as in Fig. 132, a sliding motion only can take place, but if this line passes outside the base, as in Fig. 133, an overturn ensues, and the body, therefore, is

Fig. 133.

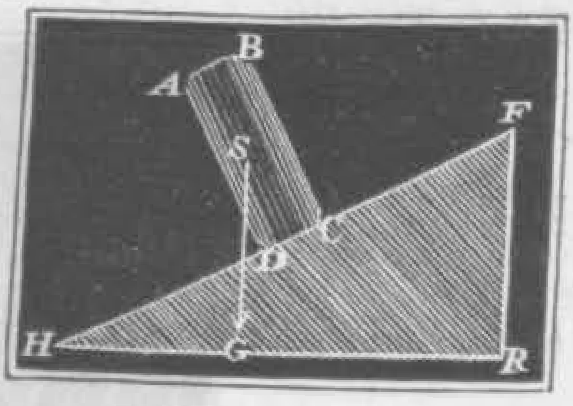
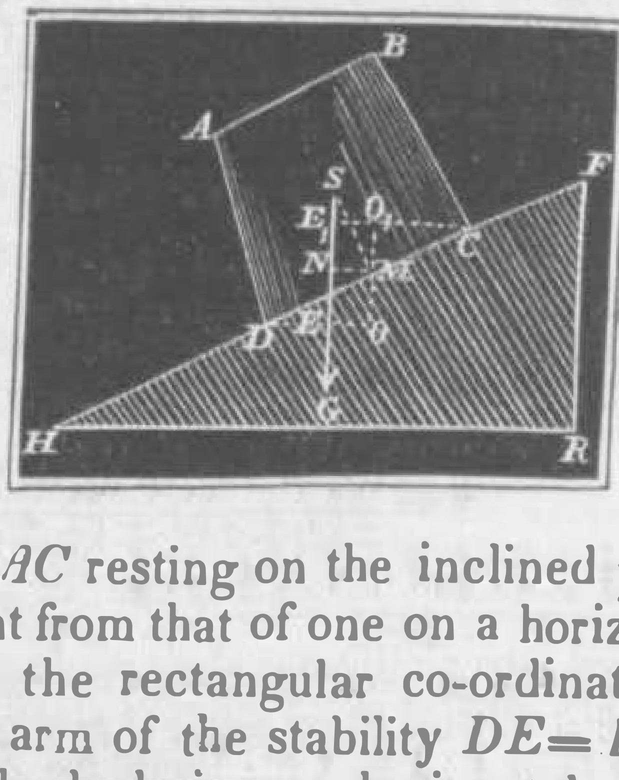


Fig. 134.



without stability. Besides, a body AC resting on the inclined plane FH, Fig. 134, has a stability different from that of one on a horizontal plane. If $DM = x$ and $MS = y$ are the rectangular co-ordinates of the centre of gravity S, we have the arm of the stability $DE = DO - MN = x \cos. \alpha - y \sin. \alpha$, while, if the body is on a horizontal plane, it is $= x$. Since $x > x \cos. \alpha - y \sin. \alpha$, the stability with reference to the lower edge D comes out less for the inclined than for the horizontal plane; it is null for $x \cos. \alpha = y \sin. \alpha$, *i. e.* for $tang. \alpha = \frac{x}{y}$. When a body that is stable $G x$ on a horizontal plane is transferred to an inclined one, whose angle of inclination corresponds to the expression $tang. \alpha = \frac{x}{y}$, it will lose its stability. On the other hand, a body may acquire on an inclined plane the stability which is wanting to it on a horizontal one. For a turning about the upper edge C, the arm $CE_1 = CO_1 + MN = x_1 \cos. \alpha + y \sin. \alpha$, whilst in its position on the horizontal plane it is $= x_1$. If now x_1 is negative, the body has no stability so long as it remains on a horizontal plane, but if it rests on an inclined one, for whose angle of inclination $tang. \alpha$ isi> $\frac{x_1}{y}$, the body is stable.

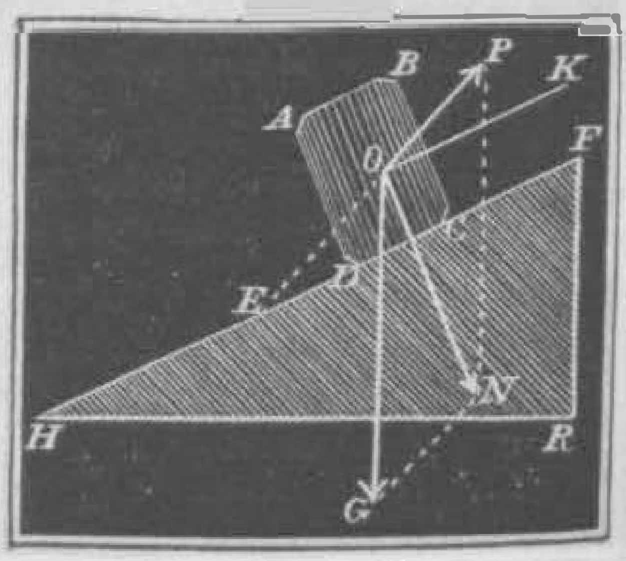
If another force besides gravity acts upon the body $ABCD$, Fig. 135, its stability continues if the direction of the resultant N of the weight G and the force P intersects the base CD of the body.

Example. The obelisk in the example of the preceding paragraphs has $x = \frac{7}{4}$ ft. and $y = 10{,}342$ ft., and will lose its stability, consequently, if transferred to an inclined plane, for whose angle of inclination:

$tang. \alpha = \frac{7}{4 \,.\, 10{,}342} = \frac{7000}{41368} = 0{,}16922$, and inclination $\alpha = 9^\circ\ 36'$.

§ 135. As the inclined plane only counteracts that pressure which is directed perpendicularly against it, the force P which is necessary to prevent a body supported upon an inclined plane from overturning, is determined by the condition that the resultant N of P and G, Fig. 135, must be at right angles to the inclined plane. From the theory of the parallelogram of forces we have $\frac{P}{G} = \frac{\sin. ONP}{\sin. PON}$, now the $\angle PNO = \angle GON = FHR = \alpha$, and $\angle PON = POK + KON = \beta + 90^\circ$, in so far as we represent by β the $\angle PEF = POK$, by which the direction of the force deviates from the inclined plane; hence we have

Fig. 135.



$$\frac{P}{G} = \frac{\sin. \alpha}{\sin. (90 + \beta)},\ i.\ e.\ \frac{P}{G} = \frac{\sin. \alpha}{\cos. \beta},$$

therefore the force which maintains the body on the plane is:

$$P = \frac{G \sin. \alpha}{\cos. \beta}.$$

For the normal pressure N

$\frac{N}{G} = \frac{\sin. OGN}{\sin. ONG}$, but the $\angle OGN = 90^\circ - (\alpha + \beta)$ and $ONG = PON = 90 + \beta$, hence it follows

$$\frac{N}{G} = \frac{\sin. [90^\circ - (\alpha + \beta)]}{\sin. (90^\circ - \beta)} = \frac{\cos. (\alpha + \beta)}{\cos. \beta}$$

and for the normal pressure against the plane

$$N = \frac{G \cos.(\alpha+\beta)}{\cos.\beta}.$$

If the force P is parallel to the plane, $\beta = 0$ and $\cos.\beta = 1$, since

$$P = G \sin.\alpha \text{ and } N = G \cos.\alpha.$$

If P acts vertically $\alpha+\beta = 90°$, hence

$$\cos.\beta = \sin.\alpha,\ \cos.(\alpha+\beta) = 0 \text{ and}$$

$P = G$ and $N = 0$, the inclined plane has then no control over the body.

Lastly, if the force acts horizontally, $\beta = -\alpha$, and $\cos.\beta = \cos.\alpha$, hence

$$P = \frac{G \sin.\alpha}{\cos.\alpha} = G \text{ tang.}\alpha\text{; and } N = \frac{G \cos.0}{\cos.\alpha} = \frac{G}{\cos.\alpha}.$$

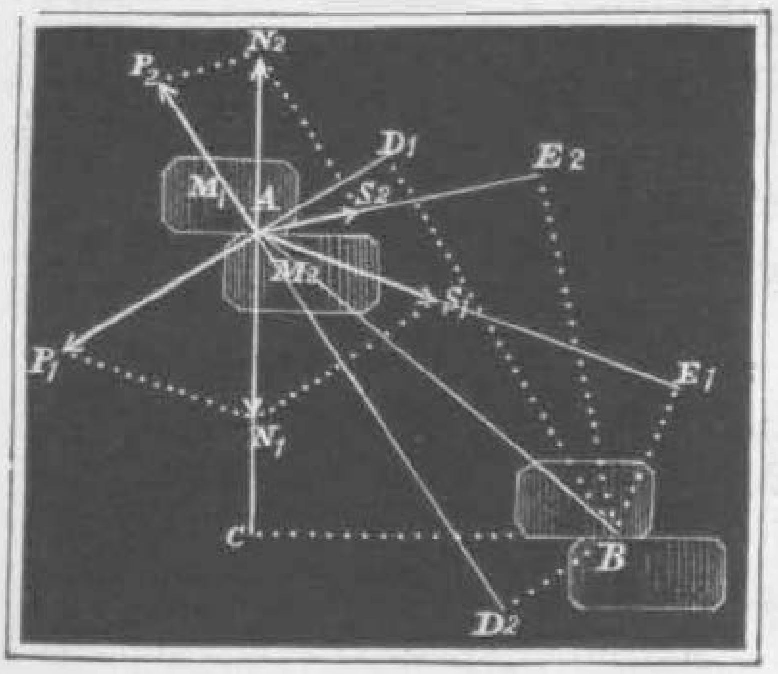
Example. To maintain a body of 500 lbs. upon an inclined plane of 50° inclination to the horizon, a force is applied whose direction makes an angle of 75° with the horizon, what is the magnitude of this force, and the pressure of the body against the plane? The force is:

$P = \frac{500 \sin. 50°}{\cos.(75-50)} = \frac{500 . \sin. 50°}{\cos. 25°} = 422{,}6$ lbs.; and the pressure on the plane:

$N = \frac{500 . \cos. 75°}{\cos. 25°} = 142{,}8$ lbs.

§ 136. *Principle of Virtual Velocities.*—If we combine the principle of the equality of action and reaction set forth in § 128, with that of virtual velocities (§ 80 and § 93), the following law transpires.

Fig. 136.



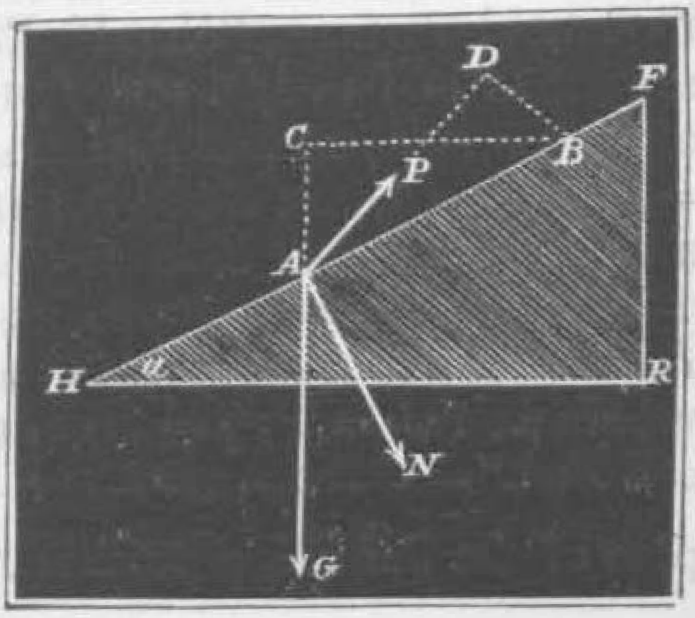
If two bodies M_1 and M_2, Fig. 136, hold each other in equilibrium, *then for a finite rectilinear or infinitely small curvilinear motion of the point of contact or pressure A, the sum of the mechanical effects of the forces of the one body is equivalent to the sum of the mechanical effects of those of the other.* If P_1 and S_1 be the forces of the one body, and P_2 and S_2 those of the other, then, for a displacement of the point of contact from A to B, the respective distances described are AD_1, AE_1, AD_2 and AE_2, and according to the above law:

$$P_1 . AD_1 + S . AE_1 = P_2 . AD_2 + S_2 . AE_2.$$

The correctness of this proposition may be proved in the following manner. As the normal pressures N_1 and N_2 are equal, there is also equilibrium between their mechanical effects, $N_1 . AC$ and $N_2 . AC$, with this difference, that the mechanical effect of the one force is positive, and that of the other negative. Now from what has preceded we have the mechanical effect $N_1 . AC$ of the resultant N_1 equivalent to the sum $P_1 AD_1 + S_1 . AE_1$ of the mechanical effects of its components P_1 and S_1, and likewise $N_2 . AC = P_2 . AD_2 + S_2 . AE_2$; hence also $P_1 . AD_1 + S_1 . AE_1 = P_2 . AD_2 + S_2 . AE_2$.

