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TECHNICAL REPORT NO. 1052
July 1993

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IN M/M/s/s+c SYSTEMS
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¹Research supported by Grant BD/645/90-RM from Junta Nacional de Investigação Científica e
Tecnológica. On leave from: Departamento de Matemática, Instituto Superior Técnico, Av.
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SOME PROPERTIES OF THE DELAY PROBABILITY IN $M/M/s/s + c$ SYSTEMS

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July 1, 1993

Abstract:
We look at an extension of the steady state delay probability in $M/M/s/s + c$ systems to nonintegral number of servers $s$ and queue capacity $c$, which we call GED function. We show that this function is increasing and concave in the queue capacity. We find that if $c \geq 1$, the reciprocal of the GED function is convex in the traffic intensity and the GED function is increasing in the traffic intensity $\rho$ if $\rho$ is below some $\rho^*_{s,c}$, and decreasing if $\rho$ is greater than $\rho^*_{s,c}$. Moreover, $\rho^*_{s,c}$ is increasing in the number of servers and, for $s \geq 1$, $\rho^*_{1,c} = 1 \leq \rho^*_{s,c} < 2$.

Keywords: Queues, delay probability, monotonicity, convexity.

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*Research supported by Grant BD/645/90 -RM from “Junta Nacional de Investigação Científica e Tecnológica”.

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1 Introduction

For the optimal design of queueing systems it is of primary importance to know the behaviour of certain performance measures of the design; among the most important such measures is the Erlang delay function,

\[ C(s, a) = \frac{a^s}{s!} \frac{1}{1-a/s}, \quad s \in \mathbb{N}, a \in (0, s). \]  (1)

Erlang [3] used \( C(s, a) \) to represent the steady state probability that a customer (call) which is a member of a Poisson stream of rate \( a \), arriving at a group of \( s \) servers (telephone trunks) with unit exponential service time, will be delayed before receiving service.

Throughout this paper, we denote by \( s, c, \lambda, \mu, a = \frac{\lambda}{\mu}, \rho = \frac{\lambda}{s\mu} = \frac{a}{s} \), repectively the number of servers, the queue capacity, the arrival rate, the service rate, the offered load and the traffic intensity. Moreover we still talk as \( s \) being the number of servers and \( c \) being the queue capacity when \( s \) and \( c \) take noninteger values. In addition, we use increasing (decreasing, convex, etc.) meaning strictly increasing (decreasing, convex, etc.).

We start by giving a brief account of some of the work on the Erlang delay function. \( C(s, s\rho) \) increases with the traffic intensity and decreases with the number of servers. With respect to second-order properties, \( C(s, a) \) was shown to be convex in the offered load (or equivalently, the arrival rate) by Lee and Cohen [11], in the service rate by Harel and Zipkin [4] and in the number of servers by Krupp [10] and subsequently by Harel [6] and Jagers and Van Doorn [9] (we note that the proofs in Krupp [10] and Jagers and Van Doorn [9] are for nonintegral number of servers whereas the proof in Harel [6] is for integer number of servers). \( C(s, a) \) is convex in the arrival rate but is not jointly convex in the arrival and service rates as shown by Harel and Zipkin [4], who also showed that \( C^{-1}(s, a) \) is convex in the arrival rate.

The Erlang delay function is related with other important performance measures such as the steady state mean number in queue and the average sojourn time in the Erlang delay system; the relevant point here being that some properties of the Erlang delay function may be carried to these measures. In particular, the fact that the steady state mean number in queue in the Erlang delay system is convex in the traffic intensity is a direct consequence of the convexity of the Erlang delay function with respect to the traffic intensity, and this was the via Lee and Cohen [11] used to prove that the mean number in queue is a convex function of the traffic intensity. Similarly, the convexity of the average sojourn time in the Erlang delay system in the number of servers, a fact proved by Dyer and Proll [2], is a direct consequence of the convexity of the Erlang delay function in the number of servers.

The Erlang delay function has been used in the construction of approximations of performance measures for \( M/G/c \) queues such as the probability that a customer has to wait, the mean number in queue and the average waiting time in queue (see Hokstad [7]). Some properties of the Erlang delay function may be carried to these approximations (for examples
of this fact see Lee and Cohen [11]). This is another important reason for the study of the properties of the Erlang delay function.