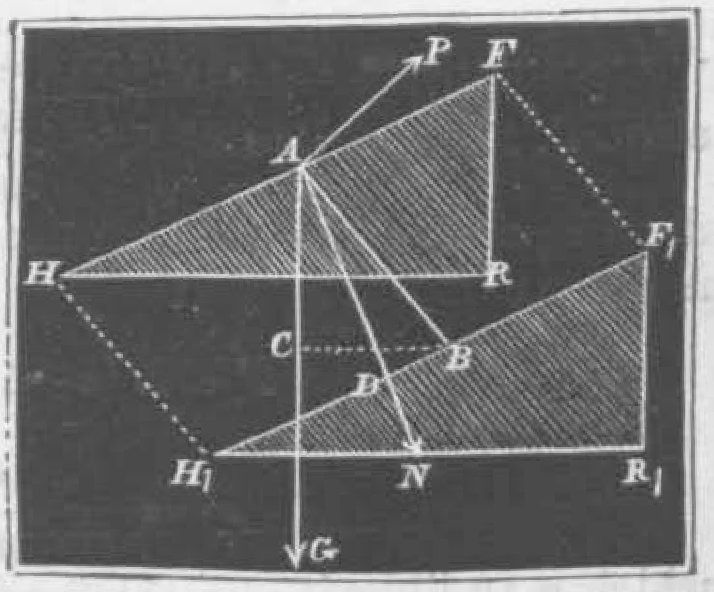
The application of the principle of virtual velocities thus made more general possesses great advantage in statical investigations, as by it the evolution of algebraical expressions becomes much simplified. If, for example, we move a body A up an inclined plane FH, Fig. 137, a distance AB, the corresponding path of the weight G, $= AC = AB\ sin.\ ABC = AB\ .\ sin.\ FHR = AB\ sin.\ \alpha$. On the other hand, the path of the force P is $AD = AB\ .\ cos.\ BAD = AB\ .\ cos.\ \beta$, and lastly, that of the normal force $N = 0$; now the mechanical effect of N is equivalent to that of $G +$ that of P, hence we have to put $N\ .\ 0 = -G\ .\ AC + P\ .\ AD$, and so we find $P = \frac{AC}{AD}\ .\ G = \frac{G\ sin.\ \alpha}{cos.\ \beta}$, quite in accordance with the former paragraph.

Fig. 137.



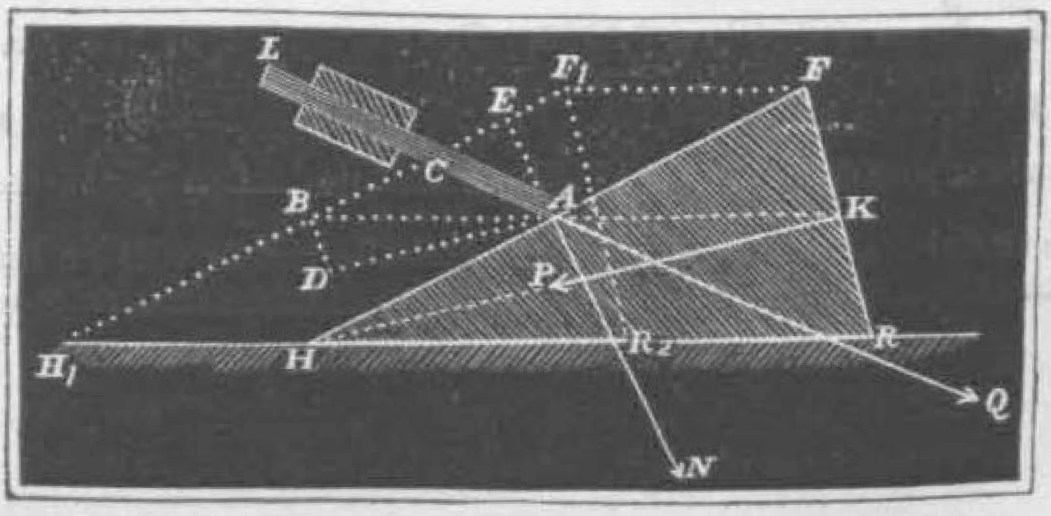
In order to find the normal pressure N, we must move forward the inclined plane HF, Fig. 138, through a space AB at right angles to the direction of the force AP, to determine the corresponding paths of the forces, and again put the mechanical effect of N equivalent to that of $G +$ the mechanical effect of P. The path of N is $AD = AB\ cos.\ BAD = AB\ cos.\ \beta$, that of G is $AC = AB\ cos.\ BAC = AB\ cos.\ (\alpha + \beta)$ and that of $P = 0$, hence the mechanical effect $N\ .\ AD = G\ .\ AC + P\ .\ 0$, and $N = \frac{G\ .\ AC}{AD} = G\ .\ \frac{cos.\ (\alpha + \beta)}{cos.\ \beta}$, just as was found in the former paragraph.

Fig. 138.



§ 137. *Theory of the Wedge.*—After this the theory of the wedge comes out very simply. The wedge is a movable inclined plane,

Fig. 139.



formed by a triangular prism FHR, Fig. 139, generally the force KP is $= P$, and at right angles to the back FR of the wedge, and holds in equilibrium another force or load $AQB = Q$, which presses against its lateral surface FH. If $FHR = \alpha$ be the angle measuring the sharpness of its edge, and further, the angle by which the direction of the force KP or AD deviates from the surface FH, therefore $FHK = HAD, = \delta$, and lastly the angle LAH, the deviation of the direction of Q from this same surface, $= \beta$, then the paths will be given which are described by the advance of the wedge from the position FHR into that of $F_1H_1R_1$, in the following manner. The path of the wedge is $AB = FF_1 = HH_1$, and that of the force is $= AD = AB \cos. BAD = AB \cos. (BAH - DAH) = AB \cos. (\alpha - \delta)$; further, the path of the bar AL or load is $AC = \frac{AB \sin. ABC}{\sin. ACB} = \frac{AB \sin. \alpha}{\sin. HAC} = \frac{AB \sin. \alpha}{\sin. \beta}$, and the simultaneous path of the normal pressure N between the wedge and the foot of the bar $= AE = AB \sin. \alpha$.

By the advance of the wedge a distance AB, the normal pressure N produces the mechanical effect $N . AE = N . AB \sin. \alpha$, the force, however, develops the mechanical effect $P . AD = P . AB \cos. (\alpha - \delta)$ and the resistance the mechanical effiect, $Q . AC = Q . AB \frac{\sin. \alpha}{\sin. \beta}$, hence $N . AB \sin. \alpha = P . AB \cos. (\alpha - \delta)$ *i. e.* $N \sin. \alpha = P \cos. (\alpha - \delta)$, as also $N . AB \sin. \alpha = Q . AB \frac{\sin. \alpha}{\sin. \beta}$, *i. e.* $N \sin. \alpha = Q \frac{\sin. \alpha}{\sin. \beta}$, and from these equations the equation between the power and resistance sought is given:

$$P \cos. (\alpha - \delta) = \frac{Q \sin. \alpha}{\sin. \beta}, \text{ or}$$

$$P = \frac{Q \sin. \alpha}{\sin. \beta \cos. (\alpha - \delta)},$$

which may likewise be obtained by the decomposition of the forces.

If the direction of the force is parallel to the base or lateral surface HR, $\delta = \alpha$, hence $P = \frac{Q \sin. \alpha}{\sin. \beta}$, and if, further, the direction of the load is perpendicular to the side FH, $\beta = 90^\circ$, and P follows $= Q \sin. \alpha$.

Example. The edge FHR of a wedge $= \alpha = 25^\circ$, the force is directed parallel to the base HR, therefore, $\delta = \alpha$, and the weight Q acts at right angles to the side FH, therefore $\beta = 90^\circ$, in what proportions are the power and weight to each other? P is $= Q \sin. \alpha$, therefore $\frac{P}{Q} = \sin. 25^\circ = 0{,}4226$. For a weight Q of 130 lbs. the power P comes out $= 130 . 0{,}4226 = 54{,}938$ lbs. In order to drive forward the weight or bar 1 foot, the wedge must pass over the space $AB = \frac{AC}{\sin \alpha} = \frac{1}{0{,}4226}$ 2,3662 feet.

Remark. The theories of the inclined plane and the wedge will be more fully developed in the fifth chapter, where the effect of friction is taken into account.

CHAPTER IV.

EQUILIBRIUM IN FUNICULAR MACHINES.

§ 138. *Funicular Machines.*—We have hitherto assumed that bodies, on which forces act, do not change their form in consequence of this action; we will now take up the equilibrium of such bodies as suffer a change in their form by the smallest forces. The former are called solid or rigid, the latter flexible bodies. In truth there is no body perfectly flexible; many of them, however, such as strings, ropes, cords, &c., and in some respects chains also, require so small a force to bend them that they may in many cases be regarded as perfectly flexible. Such bodies, which are moreover extensible, will be the subject of the following investigations.

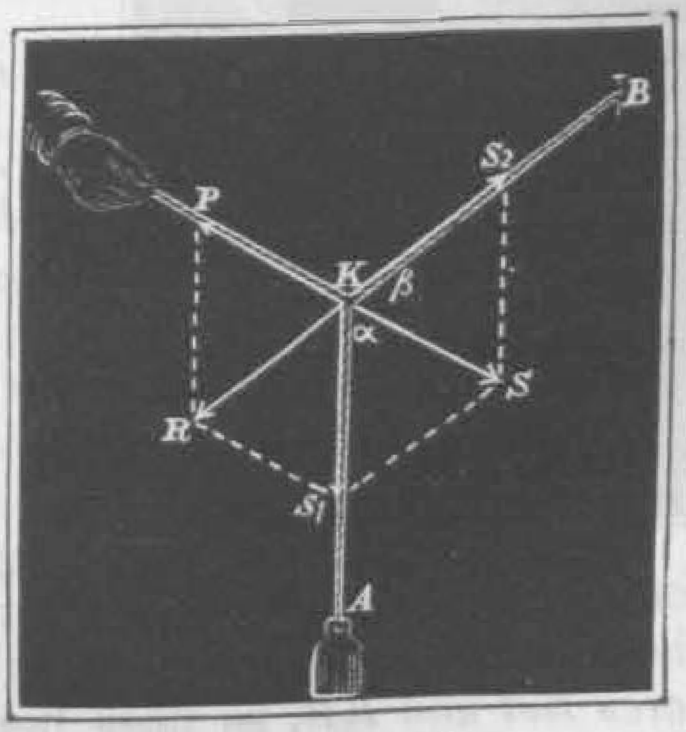
We understand by a funicular machine, a cord or a connection of cords (the word cord taken in its general sense) which becomes stretched by forces, and in this chapter we will consider the theory of the equilibrium of these machines.

That point of a funicular machine to which the force is applied, and where the cord forms an angle with the direction of the force, is called a knot or node. This may be either fixed or movable. Tension is the force which a stretched cord transmits in the direction of its axis. The tensions at the ends of a straight cord or portion of a cord are equal and opposite, § 83; also a straight cord cannot transmit other forces than the tension acting in the direction of its axis, because it must otherwise bend, and, therefore, cannot remain straight.

§ 139. *Knots or Nodes.*—Equilibrium obtains in a funicular machine, when there is equilibrium at each of its nodes. Hence we must next find what are the relations of equilibrium at any one node.

Equilibrium takes place at a node K, which a portion of a cord AKB, Fig. 140, forms, when the resultant KS of the tensions of the cord $KS_1 = S_1$ and $KS_2 = S_2$ are equal and opposite to the force P applied at the node K, for the tensions S_1 and S_2 produce the same effects as equal and opposite forces, and three forces hold each other in equilibrium, if one of them is equal to and acts opposite to the resultant of the other two (§ 75). The resultant R of the force P and the first tension S_1 is equal and opposite to the second tension S_2, &c. In every case this equation may be used to find out two of the quantities to be determined, viz. the tension of the cord and

Fig. 140.



its direction. Let, for example, the force be P, the tension S_1 and the $\angle$ between the two $AKP = 180° - AKS = 180° - \alpha$, we have for the other tension.

$$S_2 = \sqrt{P^2 + S_1^2 - 2\, PS_1 \cos. \alpha}.$$

and for its direction or deviation from KS, $BKS = \beta$, and

$$\sin. \beta = \frac{S_1 \sin. \alpha}{S_2}.$$

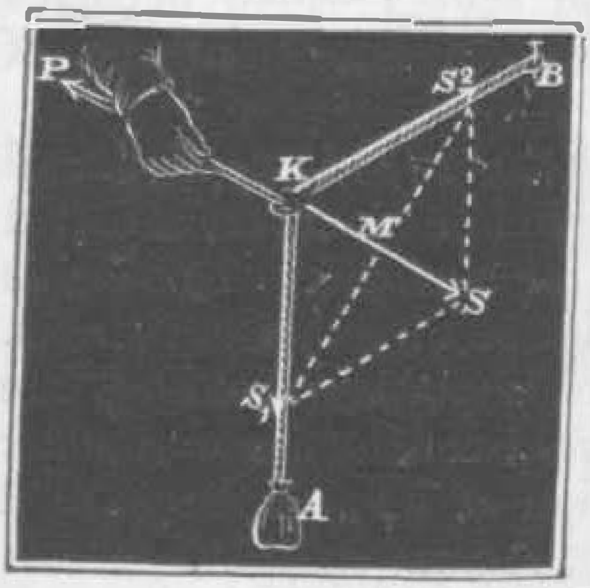
Example. If the cord AKB, Fig. 140, is fixed at the extremity B, and at the extremity A stretched by a weight $G = 135$ lbs., and the middle K by a force $P = 109$ lbs., which pulls upwards under an angle of 25°; required the direction and tension of the portion of cord KB. The magnitude of the tension is:

$$S_2 = \sqrt{109^2 + 135^2 - 2 . 109 . 135 \cos. (90° - 25°)}$$
$$= \sqrt{11881 + 18225 - 29430 . \cos. 65°} = \sqrt{17668,36} = 132,92 \text{ lbs.}$$

For the angle β, $\sin. \beta = \frac{S_1 \sin. \alpha}{S_2} = \frac{135 . \sin. 65°}{132,92}$, $Log. \sin. \beta = 0,964017 - 1$, hence $\beta = 67° 0'$, and the inclination of the portion of the cord to the horizon $= \alpha + \beta - 90° = 65° + 67° - 90° = 42°$.

§ 140. If the node K is a running or movable one, or the force P acts by means of a ring running along the cord AKB, Fig. 141, the resultant S of the tensions S_1 and S_2 is equal and opposite to the force P at the ring; besides this, the tensions are equal, for if the cord be drawn a certain space s through the ring, each of the tensions S_1 and S_2 will pass over the space s, and the force P over a space $= 0$; consequently, provided there is perfect flexibility, the mechanical effect $P . 0 = S_1 . s - S_1 . s$, *i. e.* $S_1 s = S_2 s$ and $S_1 = S_2$. From this equality of the tensions there follows the equality of the angles AKS and BKS, by which the resultant S deviates from the directions of the cords. If we put these angles $= \alpha$, the resolution of the rhomb $KS_1 SS_2$, gives

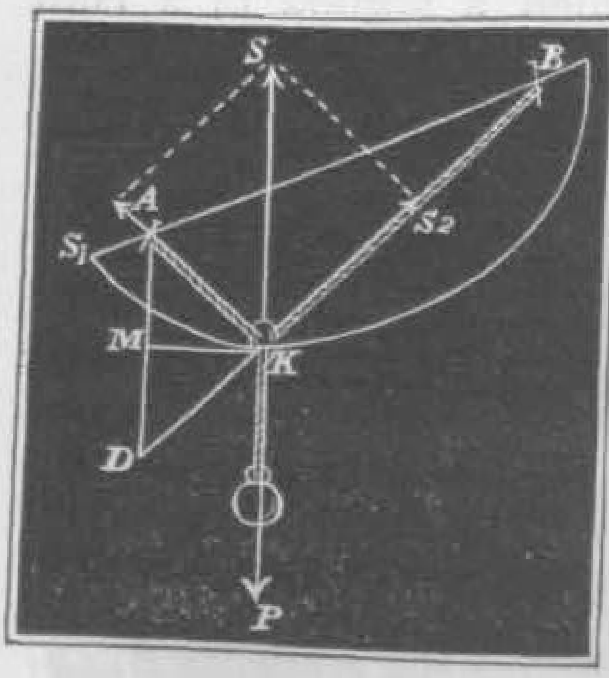
Fig. 141.



$$S = P = 2\, S_1 \cos. \alpha \text{ and inversely}$$

$$S_1 = S_2 = \frac{P}{2 \cos. \alpha}.$$

Fig. 142.

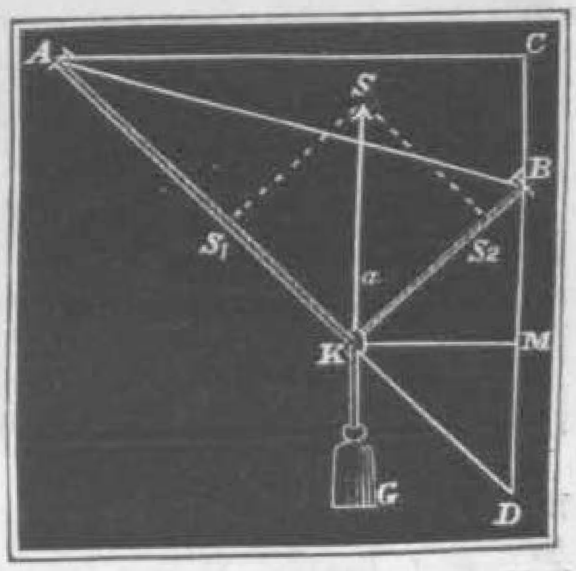


A and B are the fixed points of a cord AKB of given length ($2\, a$) with a movable node K, the place of this node may be found by constructing an ellipse, whose foci are A and B, and whose major axis is equal to the length of the cord $2\, a$, and if a tangent is drawn to this curve at right angles to the given direction of the force, the resulting point of contact is the place of the node, because the normal to the ellipse KS makes equal angles with the radii vectores KA and KB, as does the resultant S with the tensions of the cord S_1 and S_2.

If AD be drawn parallel to the given direction of the force, and BD be made equal to the given length of the cord, AD bisected at M and the perpendicular MK be raised, the place of the node K may likewise be obtained without the construction of an ellipse, for since the $\angle AKM = \angle DKM$ and $AK = DK$, it follows that $\angle AKS$ also $= \angle BKS$ and $AK + KB = DK + KB = DB$.

Example. Between the points A and B, Fig. 143, a rope of 9 feet in length is stretched by a weight G of 170 lbs. suspended to it by a ring; the horizontal distance AC of the two points is $6\frac{1}{2}$ ft., and the vertical distance $BC = 2$ ft.; to find the position of the node, the tensions and directions of the rope. From the length $AD = 9$ ft. as hypothenuse and the horizontal line $AC = 6\frac{1}{2}$ ft.; it follows that the vertical $CD = \sqrt{9^2 - 6{,}5^2} = \sqrt{81 - 42{,}25} = \sqrt{38{,}75} = 6{,}225$ feet; and from this the base BD of the equilateral triangle BDK, $= CD - CB = 6{,}225 - 2 = 4{,}225$ ft. The similarity of the triangles DKM and DAC gives $DK = BK = \frac{DM}{DC} \cdot DA = \frac{4{,}225.9}{2.6{,}225} = 3{,}054$ ft.; hence it follows, that $AK = 9 - 3{,}054 = 5{,}946$ feet; and for the angles a, by which the sides of the rope are inclined to the vertical:

Fig. 143.



$$\cos. a = \frac{BM}{BK} = \frac{2{,}1125}{3{,}054} = 0{,}6917;$$

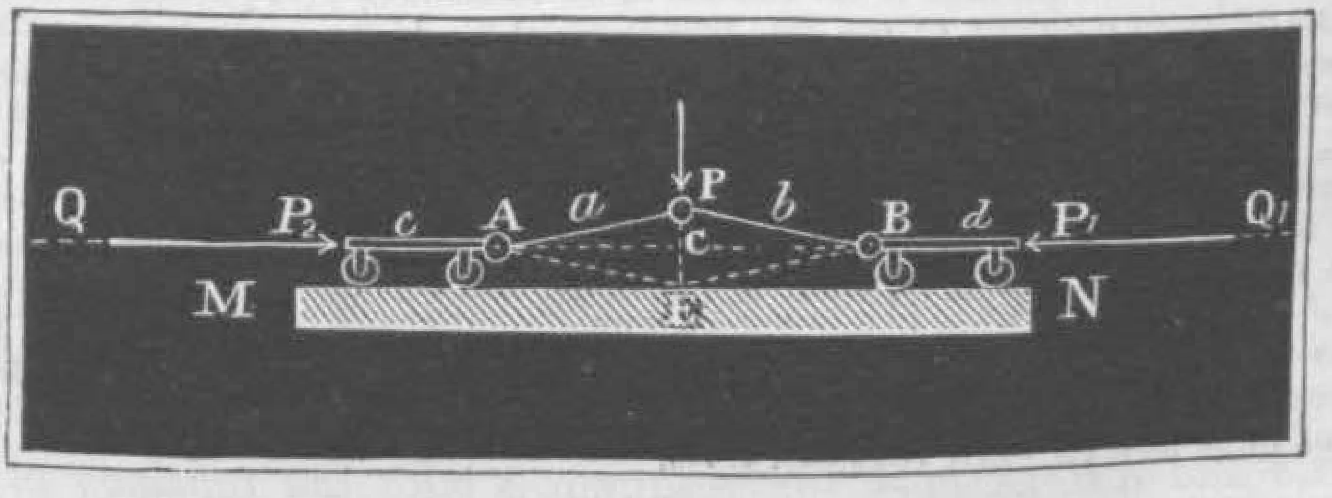
hence, $a = 46^\circ\ 14'$; and lastly, the tension of the rope $S_1 = S_2 = \frac{G}{2 \cos. a} = \frac{170}{2\ .\ 0{,}6917} = 122{,}9$ lbs.*

* If the demonstrations applied in the text to the simple funicular machine, where a single weight is represented as sustained by means of two parts of a flexible cord, attached to two fixed supports, be applied to the case of two rigid planes hinged together at a middle point, and also joined by hinges to two other planes capable of sliding to and from each other, but in opposite directions, then will the principles of the formulæ above given, be found to afford the relation between the force applied and the resistance which it is capable of overcoming, in the well-known machine called the *tricardo*, vulgarly the "toggle joint," which has been much applied of late years in the construction of printing, coining, and other presses.

When two ropes hang parallel to each other, the whole gravitating power of the *weight* is divided between them, and equally so between the points of support which sustain their upper extremities. The limit of the weight is the absolute strength of the ropes, and, in case of the tricardo, the force which could be applied to the planes would, *in that position*, be limited by the crushing force of the materials of the planes.

In the funicular machine, the question generally relates to the tension on the cords, not to the force tending to bring together the points of support, while, in the tricardo, the effort to separate the opposite extremities of the movable planes is the thing to be calculated. The following figure (143*) may render this more evident.

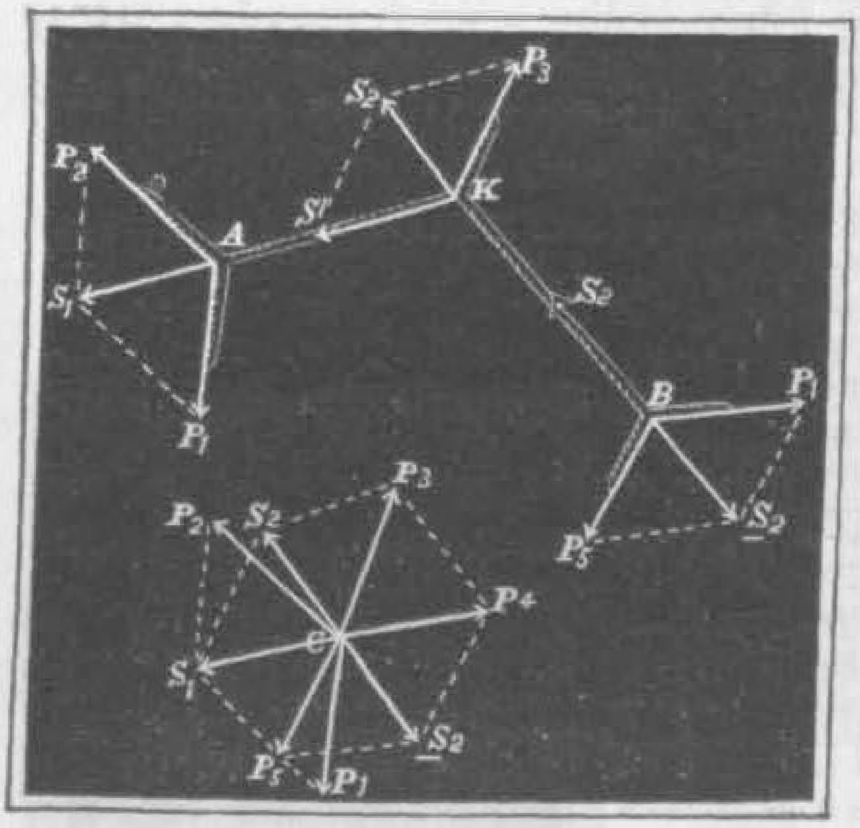
Fig. 143*.



Let a and b be the two planes of the *tricardo*, hinged at A and B to two other planes

§ 141. *Funicular Polygon.*—The relations of equilibrium in the funicular polygon, *i. e.* in a stretched cord which is acted upon by forces applied to different points, are in accordance with those of the equilibrium of forces, which are applied to one point. Let AKB, Fig. 144, be a cord stretched by the forces P_1, P_2, P_3, P_4, P_5: let P_1 and P_2 act at A, P_3 at K, and P_4 and P_5 at B. Let us put the tension of the portion $AK = S_1$ and that of $BK = S_2$, we shall then obtain S_1 for the resultant of P_1 and P_2 applied to A, and if we carry the point of application A of this tension from A to K, we shall again get S_2 for the resultant of S_1 and P_3, or of P_1, P_2, P_3; lastly, if we transport the point of application of S_2 from K to B, we shall then obtain in S_2, P_4 and P_5, or since S_2 is the resultant of P_1, P_2, P_3, also in P_1, P_2, P_3, P_4, P_5 a set of forces balancing each other. We may accordingly assert that, *when certain forces P_1, P_2, P_3, &c., hold a funicular polygon in equilibrium, they will hold each other in equilibrium also, if applied at a single point C, their direction and magnitude remaining invariable.*

Fig. 144.