Aside from the Erlang delay function, its derivatives are also important for the optimization of several systems. Akimaru and Nishimura [1] concentrated in the computation of the same derivatives and on applications of the same work. Bounds and approximations of \( C(s, a) \) have also been obtained such as in Harel [5].

In this paper we consider the Generalized Erlang Delay (GED) function

\[
D(s, a, c) = \frac{a^c \sum_{i=0}^{c-1} \left( \frac{a}{s} \right)^i}{\sum_{k=0}^{s} \frac{a^k}{k!} + \frac{a^c}{s!} \sum_{i=1}^{c} \left( \frac{a}{s} \right)^i}, \quad s \in \mathbb{N}, a \in \mathbb{R}^+, c \in \mathbb{N}_0,
\]  

which gives the steady state delay probability for the system \( M/M/s/s + c \), in which there is a waiting room with capacity \( c \), when the offered load is \( a \). It is useful not to constrain \( s \) and \( c \) to be integers. We can extend \( D(s, a, c) \) to a continuous function on \((0, +\infty)^2 \times [0, +\infty]\) as follows:

\[
D(s, a, c) = \frac{1 - \left( \frac{a}{s} \right)^c}{\left( 1 - \frac{a}{s} \right) B^{-1}(s, a) + \left[ \frac{a}{s} - \left( \frac{a}{s} \right)^{c+1} \right]}, \quad \text{with } a \neq s, c < +\infty,
\]  

where \( B(s, a) \) is the analytic continuation of the Erlang loss function, as given by Theorem 3 in Jagerman [8],

\[
B(s, a) = \left[ \int_{0}^{\infty} e^{-x} \left( 1 + \frac{x}{a} \right)^s dx \right]^{-1}.
\]

The classical Erlang delay function (1) is identical with \( D(s, a, +\infty) \) for \( 0 < a < s \), \( s \in \mathbb{N} \), as we will see later. An \( M/M/s \) system has no steady-state when \( a \geq s \). Thus \( D(s, a, +\infty) \) does not represent a delay probability when \( a \geq s \); it represents instead the fraction of customers that are delayed and served in a given \( M/M/s \) system with offered load \( a \) and unit mean service time, i.e.

\[
\lim_{t \to +\infty} \frac{\text{number of customers delayed and served in the } M/M/s \text{ system in } (0,t]}{\text{number of customers that arrive to the } M/M/s \text{ system in } (0,t]}.
\]

The identification of \( D(s, a, c) \) as being the fraction of customers that are delayed and served in an \( M/M/s/s + c \) system with offered load \( a \) and unit mean service time is valid regardless of the existence or not of a steady-state.

The advantage of the GED function over the classical Erlang delay function is that the GED function, because it has one more parameter associated with the queue capacity, gives more flexibility in terms of design of queueing systems. By changing the queue capacity \( c \) from 0 to +\( \infty \) in the GED function, we take into consideration the fraction of delayed
and served customers in a broad class of systems ranging from the Erlang loss system to the Erlang delay system. The approach now used of studying a functional of $M/M/s/s + c$ systems for which the queue capacity is one of the parameters of interest, and noninteger number of servers and queue capacities are allowed, was used by the author in [12] to study, for the same class of systems, the fraction of lost customers. The analogue of the GED function for the fraction of lost customers was named GEL function by Pacheco [12].

In our study we pay attention to two different parametrizations of the GED function, namely $D(s, a, c)$ and $D(s, s\rho, c)$. In the first parametrization, the offered load $a$ is one of the parameters of interest and in the second the traffic intensity $\rho$ is considered instead; both cases are of practical interest. In the rest of the paper we assume, unless explicitly stated, $s, a, \rho \in (0, +\infty)$ and $c \in [0, +\infty]$ and, if $A$ is a function, we write $A^{-1}$ for $\frac{1}{A}$, the reciprocal of $A$.