If the cord $AK_1 K_2 \ldots B$, Fig. 145, be stretched at the points or nodes, K_1, K_2 by weights G_1, $G_2 \ldots$ and the extremities A and B by the vertical forces V_1 and V_n, and the horizontal forces H_1 and H_n, the sum of the vertical forces will be: $V_1 + V_n - (G_1 + G_2 + G_3 + \text{\&c.})$ and of the horizontal forces: $H_1 - H_n$. The condition of equilibrium requires that both sums $= 0$; therefore

1. $V_1 + V_n = G_1 + G_2 + G_3 + \ldots$ and
2. $H_1 = H_n$; *i. e.*

In a funicular polygon stretched by weights, the sum of the vertical forces or vertical tensions at the extremities or points of suspension is

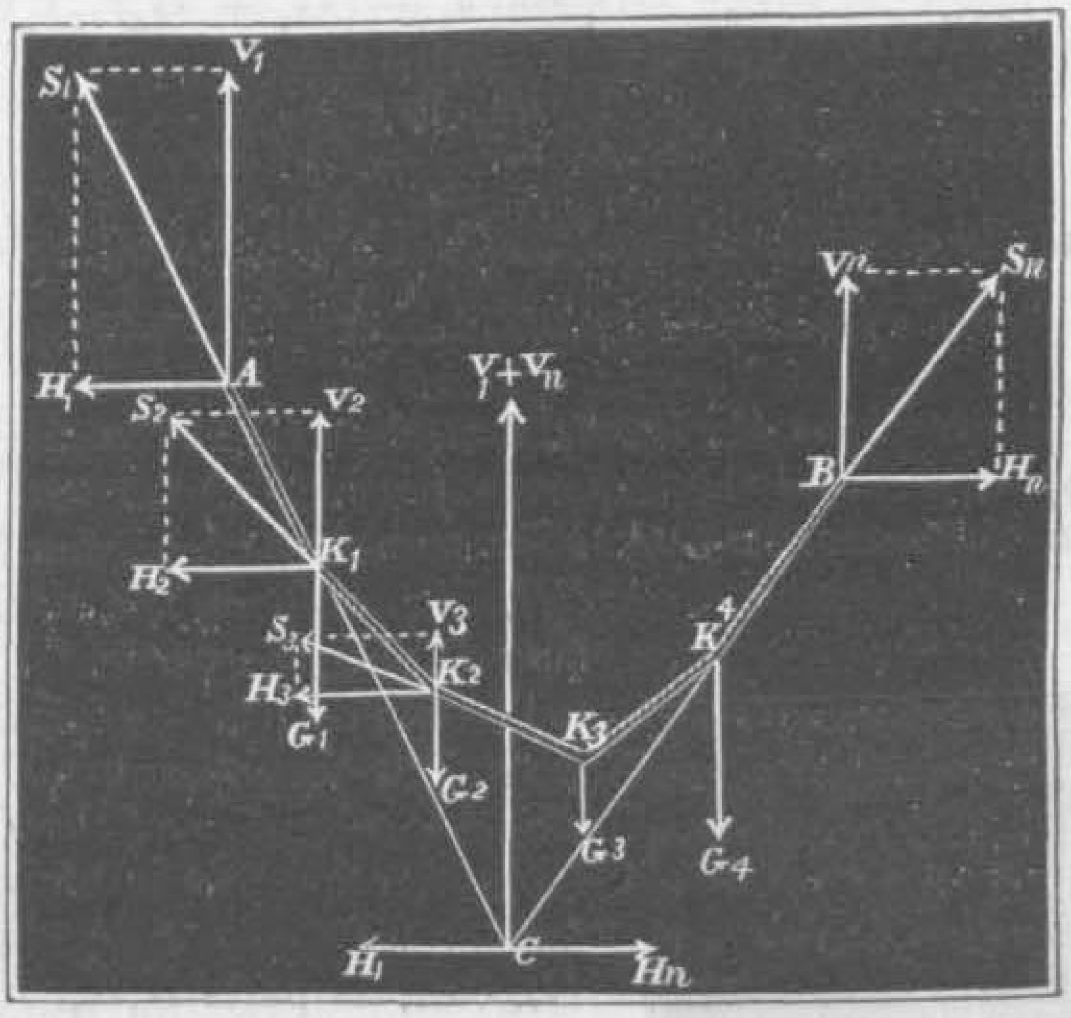
---

c and d, supposed to be capable of moving freely to and from each other along the plane MN. The hinge of a and b at P being supposed to be acted on by the small constant force P, the practical question is the relation of the resistances P_1, P_2 to this constant force P, in the different positions of the two planes a and b. If the angle PAC or $PBC = \alpha$ represent the angle of divergence of the planes a and b from the straight line QQ_1, it is evident that the force P will be represented by $CE = 2\,CP = 2 \sin. \alpha$, and the forces P_1 and P_2 each by $AB = 2 \cos. \alpha$. Hence $P : P_1 = \sin. \alpha : \cos. \alpha$; or as $\tan g.\ \alpha : rad.$ It is thus shown that the force applied at the central hinge of the "toggle joint" has to the force which resists the thrust of the planes, the relation of the sine to the cosine of the angle of inclination, or, what is the same, that of tangent to radius; or, in the case of the movable planes forming one and the same plane, that of 0 to 1, or 1 to ∞.

This subject will be found more fully treated of, and illustrated with figures of the tricardo, in a paper by the writer of this note in the *Journal of the Franklin Institute*, vol. iii. p. 354, for May, 1829.—AM. ED.

equivalent to the sum of the suspended weights, and the horizontal tension at the one extremity is equal and oppositely directed to the horizontal tension at the other extremity.

Fig. 145.



If the directions of the tensions S_1 and S_n at the cords A and B be prolonged to their intersection C, and the points of application of these tensions be transferred to this point, we shall then have the single force $P = V_1 + V_n$, because the horizontal forces H_1 and H_n counteract each other. Since this force holds in equilibrium the sum $G_1 + G_2 + G_3 + \ldots$ of the suspended weights, the point of application, or centre of gravity of these weights must, therefore, lie in the direction of the same, *i. e.*, in the vertical line passing through the point C.

§ 142. From the tension S_1 of the first portion AK_1 whose angle of inclination $S_1 AH_1 = \alpha_1$, the vertical tension follows; $V_1 = S_1 \sin. \alpha_1$, and the horizontal $H_1 = S_1 \cos. \alpha_1$. If, now, we transfer the point of application of these forces from A to the first node K_1, the weight G_1 acting vertically downwards meets these tensions, and now for the following portion $K_1 K_2$, the vertical tension $V_2 = V_1 - G_1 = S_1 \sin. \alpha_1 - G_1$, for which the horizontal tension $H_2 = H_1 = H$ remains unchanged. Both forces united give the tension of the axis of the second portion $S_2 = \sqrt{V_2^2 + H^2}$ and its inclination α_2 by the formula

$$tang. \alpha_2 = \frac{V_2}{H} = \frac{S_1 \sin. \alpha_1 - G_1}{S_1 \cos. \alpha_1}, \; i.\, e.$$

$$tang. \alpha_2 = tang. \alpha_1 - \frac{G_1}{H}.$$

If the point of application of the forces V_2 and H_2 is transferred from K_1 to K_2, we obtain in the weight G_2 meeting them another new vertical force, and therefore the vertical force of the third portion of the cord

$$V_3 = V_2 - G_2 = V_1 - (G_1 + G_2) = S_1 \sin. \alpha_1 - (G_1 + G_2),$$

whilst the horizontal force H_3 remains $= H$. The whole tension of the third portion is

$S_3 = \sqrt{V_3^2 + H^2}$, and for its angle of inclination α_3, we have

$$tang.\, \alpha_3 = \frac{V_3}{H} = \frac{S_1 \sin.\, \alpha_1 - (G_1 + G_2)}{S_1 \cos \alpha_1}, \text{ i.e.}$$

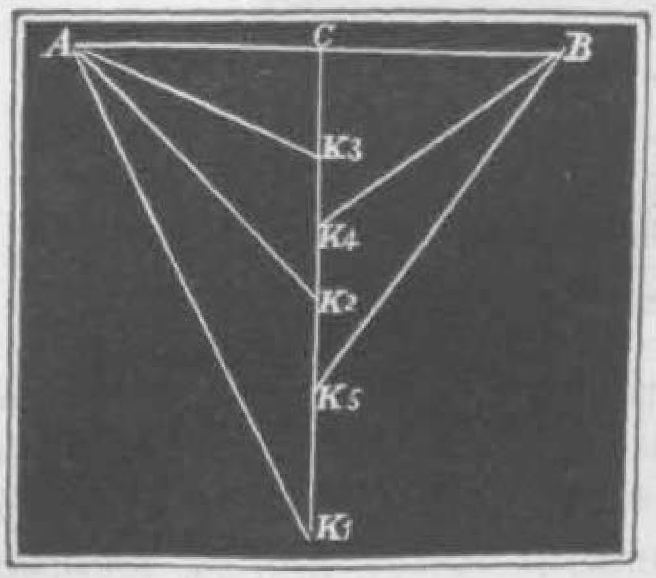
$$tang\, \alpha_3 = tang.\, \alpha_1 - \frac{G_1 + G_2}{H}.$$

For the angle of inclination of the fourth portion of the cord,

$$tang.\, \alpha_4 = tang.\, \alpha_1 - \frac{G_1 + G_2 + G_3}{H}, \text{ \&c.}$$

Besides, the tensions S_1, S_2, S_3, &c., as well as the angles of inclination α_1, α_2, α_3, &c., of the separate portions of the cord may easily be represented geometrically. If we make the horizontal line $CA = CB$, Fig. 146, = the horizontal tension H and the vertical CK_1 = vertical tension V_1 at the point of suspension A, the hypothenuse AK_1 gives the whole tension S_1 and the $\angle CAK_1$, also its inclination to the horizon; if now further we apply the weights G_1, G_2, G_3, &c., as parts K_1K_2, K_2K_3, &c., of CK, and draw the transversal lines AK_2, AK_3, &c., we shall have in them the tensions of the successive portions of the cord, and in the angles K_2AC, K_3AC, &c., the angles of inclination α_2, α_3, &c. of these portions.

Fig. 146.



§ 143. From the investigations of the preceding paragraph, the law for the equilibrium of cords stretched by weights, comes out thus:

1. *The horizontal tension is at all points of the cord one and the same, viz.:*

$$H = S_1 \cos.\, \alpha_1 = S_n \cos.\, \alpha_n.$$

2. *The vertical tension at any one point is equal to the vertical tension at the other extremity above it, less the sum of the intermediate suspended weights, therefore*

$$V_m = V_1 - (G_1 + G_2 + \ldots G_{m-1}).$$

If the angle α_1 be known and the horizontal tension H, the vertical tension at the extremity A is known; $V_1 = H \,.\, tang.\, \alpha_1$, and accordingly that at the extremity B: $V_n = (G_1 + G_2 + \ldots + G_n) - V_1$.

If, on the other hand, the angles of inclination α_1 and α_n at both points of suspension A and B are known, the horizontal and vertical tensions are given at the same time, viz.:

$$\frac{V_n}{V_1} = \frac{tang.\, \alpha_n}{tang.\, \alpha_1}, \text{ and, therefore,}$$

$$V_n = \frac{V_1\, tang.\, \alpha_n}{tang\, \alpha_1}.$$

Since $V_1 + V_n = G_1 + G_2 + \ldots$, *i. e.*

$$\left(\frac{tang.\, a_1 + tang.\, a_n}{tang.\, a_1}\right) V_1 = G_1 + G_2 \ldots, \text{ it follows thath}$$

$$V_1 = \frac{(G_1 + G_2 + \ldots)\, tang.\, a_1}{tang.\, a_1 + tang.\, a_n},$$

$$V_n = \frac{(G_1 + G_2 + \ldots)\, tang.\, a_n}{tang.\, a_1 + tang.\, a_n}, \text{ and from this:}$$

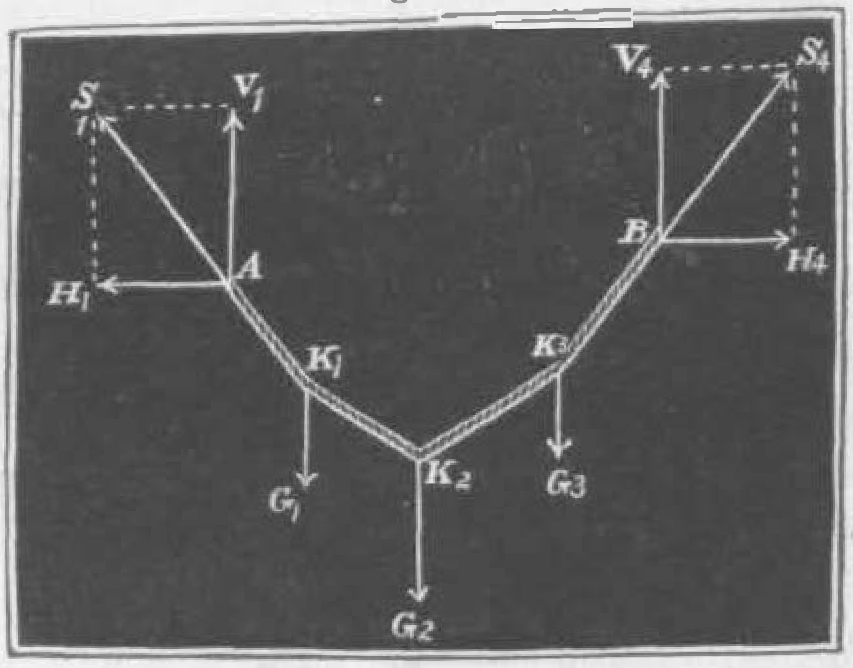
$$H = V_1\, cotg.\, a_1 = V_n\, cotg.\, a_n.$$

If both sides have the same inclination $a_n = a_1$, then $V_1 = V_n = \dfrac{G_1 + G_2 + \ldots + G_n}{2}$, and the one extremity A supports as much as the other B.

For the rest, these laws hold good also for the funicular polygon, especially when stretched by forces, if the directions of the forces are substituted for the verticals.

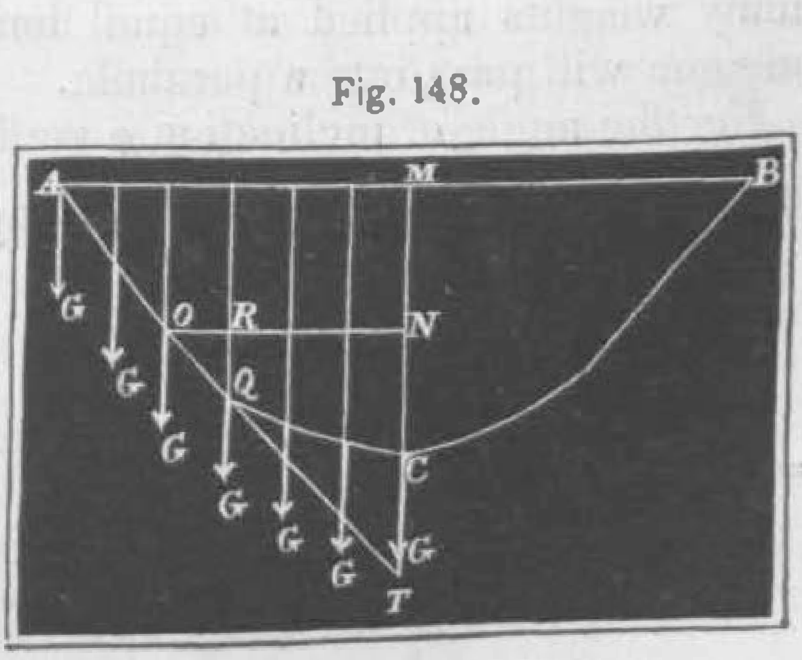
Example. The funicular polygon $AK_1 K_2 K_3 B$, Fig. 147, is stretched by three weights $G_1 = 20$, $G_2 = 30$, and $G_3 = 16$ lbs., as well as by the horizontal force $H_1 = 25$ lbs.; required to find the tensions of the axis and the angles of inclination of the sides, in the hypothesis that the ends of the string have the same inclination. Here the vertical tensions are equal, viz., $V_1 = V_4 = \dfrac{G_1 + G_2 + G_3}{2} = \dfrac{20 + 30 + 16}{2} = 33$ lbs. The vertical tension of the second portion of the string is $V_2 = V_1 - G_1 = 33 - 20 = 13$ lbs., that of the third $V_3 = V_4 - G_3$ or $(G_1 + G_2 - V_1) = 33 - 16 = 17$ lbs.; the angles of inclination a_1 and a_4 of the ends are determined by $tang.\, a_1 = tang.\, a_4 = \dfrac{V_1}{H} = \dfrac{33}{25} = 1{,}32$; that of the second and third portions by the $tang.\, a_2 = tang.\, a_1 - \dfrac{G_1}{H} = 1{,}32 - \dfrac{20}{25} = 0{,}52$, and $tang.\, a_3 = tang.\, a_4 - \dfrac{G_3}{H} = 1{,}32 - \dfrac{16}{25} = 0{,}68$; hence $a_1 = a_4 = 52^\circ\, 51'$; $a_2 = 27^\circ 28'$ $a_3 = 34^\circ\, 13'$; lastly, the tensions of the axis are $S_1 = S_4 = \sqrt{V_1^2 + H^2} = \sqrt{33^2 + 25^2} = \sqrt{1714} = 41{,}40$ lbs., $S_2 = \sqrt{V_2^2 + H^2} = \sqrt{13^2 + 25^2} = \sqrt{794} = 28{,}18$ lbs., and $S_3 = \sqrt{V_3^2 + H^2} = \sqrt{17^2 + 25^2} = 30{,}23$ lbs.

Fig. 147.



§ 144. *The Parabola as Catenary.*—Let us suppose that the string ACB, Fig. 148, is stretched by equal weights G_1, G_2, &c., suspended at equal horizontal distances from each other. Let us represent by b_1 the horizontal distance AM between the point of suspension A and the lowest C, but the vertical distance CM by a. Let

Fig. 148.



us put further for another point O of the polygon, the corresponding co-ordinates $ON = y$ and $CN = x$. If, now, the vertical tension of A be $= V$, that of O will be $= \frac{y}{b} \cdot V$, and hence for the angle of inclination to the horizon, $NOT = ROQ = \phi$ of the portion of the string OQ, we shall have $tang. \phi = \frac{y}{b} \cdot \frac{V}{H}$, where H is the constant of the horizontal tension.

Hence $QR = OR \cdot tang. \phi = OR \cdot \frac{y}{b} \cdot \frac{V}{H}$ is the vertical distance of two adjacent angles of the funicular polygon. If we substitute for y OR, $2\ OR$, $3\ OR$, &c., the last equation will give the corresponding vertical distances of the first, second, and third angles, &c., reckoned from below upwards; then, if we add together all these values, whose amount may be $= m$, we shall obtain the height CN of the point O vertically above the lowest point C, viz.:

$$x = CN = \frac{V}{H} \cdot \frac{OR}{b} (OR + 2\ OR + 3\ OR + \ldots + m \cdot OR)$$

$$\frac{V}{H} \cdot \frac{\overline{OR}^2}{b} (1 + 2 + 3 + \ldots + m) = \frac{V}{H} \cdot \frac{m(m+1)}{1 \cdot 2} \cdot \frac{OR^2}{b},$$

in accordance with the theory of arithmetical series.

Lastly, if OR be put $= \frac{y}{m}$, we shall have:

$$x = \frac{V}{H} \cdot \frac{m(m+1)}{2m^2} \cdot \frac{y^2}{b}.$$

If the number of weights be very great, $m + 1$ may be taken $= m$, whence we shall have:

$$x = \frac{V}{H} \cdot \frac{y^2}{2b}.$$

For $x = a$, $y = b$, hence also:

$$a = \frac{V}{H} \cdot \frac{b}{2}, \text{ and more simply:}$$

$$\frac{x}{a} = \frac{y^2}{b^2}, \text{ which is the equation to a parabola.}$$

If, therefore, a string devoid of weight be stretched by infinitely many weights applied at equal horizontal distances, the funicular polygon will pass into a parabola.

For the angle of inclination ϕ we have besides:

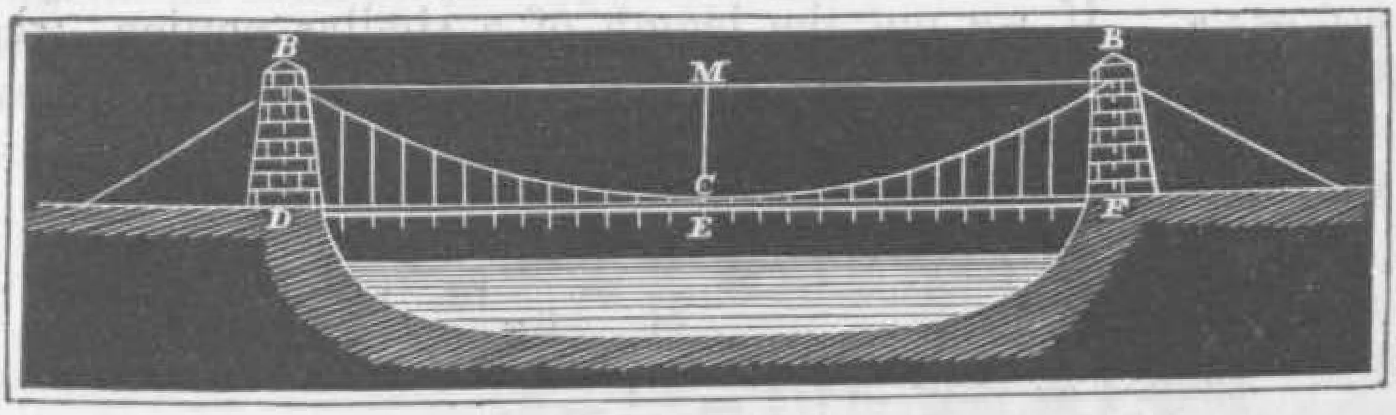
$$tang. \phi = \frac{y}{b} \cdot \frac{2a}{b} = 2y \cdot \frac{a}{b^2} = 2y \cdot \frac{x}{y^2} = \frac{2x}{y}, \text{ as also}$$

$$tang. \alpha = \frac{2a}{b}.$$

Therefore the tangent OT cuts the axis of the abscissæ, so that $CT = CN = x$.

If the chains and rods of a chain bridge, Fig. 149, were without weight, or light enough in respect to the weight of the loaded bridge *DEF*, which only is to be taken into consideration, then the chain *ACB* would form a parabola.

Fig. 149.



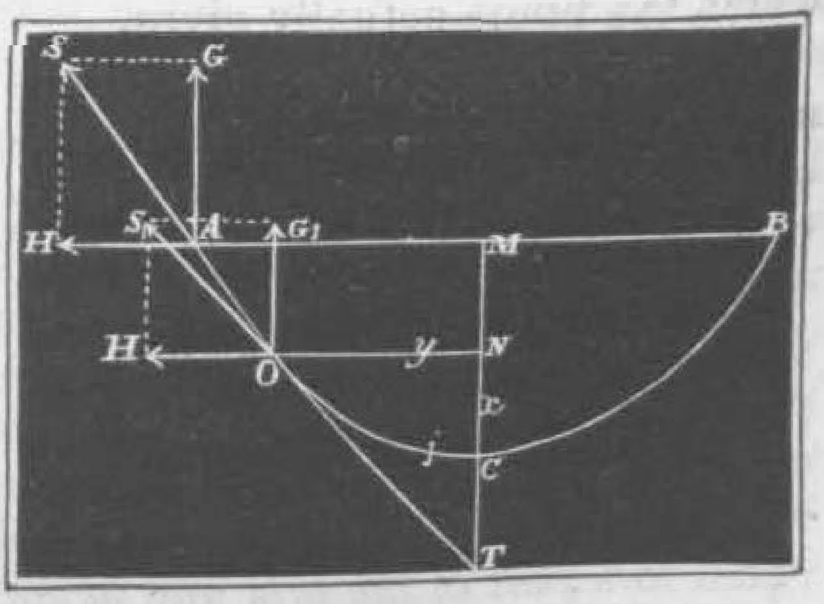
Example. The whole load of a chain-bridge in Fig. 149, $= 320000$ lbs.; the span $AB = 2b = 150$ feet, and the height of the arch $CM = a = 15$ feet; to find the tensions and other relations of the chains. The inclinations of the ends of the chain to the horizon is determined by the formula, $tang.\ \alpha = \frac{2a}{b} = \frac{30}{75} = \frac{2}{5} = 0{,}4$, therefore, $\alpha = 21^\circ$ $48'$. The vertical tension at each point of suspension is $V_1 = \frac{1}{2}$ the weight $= 160000$ lbs.; the horizontal, $H = V_1\, cotg.\ \alpha = 160000 \cdot \frac{1}{0{,}4} = 400000$ lbs.; lastly, the whole tension at one end:

$$S = \sqrt{V^2 + H^2} = V\sqrt{1 + cotg.\ \alpha^2} = 160000 \sqrt{1 + \left(\frac{1}{0{,}4}\right)^2}$$

$$= 160000 \sqrt{\frac{29}{4}} = 80000\sqrt{29} = 430813 \text{ lbs.}$$

§ 145. *Catenary.*—When a perfectly flexible and extensible string suspended from two points, or a chain consisting of short links, is stretched by its own weight, its axis forms a curved line, to which the name of catenary has been given. The imperfectly elastic and extensible cords, ropes, bands, chains, &c., met with in practice, give curved lines which approximate to the catenary only, but may usually be treated as such. From the foregoing, the horizontal tension of the catenary is equally great at all points, on the other hand, the vertical tension is equivalent to the vertical tension of the points of suspension lying above it, less the weight of the portions of the chain above. Since the tension at the vertex, where the catenary is horizontal, is null, the vertical tension, therefore, at the point of suspension, is equivalent to the weight of the chain from that point to the vertex; and the vertical tension at each place also equivalent to the weight of the portion of the rope or chain lying below it.