We start, in section 2, by deriving some preliminary results that will be used later in the paper. In section 3 we first study monotonicity properties of the GED function as well as the limit values of the same function when its parameters go to 0 and $+\infty$. We prove that the GED function is increasing and concave with respect to the queue capacity, and therefore that its reciprocal is decreasing and convex in the same queue capacity. We find that if $c \geq 1$, $D^{-1}(s, s\rho, c)$ is convex (non-strictly if $c = +\infty$) in the traffic intensity and $D(s, s\rho, c)$ is increasing in the traffic intensity $\rho$ if $\rho$ is below some $\rho_{s,c}^*$ and decreasing if $\rho$ is greater than $\rho_{s,c}^*$. Moreover, $\rho_{s,c}^*$ is increasing in the number of servers and, for $s \geq 1$, $\rho_{1,c}^* = 1 \leq \rho_{s,c}^* < 2$. These results are of obvious importance for the characterization of the delay probability in $M/M/s/s + c$ systems. In section 4 we present tables of the values $\rho_{s,c}^*$ as an illustration of the results obtained in section 3.

2 Preliminary Results

We start this section by stating in Lemma 1 some results on the monotonicity properties of the Erlang loss function $B(s, a)$; the reason for this is that we base the monotonicity results for the GED function $D(s, a, c)$ on similar results for $B(s, a)$ and the relationship between these two functions. After that we give some additional properties of the Erlang loss function in Lemma 2, and of a function that is important for the developments in section 3 in Lemma 3. Lemma 1 is Lemma 1 in Pacheco [12].

**Lemma 1** For $s, a, \rho \in (0, +\infty)$, the functions $B(s, a)$ and $B(s, s\rho)$ have the following properties:

(i) $B(s, a)$ increases with $a$; $B(s, a) \xrightarrow{a \to 0^+} 0$; $B(s, a) \xrightarrow{a \to +\infty} 1$.

(ii) $B(s, a)$ decreases with $s$; $B(s, a) \xrightarrow{s \to 0^+} 1$; $B(s, a) \xrightarrow{s \to +\infty} 0$.

(iii) $B(s, s\rho)$ increases with $\rho$; $B(s, s\rho) \xrightarrow{\rho \to 0^+} 0$; $B(s, s\rho) \xrightarrow{\rho \to +\infty} 1$. 

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(iv) $B(s, s\rho)$ decreases with $s$; $B(s, s\rho) \overset{s \to 0^+}{\longrightarrow} 1$; $B(s, s\rho) \overset{s \to +\infty}{\longrightarrow} \left\{ \begin{array}{ll} 0 & \text{if } \rho < 1 \\ 1 - 1/\rho & \text{if } \rho \geq 1 \end{array} \right.$.

**Lemma 2** For $s, \rho \in (0, +\infty)$, we have:

(i) For fixed $s$, $B^{-1}(s, s\rho)$ is decreasing and convex in $\rho$.

(ii) For fixed $\rho$, $B^{-1}(s, s\rho)$ increases with $s$ and $\frac{\partial B^{-1}}{\partial \rho}(s, s\rho)$ decreases with $s$.

(iii) $B^{-1}(1, \rho) = 1 + \frac{1}{\rho}$ and $\frac{\partial B^{-1}}{\partial \rho}(1, \rho) = -\frac{1}{\rho^2}$.

(iv) If $\rho > 1$, $1 < B^{-1}(s, s\rho) < \frac{\rho}{\rho - 1}$ and $-\frac{1}{(\rho - 1)^2} \leq \frac{\partial B^{-1}}{\partial \rho}(s, s\rho) < 0$, $\forall s$.

(v) If $A(s, \rho, c) = (1 - \rho)B^{-1}(s, s\rho) + (\rho - \rho^{c+1})$,

\[ A(s, \rho, c) > (\ldots, <) 0 \iff \rho < (\ldots, >) 1. \]

\[ A(s, \rho, c) + \rho^{c+1} > 0. \]