Fig. 150.



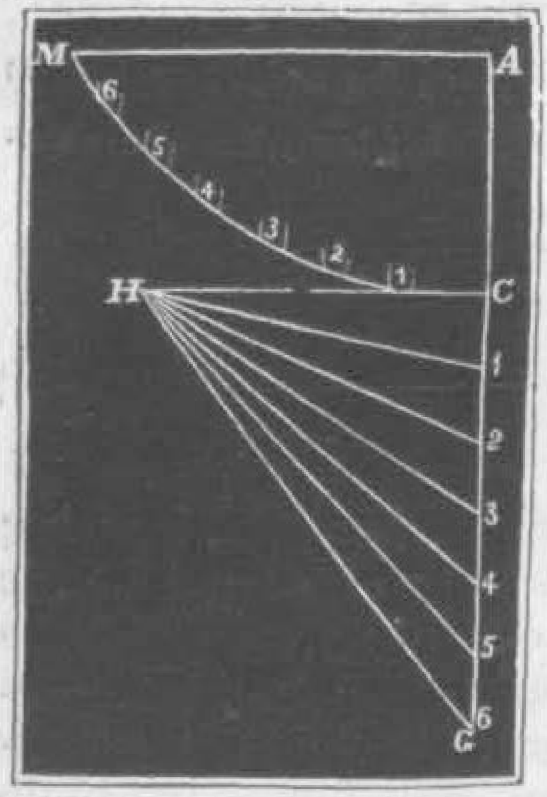
If equal lengths of the chain be equally heavy, we have then the common catenary, which only we will now consider. If a portion of the rope, or chain one foot in length, weighs γ, and if the arc corresponding to the co-ordinates $CM = a$ and $MA = b$, Fig. 150, $AOC = l$, we then have the weight of the portion of the chain $AOC = l\gamma$; if, on the other hand, the length of the arc (l) corresponding to the co-ordinates ($CN = x$, and $NO = y$) $= s$, we have the weight of this arc $= s\,\gamma$. If we put the length of a similar portion, whose weight $= H, = c$, (the horizontal tension,) we have further $H = c\,\gamma$, and, therefore, for the angles of inclination α and ϕ at the points A and O:

$$tang.\ \alpha = tang.\ SAH = \frac{G}{H} = \frac{l\,\gamma}{c\,\gamma} = \frac{l}{c}, \text{ and}$$

$$tang.\ \phi = tang.\ NOT = \frac{s\,\gamma}{c\,\gamma} = \frac{s}{c}.$$

§ 146. If we make the horizontal line CH, Fig. 151, = the length c of the portion of chain measuring the horizontal tension, and $CG =$ the length l of the arc of the chain on one side, we have, in accordance with § 142, in the hypothenuse GH, the measure and direction of the funicular tension at the point A, for

Fig. 151.



$$tang.\ CHG = \frac{CG}{CH} = \frac{l}{c} \text{ and}$$

$$\overline{GH} = \sqrt{\overline{CG}^2 + \overline{CH}^2} = \sqrt{l^2 + c^2},$$

$$\text{or } S = \sqrt{G^2 + H^2} = \sqrt{l^2 + c^2}\ .\ \gamma$$

$$= \overline{GH}\ .\ \gamma.$$

If now we divide CG into equal parts and draw from H to the points 1, 2, 3, &c., straight lines, these will give the measure and directions of the tensions of those points of the catenary which we obtain when we divide the length of the catenary arc AC into as many equal parts. So, for example, the line $H3$ gives the measure and direction of the tension or the tangents at the point (3) to the arc AC, because in this point the vertical tension $= \overline{C3}\ .\ \gamma$, whilst the horizontal tension remains the same $= c\ .\ \gamma$, therefore for this point $tang.\ \phi = \frac{\overline{C3}\ .\ \gamma}{c\,\gamma} = \frac{\overline{C3}}{\overline{CH}}$, which the figure actually gives.

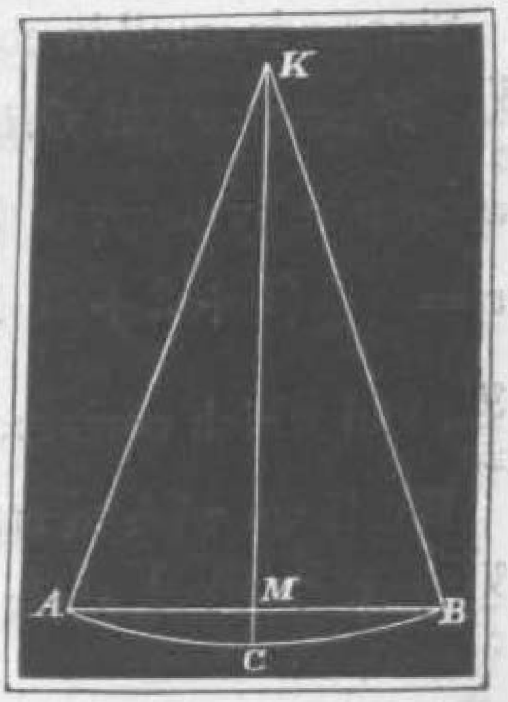
This peculiarity of the catenary is of use in constructing this curve mechanically, with an approximation to correctness After the given length CG of the catenary arc for construction has been divided into very many equal parts, the line $CH = c$ measuring the horizontal tension is applied to it, and the transversal lines $H1$, $H2$, $H3$, &c., drawn; if a part $C1$ of the arc be placed npon CH, and through the point of division obtained (1) a parallel to $H1$ be drawn, which cuts off from it a part (12); and likewise through the point (2) another line parallel to $H2$ be drawn, and which cuts off from it a point (23) equal to a part of the arc, and again through this (3) another, parallel to $H3$, and (34) be made equal to another part of the arc, and we proceed in this manner, we shall obtain a polygon (C 1 2 3 4 . . .); as we have taken these sides very small, we may consider it as a curve and easily find the curve to it, if we connect the middle points of the small sides (C 1), (12), (23), by a trace or line.

For practical purposes, a finely linked chain suspended against a perpendicular wall enables us to determine accurately enough a catenary answering certain conditions, as those of given length and height, or of given width or length of the arc.

§ 147. In many cases, and also in applications to architecture and to machines, the horizontal tension of the catenary is very great, and the height of the arc small in comparison with the width. Under this supposition, an equation to this curve is obtained in the following manner.

Let s be the length, $x = CM$ the absciss, and $y = AM$ the ordinate of a very compressed arc AC, Fig. 152. If we make $AK = CK$, we may consider this arc as a circular one described from K as a centre. Since from the known equation of the circle $y^2 = x(2r - x)$, it follows that the radius CK of the circle,

Fig. 152.



$r = \frac{y^2}{2x} + \frac{x}{2}$, or more simply, if we neglect $\frac{x}{2}$ as small in comparison with $\frac{y^2}{2x}$, $r = \frac{y^2}{2x}$. For the angle $AKC = \phi^0$, subtended at the centre by AB $sin.\ \phi 1 = \frac{AM}{AK} = \frac{y}{r} = \frac{2x}{y}$, and the arc $\phi = sin.\ \phi + \frac{1}{6}\ sin.\ \phi^3 + \frac{3}{40}\ sin.\ \phi^5 + \ldots$; if we have regard only to the two first members, it therefore follows that:

$$\phi = \frac{2x}{y} + \frac{1}{6} \cdot \left(\frac{2x}{y}\right)^3 = \frac{2x}{y} + \frac{4}{3i} \cdot \left(\frac{x}{y}\right)^3.$$

Now the arc $AC = s = r\,\phi = \frac{y^2}{2xi} \cdot \phi$; hence:

$$s = y + \frac{2}{3} \cdot \frac{x^2}{y} = y\left[1 + \frac{2}{3}\left(\frac{x}{y}\right)^2\right].$$

But inversely, $y = \frac{s}{1 + \frac{2}{3}\left(\frac{x}{y}\right)^2}$, which may be put:

$$y = s\left[1 - \frac{2}{3}\left(\frac{x}{s}\right)^2\right], \text{ and on the other handi:}$$

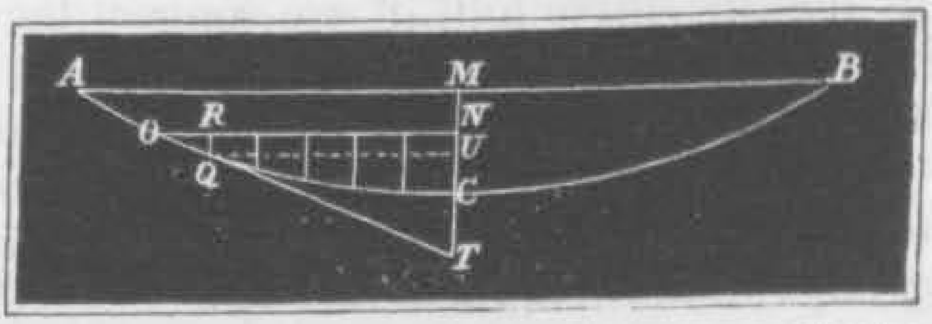
$$x = \sqrt{\frac{3}{2}y(s - y)}.$$

Example. The width of a very compressed arc, whose law for the rest is not known, is $2\,b = 3{,}5$ feet, and the height $a = 0{,}25$ feet; its length, therefore, is:

$$2\,l = 3{,}5\left[1 + \frac{2}{3} \cdot \left(\frac{0{,}25}{1{,}75}\right)^2\right] = 3{,}5\left(1 + \frac{2}{3} \cdot 0{,}143^2\right) = 3{,}5 + 3{,}5 \cdot 0{,}0136 = 3{,}548 \text{ ft.}$$

§ 148. We will now apply the formula $s = y\left[1 + \frac{2}{3}\left(\frac{x}{y}\right)^2\right]$ for

Fig. 153.



the length of a compressed arc to a strongly stretched catenary ACB, Fig. 153, while we put the vertical tension at a point O, $= V = s\gamma = y\left[1 + \frac{2}{3}\left(\frac{x}{y}\right)^2\right] \cdot \gamma$, and therefore for the angle made by the tangent $TON = \phi$, $tang.\ \phi = \frac{s}{c} = \frac{y}{c}\left[1 + \frac{2}{3}\left(\frac{x}{y}\right)^2\right]$.

If we divide the ordinate y into m equal parts, we find the portion $RQ = NU$ of the absciss x corresponding to one such part OR, when we put $RQ = OR \,.\, tang.\ \phi = OR \,.\, \frac{y}{c}\left[1 + \left(\frac{x}{y}\right)^2\right]$.

Since x is small in comparison with y, RQ is approximately $= OR \,.\, \frac{y}{c}$. If now we put $OR = \frac{y}{m}$ and successively for y: $\frac{y}{m}, \frac{2y}{m}, \frac{3y}{m}$, &c., we obtain by degrees the several parts of x, whose sum, therefore, is $x = \frac{y^2}{cm^2}(1 + 2 + 3 + \ldots + m) = \frac{y^2}{cm^2} \cdot \frac{m(m+1)}{2}$ (§ 144) $= \frac{y^2}{2c}$, and which corresponds with the equation to the parabola.

But if we wish to attain greater accuracy, we must put $QR = OR \,.\, \frac{y}{c}\left[1 + \frac{2}{3}\left(\frac{x}{y}\right)^2\right]$, substitute for x its value last found $\frac{y^2}{2c}$, and we shall then obtain:

$$QR = OR \,.\, \frac{y}{c}\left(1 + \frac{1}{6} \cdot \frac{y^2}{c^2}\right) = \frac{OR}{c}\left(y + \frac{1}{6} \cdot \frac{y^3}{c^2}\right).$$

Let us again successively put $y = \frac{y}{m}, \frac{2y}{m}, \frac{3y}{m}$, &c., and for OR likewise $\frac{y}{m}$, we shall then find the several values of x, and the sum itself:

$$x = \frac{y}{cm}\left[\frac{y}{m}(1+2+3+\ldots+m) + \frac{1}{6\,c^2}\cdot\left(\frac{y}{m}\right)^3(1^3+2^3+3^3+\ldots+m^3)\right].$$

Now for a very great number of members, the sum of the natural numbers from 1 to $m = \frac{m^2}{2}$, and the sum of their cubes $= \frac{m^4}{4}$, accordingly

$$x = \frac{y}{c}\left(\frac{y}{2} + \frac{1}{6\,c^2} \cdot \frac{y^3}{4}\right)\ i.\ e.$$

1. $x = \frac{y^2}{2c} + \frac{y^4}{24c^3} = \frac{y^2}{2c}\left[1 + \frac{1}{12}\cdot\left(\frac{y}{c}\right)^2\right]$, the equation of a strongly stretched catenary.

By inversion it follows that $y^2 = 2cx - \frac{y^4}{12c^2} = 2cx - \frac{4c^2x^2}{12c^2}$ $= 2cx - \frac{x^2}{3}$, therefore:

2. $y = \sqrt{2cx - \frac{x^2}{3}}$, or approximately $y = \sqrt{2cx}\left(1 - \frac{x}{12c}\right)$.

The measure of the horizontal tension is further given:

$$c = \frac{y^2}{2x} + \frac{y^4}{2x \cdot 12c^2} = \frac{y^2}{2x} + \frac{y^4}{24x} \cdot \frac{4x^2}{y^4}, \text{ i. e.}$$

3. $c = \frac{y^2}{2x} + \frac{x}{6}$.

The angle of the tangent ϕ is determined by:

$$\text{tang. } \phi = \frac{y}{c}\left[1 + \frac{2}{3}\left(\frac{x}{y}\right)^2\right] = \frac{y\left[1 + \frac{2}{3}\left(\frac{x}{y}\right)^2\right]}{\frac{y^2}{2x}\left[1 + \frac{1}{3}\left(\frac{x}{y}\right)^2\right]}$$

$$= \frac{2x}{y}\left[1 + \frac{2}{3}\left(\frac{x}{y}\right)^2\right]\left[1 + \frac{1}{3}\left(\frac{x}{y}\right)^2\right], \text{ i. e.}$$

4. $\text{tang. } \phi = \frac{2x}{y}\left[1 - \frac{1}{3}\left(\frac{x}{y}\right)^2\right]$.

Lastly, we must here place the formula of rectification found in the former paragraph:

5. $s = y\left[1 + \frac{2}{3}\left(\frac{x}{y}\right)^2\right] = y\left[1 + \frac{1}{6}\left(\frac{y}{c}\right)^2\right]$.

Example.—1. For a span $2b = 16$ feet and height of arc $a = 2\frac{1}{2}$ feet, the length $2l$ is $= 16\left[1 + \frac{2}{3}\left(\frac{2{,}5}{8}\right)^2\right] = 16 + 16 \cdot 0{,}065 = 17{,}04$ feet, the length of the portion of chain which measures the horizontal tension: $c = \frac{b^2}{2a} + \frac{a}{6} = \frac{64}{5} + \frac{5}{12} = 12{,}8 + 0{,}417$ $= 13{,}217$ feet; the tangent of the angle of suspension: $\text{tang. } \alpha = \frac{2a}{b}\left[1 + \frac{1}{3}\left(\frac{a}{b}\right)^2\right]$ $= \frac{5}{8}\left[1 + \frac{1}{3}\left(\frac{5}{16}\right)^2\right] = \frac{5 \cdot 1{,}03255}{8} = 0{,}6453\ldots$, the angle of suspension, therefore, $\alpha = 32^\circ\ 50'$.—2. A chain of 10 feet length and $9\frac{1}{2}$ span, has the height of its arc $a = \sqrt{\frac{3}{2}(l - b)b} = \sqrt{\frac{3}{2}\frac{(10 - 9\frac{1}{2})}{2}\frac{9\frac{1}{2}}{2}} = \sqrt{\frac{3}{2}\cdot\frac{19}{16}} = \sqrt{\frac{57}{32}}$ $= \sqrt{1{,}7812} = 1{,}335$ feet, and the measure of the horizontal tension:

$$c = \frac{b^2}{2a} + \frac{a}{6} = \frac{4{,}75^2}{2 \cdot 1{,}335} + \frac{1{,}335}{6} = 8{,}673 \text{ feet.}$$

3. If a line 30 feet long and weighing 8 lbs., be stretched horizontally by a force of 20 lbs., the vertical tension $V = \frac{1}{2}G = 4$ lbs.; the horizontal force $H = \sqrt{S^2 - V^2}$ $= \sqrt{20^2 - 4^2} = \sqrt{384} = 19{,}596$ lbs. The tangent of the angle of suspension: $\text{tang. } \phi = \frac{V}{H} = \frac{4}{19{,}596} = 0{,}20412$, the angle ϕ itself $= 11^\circ\ 32'$; the measure of

the horizontal tension $c = \frac{H}{\gamma} = H \div \frac{8}{30} = \frac{30}{8} H = 73{,}485$ feet; the span $2\,b$ $= 2\,l \left[1 - \frac{1}{6} \cdot \left(\frac{l}{c}\right)^2\right] = 30 \cdot \left[1 - \frac{1}{6} \cdot \left(\frac{15}{73{,}48}\right)^2\right] = 30 - 0{,}208 = 29{,}792$ ft., and the height of the arc $a = \sqrt{\frac{3}{2}\, b\,(l - b)} = \sqrt{\frac{3}{2}\,\frac{29{,}792 \,.\, 0{,}208}{2 \,.\, 2}}$ $= \sqrt{29{,}792 \,.\, 0{,}078} = 1{,}524$ feet.

§ 149. The higher calculus gives the following general formulæ for the catenary, and which hold good for all tensions.

1. $s = \sqrt{2\,c\,x + x^2}$, and inversely, $x = \sqrt{c^2 + s^2} - c$ and $c = \frac{s^2 - x^2}{2x}$.

2. $s = \frac{c}{2}\left(e^{\frac{y}{c}} - e^{-\frac{y}{c}}\right)$, inversely $y = c\,L\,n\left(\frac{s + \sqrt{c^2 + s^2}}{c}\right)$, where e is the base 2,71828 of the natural system of logarithms, and $L\,n$ the logarithm = 2,30258 times the common logarithm.

3. $y = c\,L\,n\left(\frac{c + x + \sqrt{2\,c\,x + x^2}}{c}\right)$, inversely $x = \frac{c}{2}\left(e^{\frac{y}{c}} + e^{-\frac{y}{c}}\right) - c$,

4. $y = \frac{s^2 - x^2}{2\,x}\, L\,n\left(\frac{s + x}{s - x}\right)$.

The use of these formulæ is very troublesome, especially in complicated problems, where a direct solution is generally not possible.

Example. The two co-ordinates of a catenary are $x = 2$ feet, and $y = 3$ feet; required the horizontal tension c of this curve? Approximately from No. 3 of the former paragraphs $c = \frac{y^2}{2x} + \frac{x}{6} = \frac{9}{4} + \frac{2}{6} = 2{,}58$. From No. 3 of the present paragraphs y is exactly $= c\,Ln\left(\frac{c + x + \sqrt{2\,c\,x + x^2}}{c}\right)$, i. e. $3 = c\,Ln\left(\frac{c + 2 + \sqrt{4\,c + 4}}{c}\right)$.

If c be here put $= 2{,}58$, we then have the error $f = 3 - 2{,}58\,Ln\left(\frac{4{,}58 + 2\sqrt{3{,}58}}{2{,}58}\right)$ $= 3 - 2{,}58\,Ln\left(\frac{8{,}3642}{2{,}58}\right) = 3 - 3{,}035 = -0{,}035$; but if c be put $= 2{,}53$, we then have the error $f_1 = 3 - 2{,}53\,Ln\left(\frac{4{,}53 + 2\sqrt{3{,}53}}{2{,}53}\right) = 3 - 2{,}53\,Ln\left(\frac{8{,}2876}{2{,}53}\right)$ $= 3 - 3{,}002 = -0{,}002$. In order now to find the true value of c, if, according to a known rule, we put

$$\frac{c - 2{,}58}{c - 2{,}53} = \frac{f}{f_1} = \frac{0{,}035}{0{,}002} = 17{,}5;$$ in this manner it will follow that:

$$16{,}5 \,.\, c = 17{,}5 \,.\, 2{,}53 - 2{,}58 = 41{,}69;$$ therefore:

$$c = \frac{41{,}69}{16{,}5} = 2{,}527 \text{ feet.}$$

Remark. Practical applications of the catenary will be given when, in the Second Part, we come to treat of the construction of vaults, chain-bridges, &c.

§ 150. *The Pulley.*—Ropes, cords, &c., are the usual means by which forces are transmitted over the wheel and axle. We will here develop what is most general in the theories of these two arrangements, without, however, taking into account friction and rigidity of cords.

A pulley is a circular disc, *ABC*, Fig. 154 and Fig. 155, turning about an axis on whose circumference lies a cord or string, and whose extremities are stretched by the forces P and Q. In a fixed pulley, the block in which the axis or pivot reposes is immovable; in a free pulley, on the other hand, it is movable.

Fig. 154.

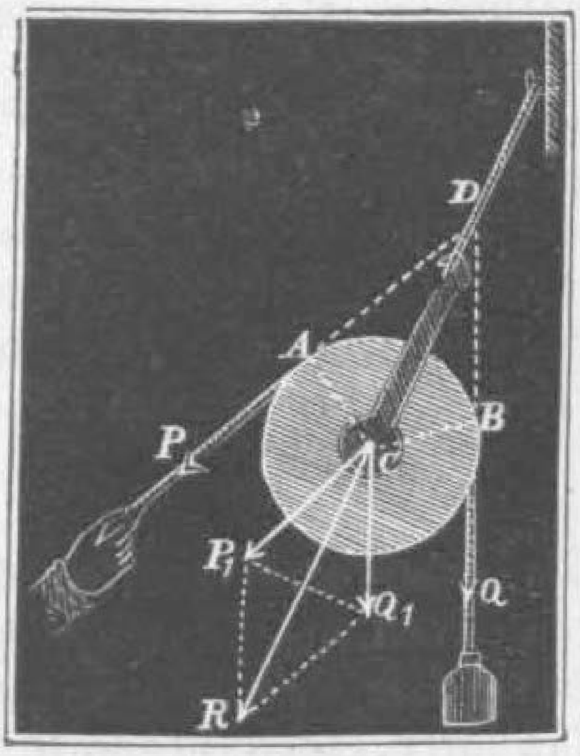
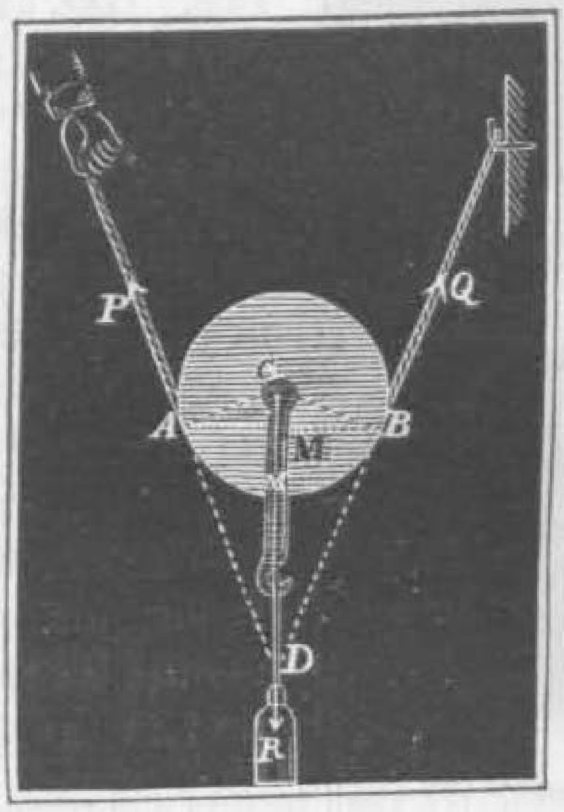


Fig. 155.



In the condition of equilibrium of a pulley, the forces P and Q at the extremities of the string are equal; for every pulley is a bent lever, the arms of which are equal in length, which we may obtain if we let fall perpendiculars CA and CB from the axis C on the directions of the forces, or of the strings DP and DQ. It is clear that the forces P and Q in any revolution about C describe the same space, viz. $r\,\phi$, if r be the radius $CA = CB$ and ϕ° the angle of revolution; and that from this we may infer the equality between P and Q. From the forces P and Q there arises the resultant $CR = R$, which is taken up by the block and is dependent on the angle $ADB = \alpha$, which the directions of the string include; and moreover it gives as the diagonal of the rhomb CP_1RQ_1 constructed from P and α : $R = 2\,P\,cos.\,\frac{\alpha}{2}$.

§ 151. In the fixed pulley, Fig. 154, the force Q consists of the weight to be overcome or raised at one extremity of the string; here, therefore, the force is equal to the weight, and the application of this pulley effects nothing but a change of direction. In the movable pulley, Fig. 155, on the other hand, the weight on the hook R acts at the extremity of the block, whilst the one extremity of the string is fastened to a fixed object; here, therefore, the force P is to be put $= \frac{R}{2\,cos.\,\frac{\alpha}{2}}$. If we represent the chord AMB, which corresponds to the arc over which the string passes, by a, the radius $CA = CB$, as before $= r$, then $a = 2\,AM = 2\,.\,CA\,cos.\,CAM = 2\,CA\,cos.\,ADM = 2\,r\,cos.\,\frac{\alpha}{2}$, hence $\frac{r}{a}$ may be put $= \frac{1}{2\,cos.\,\frac{\alpha}{2}}$, and likewise $\frac{P}{R} = \frac{r}{a}$. From this, therefore, *the power in the movable pulley is to the weight as the radius of the pulley to the chord of the arc over which the string passes.*

Fig. 156.