**Proof:** (i) From (4), we have

$$B^{-1}(s, s\rho) = \int_0^\infty e^{-x} \left(1 + \frac{x}{s\rho}\right)^s dx,$$  \hspace{1cm} (5)

so that,

$$\frac{\partial B^{-1}}{\partial \rho}(s, s\rho) = \int_0^\infty e^{-x} \left(1 + \frac{x}{s\rho}\right)^s \left[ -\frac{x}{\rho(x + s\rho)} \right] dx < 0, \forall \rho,$$  \hspace{1cm} (6)

and

$$\frac{\partial^2 B^{-1}}{\partial \rho^2}(s, s\rho) = \int_0^\infty e^{-x} \left(1 + \frac{x}{s\rho}\right)^s \left[ \frac{(xs)^2 + xs(x + 2s\rho)}{[\rho(x + s\rho)]^2} \right] dx > 0, \forall \rho.$$  \hspace{1cm} (7)

(ii) The statement follows directly from (5) and (6) since for $x, \rho \in (0, +\infty)$,

$$\left(1 + \frac{x}{s\rho}\right)^s \text{ and } \frac{s}{x + s\rho} \text{ increase with } s.$$

(iii) The statement is a direct consequence of (5), (6) and the fact that for $n \in \mathbb{N}$,

$$\int_0^\infty x^n e^{-x} dx = n!.$$

(iv) Let $\rho > 1$. The fact that $1 < B^{-1}(s, s\rho) < \frac{\rho}{\rho - 1}$ follows from Lemma 1(iv). From (6), it follows that

$$0 > \frac{\partial B^{-1}}{\partial \rho}(s, s\rho) = \int_0^\infty e^{-x} \left(1 + \frac{x}{s\rho}\right)^s \left[ -\frac{x}{\rho(x + s\rho)} \right] dx$$

$$\geq -\frac{1}{\rho^2} \int_0^\infty x e^{-x(1-1/\rho)} dx = -\frac{1}{(\rho - 1)^2}.$$
(v) Noting that $B^{-1}(s, s\rho) - 1 > 0, \forall (s, \rho)$, by Lemma 1(iii), and writing

$$A(s, \rho, c) = (1 - \rho)[B^{-1}(s, s\rho) - 1] + (1 - \rho^{c+1}),$$

the first part of (v) can be easily verified. The second part of the same result follows from the fact that

$$A(s, \rho, c) + \rho^{c+1} = (1 - \rho)B^{-1}(s, s\rho) + \rho,$$

$B^{-1}(s, s\rho) \geq 1$, and $B^{-1}(s, s\rho) \leq \rho/\rho - 1$ for $\rho > 1$, by Lemma 1 (iv). □

**Lemma 3** For $\rho, c \in (0, +\infty)$, let $f(c, \rho)$ be the positive continuous functions given by

$$f(c, \rho) = \frac{1 - \rho}{1 - \rho^c}, \quad \rho \neq 1.$$  

(8)

We have:

(i) If $c > 1$ is fixed, then $f(c, \rho)$ is decreasing and convex in $\rho$.

(ii) If $c < 1$ is fixed, then $f(c, \rho)$ is increasing and concave in $\rho$.

(iii) $f(c, 1) = \frac{1}{c}$, \quad $\frac{\partial f}{\partial \rho}(c, 1) = -\frac{c-1}{2c}$ and $\frac{\partial^2 f}{\partial \rho^2}(c, 1) = \frac{c^2-1}{6c}$.

(iv) If $\rho$ is fixed, then $f(c, \rho)$ is decreasing and convex in $c$.

**Proof:** (iii) Since $f(1, \rho) \equiv 1$, the statement is true if $c = 1$. We consider thus $c \neq 1$. The fact that $f(c, 1) = 1/c$ follows from the continuity of $f$ using L'Hôpital's rule. If we let for $\rho \in (0, +\infty)$

$$h_c(\rho) = -1 + c\rho^{c-1} - (c - 1)\rho^c,$$

then

$$\frac{\partial f}{\partial \rho}(c, 1) = \frac{h_c(\rho)}{(1 - \rho^c)^2},$$  

and using L'Hôpital's rule twice we get

$$\frac{\partial f}{\partial \rho}(c, 1) = \lim_{\rho \to 1} \frac{h_c(\rho)}{(1 - \rho^c)^2} = -\frac{c - 1}{2c}.$$  

If we let

$$g_c(\rho) = (c - 1) - (c + 1)\rho + (c + 1)\rho^c - (c - 1)\rho^{c+1},$$  

then

$$\frac{\partial^2 f}{\partial \rho^2}(c, \rho) = c\rho^{c-2} \frac{g_c(\rho)}{(1 - \rho^c)^3},$$  

and using L'Hôpital's rule three times we get

$$\frac{\partial^2 f}{\partial \rho^2}(c, 1) = \lim_{\rho \to 1} \frac{c g_c(\rho)}{(1 - \rho^c)^3} = \frac{c^2 - 1}{6c}.$$  

(14)
(i) Let $c > 1$ be fixed. We have, from (9),
\[ h'_c(\rho) = c(c - 1)\rho^{c-2}(1 - \rho) \begin{cases} > 0 & , \rho < 1 \\ < 0 & , \rho > 1 \end{cases} ; \]
now, since from (9) $h_c(1) = 0$, it follows that
\[ h_c(\rho) < 0, \forall \rho \neq 1. \tag{15} \]
From (10), (11) and (15) we conclude (since $c > 1$) that
\[ \frac{\partial f}{\partial \rho}(c, \rho) < 0, \forall \rho. \]
From (12), we have
\[ g'_c(\rho) = (c + 1) \left[ - (1 - \rho^{c-1}) + (c - 1)\rho^{c-1}(1 - \rho) \right], \tag{16} \]
and
\[ g''_c(\rho) = c(c + 1)(c - 1)\rho^{c-2}(1 - \rho), \]
therefore
\[ g''_c(\rho) > (=, <) 0 \iff \rho < (=, >) 1. \]
This implies, since from (16) $g'_c(1) = 0$, that
\[ g'_c(\rho) = (<) 0 \iff \rho = (\neq) 1, \]
and, since from (12) $g_c(1) = 0$,
\[ g_c(\rho) > (=, <) 0 \iff \rho < (=, >) 1. \tag{17} \]
From (13), (14) and (17) we conclude (since $c > 1$) that
\[ \frac{\partial^2 f}{\partial \rho^2}(c, \rho) > 0, \forall \rho. \]
This concludes the proof that $f(c, \rho)$ is decreasing and convex in $\rho$, for fixed $c > 1$.

(ii) If $c < 1$, following the steps in (i) (correspondent to the case $c > 1$) we conclude that $f(c, \rho)$ is increasing and concave in $\rho$.

(iv) From (8) we have
\[ \frac{\partial f}{\partial c}(c, \rho) = \rho^c \frac{\ln \rho}{1 - \rho^c} f(c, \rho), \quad \rho \neq 1, \tag{18} \]
and from (iii) $\frac{\partial f}{\partial c}(c, 1) = -c^{-2}$, so that $\frac{\partial f}{\partial c}(c, \rho) < 0$, for all $\rho$. This shows that $f(c, \rho)$ is decreasing in $c$ for fixed $\rho$. 

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From (18) it follows that
\[
\frac{\partial^2 f}{\partial c^2}(c, \rho) = \rho^c (1 + \rho^c) \left[ \frac{\ln \rho}{1 - \rho^c} \right]^2 f(c, \rho), \quad \rho \neq 1,
\]
and \( \frac{\partial^2 f}{\partial c^2}(c, 1) = 2c^{-3} \), so that \( \frac{\partial^2 f}{\partial c^2}(c, \rho) > 0, \forall \rho \). This shows that \( f(c, \rho) \) is convex in \( c \) for fixed \( \rho \). □

3 Properties of the GED Function

The GED function as given by equation (3) is uniquely defined, nevertheless the values of \( D(s, a, c) \) are only implicitly given there when the number of servers and the offered load are equal (\( a = s \)) and when the queue capacity is infinite (\( c = +\infty \)); the next represents a more explicit definition of the GED function:

**Lemma 4** For \( (s, \rho, c) \in (0, +\infty)^2 \times [0, +\infty] \), and \( D(s, a, c) \) defined by equation (3),

\[
D(s, a, c) = \begin{cases} 
0, & c = 0, (a, s) \in (0, +\infty)^2 \\
\frac{1 - \left(\frac{a}{s}\right)^c}{(1 - \frac{a}{s})B^{-1}(s, a) + \left[ \frac{a}{s} - \left(\frac{a}{s}\right)^{c+1} \right]^{1/2}}, & (s, a, c) \in (0, +\infty)^3, a \neq s \\
\frac{c}{c + \frac{1}{B^{-1}(s, a)}}C(s, a) & , c \in (0, +\infty), a = s \\
\frac{s}{a}, & c = +\infty, a < s \\
& , c = +\infty, a \geq s 
\end{cases}, \tag{19}
\]

where
\[
C(s, a) = \left[ \left(1 - \frac{a}{s}\right)B^{-1}(s, a) + \frac{a}{s} \right]^{-1}, \text{ for } 0 < a < s < +\infty \tag{20}
\]
is a continuous extension of the Erlang delay probability, given by equation (1), to noninteger number of servers.

**Proof:** Assume along the proof that \( (s, a) \in (0, +\infty)^2 \). For \( c \in (0, +\infty) \), \( a \neq s \) the theorem restates the definition of \( D(s, a, c) \) as given by equation (3). Since we assumed \( D(s, a, c) \) to be continuous, and using equation (3),

\[
D(s, a, 0) = \lim_{c \to 0^+} D(s, a, c) = 0.
\]

The fact that, for \( c \in (0, +\infty) \), \( D(s, s, c) = \frac{c}{B^{-1}(s, s) + c} \) follows again from equation (3) using L'Hôpital's rule. Finally, if \( c = +\infty \), by the continuity of \( D(s, a, c) \), and using equation (3) and the fact that \( D(s, s, c) = \frac{c}{B^{-1}(s, s) + c} \),

\[
D(s, a, +\infty) = \lim_{c \to +\infty} D(s, a, c) = \begin{cases} 
C(s, a) & \text{if } a < s \\
\frac{s}{a} & \text{if } a \geq s 
\end{cases},
\]

by the use of L'Hôpital's rule in case \( a \geq s \). □
Remark 1 When there is no queue room customers do not wait, $D(s,a,0) \equiv 0$ and thus there is nothing to be studied if the queue capacity is zero. When the queue capacity is infinite the GED function $D(s,a,+\infty)$ is equal to the Erlang delay function $C(s,a)$ if the offered load is smaller than the number of servers (i.e. if steady-state can be reached or equivalently if the traffic intensity is less than 1) and it is equal to $1/\rho$ otherwise. $1/\rho$ represents the long run proportion of customers that are delayed and served in $M/M/s/s+c$ systems with traffic intensity $\rho \geq 1$. If $\rho \geq 1$ as the queue capacity grows larger and larger ($c \to +\infty$) the fraction of delayed customers $D(s,s\rho,c)$ goes to $1/\rho$. □

We look now at the monotonicity properties of the GED function. Lemma 5 is a direct consequence of Lemma 4 (via the relation between the GED function with the Erlang loss function it establishes) and Lemma 1, which describes the monotonicity properties of the Erlang loss function.

Lemma 5 For $(s,a,\rho,c) \in (0, +\infty) \times [0, +\infty], D(s,a,c)$ and $D(s,s\rho,c)$ have the following properties:

(i) $D(s,a,c) \xrightarrow{a \to 0^+} 0; \quad D(s,a,c) \xrightarrow{a \to +\infty} 0.$

(ii) $D(s,a,c) \xrightarrow{s \to 0^+} 0; \quad D(s,a,c) \xrightarrow{s \to +\infty} 0.$

(iii) $D(s,s\rho,c) \xrightarrow{\rho \to 0^+} 0; \quad D(s,s\rho,c) \xrightarrow{\rho \to +\infty} 0.$

(iv) $D(s,s\rho,c)$ decreases with $s$; in addition, if $c < +\infty$,

$$D(s,s\rho,c) \xrightarrow{s \to 0^+} \begin{cases} \frac{1-\rho^c}{1-\rho^{c+1}} & \text{if } \rho \neq 1, \\ \frac{c}{c+1} & \text{if } \rho = 1, \end{cases} \quad D(s,s\rho,c) \xrightarrow{s \to +\infty} \begin{cases} 0 & \text{if } \rho < 1, \\ \frac{1}{\rho} - \frac{1}{\rho^{c+1}} & \text{if } \rho \geq 1, \end{cases}$$

and $D(s,s\rho, +\infty) \xrightarrow{s \to 0^+} 1$, $D(s,s\rho, +\infty) \xrightarrow{s \to +\infty} \begin{cases} 0 & \text{if } \rho < 1, \\ \frac{1}{\rho} & \text{if } \rho \geq 1. \end{cases}$

In Lemma 5 we did not characterize the behaviour of the GED function with respect to the queue capacity; this characterization is done in Theorem 1.