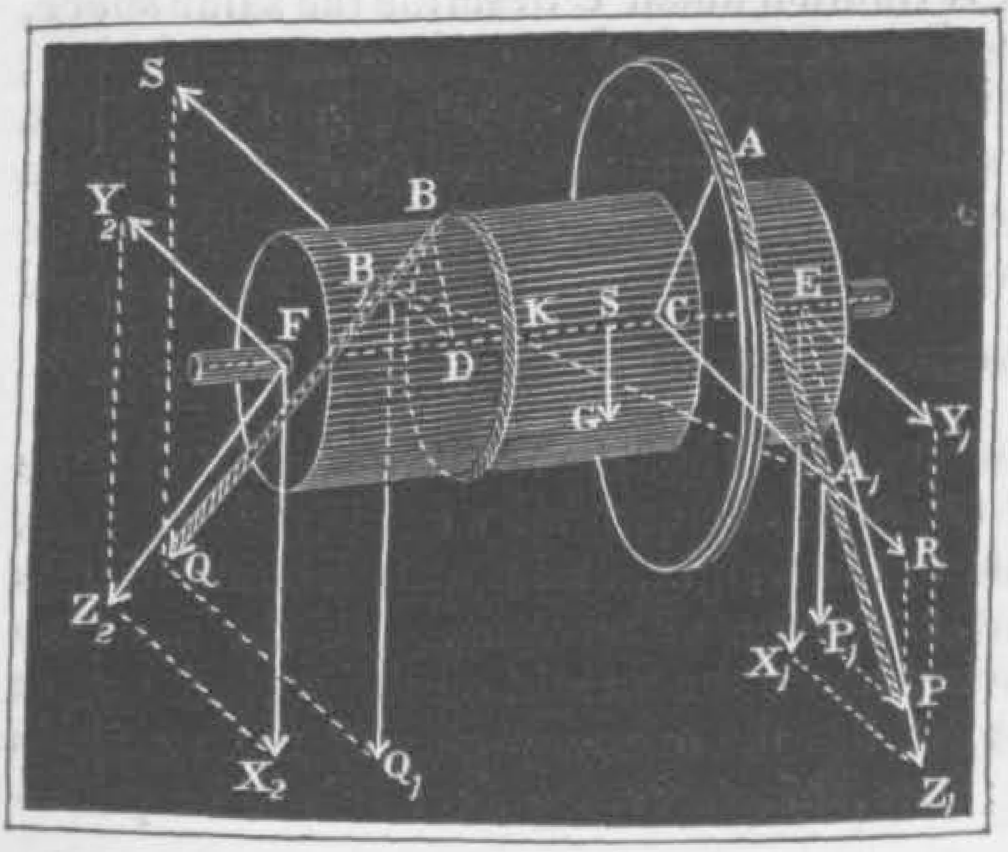
If $a = 2r$, the string passes over a semicircle, Fig. 156, the force then is at a minimum; viz. $P = \frac{1}{2} R$; if $a = r$, that is 60^0 of the part of the pulley over which the string passes, we have $P = R$; the smaller, therefore, a becomes, the greater is P, and for a infinitely small, the force P becomes infinitely great. An inverse proportion takes place in the spaces; if s is the space of P, which corresponds to a space $R = h$, we have then $Ps = Rh$, therefore, $\frac{s}{h} = \frac{a}{r}$.

The movable pulley is thus a means of modifying force; for example, a given weight may by this means be raised by a smaller force, but in proportion as there is gain in force, there is loss in space.

Remark. We shall treat of the composition of pulleys and systems of pulleys, as well as of the resistances arising from friction and rigidity, more fully in a subsequent Part.

§ 152. *The Wheel and Axle.*—The wheel and axle is a rigid connection of two fixed pulleys or wheels, capable of revolving about a common axis $ABFE$, Fig. 157. The smaller of these wheels is called the axle, the greater one the wheel. The round extremities E and F, on which this arrangement rests, are called gudgeons. The axis of revolution of the wheel and axle is either horizontal, or vertical, or inclined. Here we shall only speak of the wheel and axle which revolves about a horizontal axis. We shall also here suppose, that the forces P and Q, or the power P and the weight Q act at the extremities of a perfectly flexible string, which passes round the circumference of the wheel and axle. The questions to be answered are, in what relations the powers and weights are to each other, and what pressures the gudgeons E and F have to sustain?

Fig. 157.



Let us imagine a horizontal plane passed through the axis CD and the points of application A and B of the power P, and the weight Q transferred to this plane, and therefore P and Q applied at A_1 and B_1. If the angles AA_1C and BB_1D, which both forces make with the horizon $= \alpha$ and β, these forces may be replaced by the horizontal forces $R = P \cos. \alpha$, $S = Q \cos. \beta$, and by the vertical forces $P_1 = P \sin. \alpha$, $Q_1 = Q \sin. \beta$. The horizontal forces are directed towards the axis, and being applied at C and D, become perfectly counteracted by the axis. The vertical forces P_1 and Q_1, on the other hand, tend to turn the wheel and axle about its axis. If K be the intersection with the axis of the line connecting the points A_1 and B_1, KA_1 and KB_1 are the arms of P_1 and Q_1, and equilibrium subsists about K, and also about CD, ifi:

$$P_1 . KA_1 = Q_1 . KB_1, \text{ or, since } \frac{KA_1}{KB_1} = \frac{CA_1}{DB_1}, \text{ if}$$

$$P_1 . CA_1 = Q_1 . DB_1, \text{ or, as } \frac{P_1}{P} = \frac{CA}{CA_1}, \text{ and}$$

$$\frac{Q_1}{Q} = \frac{DB}{DB_1},$$

$$\frac{P . CA}{CA_1} . CA_1 = \frac{Q . DB}{DB_1} . DB_1, \text{ i. e.}$$

$$P . CA = Q . DB, \text{ or } Pa = Qb,$$

if a and b represent the arms of the power and weight, or the radii of the wheel and axle. In the wheel and axle, therefore, as in every lever, the moment of the power is equivalent to the moment of the weight.

§ 153. The forces P_1 and Q_1 give at K a vertical pressure $P_1 + Q_1$, with which must also be associated the weight G of the whole wheel and axle applied at the centre of gravity S. The supports of the gudgeons at E and F have also to sustain the vertical pressure $P_1 + Q_1 + G = P \sin. \alpha + Q \sin. \beta + G$. If we put the whole length of the wheel and axle measured from E to $F = L$, the part $EC = l_1$, $CD = l$, $DF = l_2$, therefore $L = l + l_1 + l_2$, and the distances ES and FS of the centre of gravity S from the supports d_1 and d_2, therefore also $L = d_1 + d_2$, we shall obtain since

$$\frac{DK}{DC} = \frac{P_1}{P_1 + Q_1}, \text{ as } DK = \frac{P_1 l}{P_1 + Q_1}$$

for the vertical pressure X_1 at the gudgeon E:

$$X_1 . EF = G . FS + (P_1 + Q_1) FK,$$

$$X_1 = \frac{Gd_2 + (P_1 + Q_1)\left(l_2 + \frac{P_1}{P_1 + Q_1} . l\right)}{L}, \text{ i. e.}$$

$$X_1 = \frac{Gd_2 + (P_1 + Q_1)\, l_2 + P_1 l}{L}.$$

On the other hand, for the vertical pressure X_2 at F:

$$X_2 . EF = G . ES + (P_1 + Q_1) EK, \text{ i. e.}$$

$$X_2 = \frac{Gd_1 + (P_1 + Q_1)\left(l_1 + \frac{Q_1}{P_1 + Q_1} . l\right)}{L}, \text{ i. e.}$$

$$X_2 = \frac{Gd_1 + (P_1 + Q_1)\, l_1 + Q_1 l}{L}.$$

The horizontal forces R and S have the moments about F, $R \,.\, FC = R\,(l + l_2)$, and $S \,.\, FD = S \,.\, l_2$, and about E : $S \,.\, ED = S\,(l + l_1)$, and $R \,.\, EC = Rl_1$; if, therefore, we put the horizontal pressures upon E and F effected by them $= Y_1$ and Y_2, we shall obtain:

$Y_1 \,.\, FE = R \,.\, FC - S \,.\, FD$, as

$$Y_1 = \frac{R\,(l + l_2) - Sl_2}{L}; \text{ and}$$

$Y_2 \,.\, FE = S \,.\, ED - R \,.\, EC$, as

$$Y_2 = \frac{S\,(l + l_1) - Rl_1}{L}.$$

From X_1 and Y_1 the total pressure at E is:

$Z_1 = \sqrt{X_1^2 + Y_1^2}$, and likewise from X_2 and Y_2, the same at F:

$Z_2 = \sqrt{X_2^2 + Y_2^2}$.

Lastly, if ϕ and ψ be the angles which the directions of these pressures make with the horizon, we shall then have

$$tang.\, \phi = \frac{X_1}{Y_1} \text{ and } tang\, \psi = \frac{X_2}{Y_2}.$$

Example. The weight Q of a wheel and axle pulls perpendicularly downwards, and amounts to 365 lbs.; the radius of the wheel $a = 1\frac{3}{4}$ ft.; that of the axle $b = \frac{3}{4}$ ft.; the weight of the machine itself is 200 lbs.; its centre of gravity S lies distant from E and F, $d_1 = 1\frac{1}{2}$, and $d_2 = 2\frac{1}{2}$ ft.; the middle of the wheel is about $l_1 = \frac{3}{4}$ ft. from the gudgeon E, and the vertical plane in which the weight acts is about $l_2 = 2$ ft. from the gudgeon F. Now if the force P necessary for restoring the equilibrium at the wheel inclined to the horizon at an angle $50^\circ = \alpha$, pulls downwards, what will this be, and what will be the pressures on the gudgeons? $Q = 365$, $\beta = 90^\circ$, consequently $Q_1 = Q\ sin.\ \beta = Q$ and $S = Q\ cos.\ \beta = 0$; further, P being unknown, and $\alpha = 50^\circ$, consequently $P_1 = P\ sin.\ \alpha = 0{,}7660 \,.\, P$ and $R = P\ cos.\ \alpha = 0{,}6428 \,.\, P$; but now $a = 1\frac{3}{4} = \frac{7}{4}$ and $b = \frac{3}{4}$, it follows, therefore, $P = \frac{b}{a}\ Q = \frac{3}{7} \,.\, 365 = 156{,}4$ lbs., $P_1 = 119{,}8$ and $R = 100{,}5$. Further, because $G = 200$, $d_1 = \frac{3}{2}$, $d_2 = \frac{5}{2}$, $l_1 = \frac{3}{4}$, $l_2 = 2$, $L = \frac{3}{2} + \frac{5}{2} = 4$, and $l = L - (l_1 + l_2) = 4 - \frac{11}{4} = \frac{5}{4}$, so that the vertical pressure at E is:

$$X_1 = \frac{200 \,.\, \frac{5}{2} + (365 + 119{,}8) \,.\, 2 + 119{,}8 \,.\, \frac{5}{4}}{4} = \frac{1619{,}35}{4} = 404{,}8 \text{ lbs.}$$

and that at F:

$$X_2 = \frac{200\,.\, \frac{3}{2} + (365 + 119{,}8) \,.\, \frac{3}{4} + 365 \,.\, \frac{5}{4}}{4} = \frac{1119{,}85}{4} = 280{,}0 \text{ lbs.}$$

Both of these forces together give:

$$X_1 + X_2 = Q + G + P_1 = 684{,}8 \text{ lbs.}$$

The horizontal force at E is:

$$Y_1 = \frac{100{,}5 \,.\, (\frac{5}{4} + 2) - 0 \,.\, 2}{4} = 81{,}7 \text{ lbs., and that at } F:$$

$$Y_2 = \frac{0 \,.\, (\frac{5}{4} + \frac{3}{4}) - 100{,}5 \,.\, \frac{3}{4}}{4} = -\ 18{,}8 \text{ lbs.}$$

the sum of these is exactly $= R + S = 100{,}5$ lbs.

The pressure at E is inclined at an angle ϕ to the horizon, for which we have:

$$tang.\, \phi = \frac{X_1}{Y_1} = \frac{404{,}8}{81{,}7},\ Log.\ tang.\ \phi = 0{,}69502,\ \phi = 78^\circ\ 35'.$$

The pressure itself: $Z_1 = \frac{X_1}{sin.\ \phi} = 413{,}0$ lbs.

On the other hand, for the inclination ψ of the pressure at F:

$$tang.\, \psi = \frac{X_2}{Y_2} = \frac{280{,}0}{18{,}8},\ Log.\ tang.\ \psi = 1{,}17300,\ \psi = 86^\circ,\ 9',\ 5;$$

and the pressure $Z_2 = \frac{Y_2}{cos.\ \psi} = 280{,}6$ lbs.