Theorem 1 For $s,a,\rho \in (0, +\infty), D(s,a,c)$ and $D(s,s\rho,c)$ are increasing and concave in $c$ in the interval $(0, +\infty)$.

Proof: It suffices to prove the result for $D(s,s\rho,c)$. Suppose that $c \in (0, +\infty)$. We consider first the case $\rho = 1$. From equation (19), $D(s,s,c) = c/[B^{-1}(s,s) + c]$, so that

$$\frac{\partial D}{\partial c}(s,s,c) = \frac{B^{-1}(s,s)}{[B^{-1}(s,s) + c]^2} > 0$$

and

$$\frac{\partial^2 D}{\partial c^2}(s,s,c) = -\frac{2B^{-1}(s,s)}{[B^{-1}(s,s) + c]^3},$$

so that the result is obviously true if $\rho = 1$. 


If \( \rho \neq 1 \), from equation (19) \( D(s, s\rho, c) = [1 - \rho^c]/A(s, \rho, c) \), where as defined in Lemma 2

\[
A(s, \rho, c) = (1 - \rho)B^{-1}(s, s\rho) + (\rho - \rho^{c+1}), \quad s, \rho, c \in (0, +\infty),
\]

so that, after a few steps, we get

\[
\frac{\partial D}{\partial c}(s, s\rho, c) = -\frac{\rho^c(\ln \rho)(1 - \rho)B^{-1}(s, s\rho)}{A^2(s, \rho, c)} > 0,
\]

by Lemma 3(v). It is also easy to derive that

\[
\frac{\partial^2 D}{\partial c^2}(s, s\rho, c) = -\frac{\rho^c(\ln \rho)^2(1 - \rho)B^{-1}(s, s\rho)[A(s, \rho, c) + 2\rho^{c+1}]}{A^3(s, \rho, c)} < 0,
\]

by Lemma 2(v). \( \Box \)

The reciprocal of the GED function is decreasing and convex in the queue capacity since, from Theorem 1, the GED function is positive, increasing and concave in the queue capacity. We will now use Lemmas (2) and (3) to show that the reciprocal of the GED function is convex in the traffic intensity and to prove in a simple and direct way, without using Theorem 1, that the same function is decreasing and convex in the queue capacity.

**Theorem 2** For \( s, a, \rho \in (0, +\infty) \) and \( c \in [0, +\infty] \), we have:

(i) If \( c \in [1, +\infty] \), \( D^{-1}(s, s\rho, c) \) is convex (non-strictly if \( c = +\infty \)) in \( \rho \).

(ii) \( D^{-1}(s, s\rho, c) \) and \( D^{-1}(s, a, c) \) are decreasing and convex in \( c \), for \( c \in (0, +\infty) \).

**Proof:** If we let \( f(c, \rho) \) be as in Lemma 3, we have from equation (3)

\[
D^{-1}(s, s\rho, c) = f(c, \rho)B^{-1}(s, s\rho) + \rho. 
\]

(21)

(i) Assume \( c \in [1, +\infty] \). (21) implies that

\[
\frac{\partial^2 D^{-1}}{\partial \rho^2}(s, s\rho, c) = \frac{\partial^2 f}{\partial \rho^2}(c, \rho)B^{-1}(s, s\rho) + 2\frac{\partial f}{\partial \rho}(c, \rho)\frac{\partial B^{-1}}{\partial \rho}(s, s\rho)
\]

\[
\quad + f(c, \rho)\frac{\partial^2 B^{-1}}{\partial \rho^2}(s, s\rho) > 0, \ \forall \rho,
\]

where the last inequality follows since, from Lemma 3(i) and (iii),

\[
f(c, \rho) > 0, \quad \frac{\partial f}{\partial \rho}(c, \rho) \leq 0, \quad \frac{\partial^2 f}{\partial \rho^2}(c, \rho) \geq 0,
\]

and from Lemma 2(i),

\[
B^{-1}(s, s\rho) > 0, \quad \frac{\partial B^{-1}}{\partial \rho}(s, s\rho) < 0, \quad \frac{\partial^2 B^{-1}}{\partial \rho^2}(s, s\rho) > 0,
\]

(22)
for all $\rho \in (0, +\infty)$. The statement is thus proved if $c < +\infty$. Consider now the case $c = +\infty$. From equation (19) it follows that

$$D^{-1}(s, s\rho, +\infty) = \begin{cases} C^{-1}(s, s\rho), & \rho < 1 \\ \rho, & \rho \geq 1 \end{cases}. \quad (24)$$

Now, from (24) and (20) and using Lemma 2(i), it follows that for $\rho < 1$

$$\frac{\partial D^{-1}}{\partial \rho}(s, s\rho, +\infty) = -\left[B^{-1}(s, s\rho) - 1\right] + (1 - \rho)\frac{\partial B^{-1}}{\partial \rho}(s, s\rho) < 0, \quad (25)$$

and

$$\frac{\partial^2 D^{-1}}{\partial \rho^2}(s, s\rho, +\infty) = (1 - \rho)\frac{\partial^2 B^{-1}}{\partial \rho^2}(s, s\rho) - 2\frac{\partial B^{-1}}{\partial \rho}(s, s\rho) > 0. \quad (26)$$

$D^{-1}(s, s\rho, +\infty)$ is thus decreasing and convex in $\rho$ for $\rho < 1$ and, from equation (24), is linear and increasing for $\rho \geq 1$. This implies, by the continuity of $D^{-1}(s, s\rho, +\infty)$ in $\rho$, that $D^{-1}(s, s\rho, +\infty)$ is non-strictly convex in $\rho$.

(ii) The statement is an immediate consequence of equation (21) and Lemma 3(iv). \(\square\)

**Theorem 3** For $s, \rho \in (0, +\infty)$ and $c \in [1, +\infty]$, we have:

(i) For fixed $s$ and $c$, $\exists \rho_{s,c}^* \in (0, +\infty)$, s.t.

$$\frac{\partial D^{-1}}{\partial \rho}(s, s\rho, c) < (>) 0 \iff \rho < (>) \rho_{s,c}^*. \quad (27)$$

(ii) If $c \in [1, +\infty)$ is fixed, then $\rho_{s,c}^*$ is increasing in $s$.

(iii) For fixed $s$ and $c \in [1, +\infty)$,

$$\rho_{s,c}^* < (=, >) 1 \iff s < (=, >) 1,$$

and $\rho_{s,+\infty}^* \equiv 1 \equiv \rho_{1,c}^*$.

(iv) $\rho_{s,c}^* < 2, \forall s, c.$

**Proof:** (i) We consider first $c = +\infty$. From equation (24), $\frac{\partial D^{-1}}{\partial \rho}(s, s\rho, +\infty) = 1$ if $\rho > 1$, so that, using (25),

$$\frac{\partial D^{-1}}{\partial \rho}(s, s\rho, +\infty) = \begin{cases} < 0, & \rho < 1 \\ > 0, & \rho > 1 \end{cases}. \quad (28)$$

We consider now $c \in [1, +\infty)$. The fact that, from Lemma 5(iii),

$$D^{-1}(s, s\rho, c) \xrightarrow{\rho \rightarrow 0^+} +\infty; \quad D^{-1}(s, s\rho, c) \xrightarrow{\rho \rightarrow +\infty} +\infty,$$
along with Theorem 2(i) proves the result in (27) for $c \in [1, +\infty)$. This and (28) show that this statement is true.

(ii) Let $c \in [1, +\infty)$, then from (21) it follows that

\[ \frac{\partial D^{-1}}{\partial \rho}(s, s\rho, c) = 1 + \frac{\partial f}{\partial \rho}(c, \rho) B^{-1}(s, s\rho) + f(c, \rho) \frac{\partial B^{-1}}{\partial \rho}(s, s\rho). \]  
\[ (29) \]

From (29), (22) and Lemma 2(ii), it follows that $\frac{\partial D^{-1}}{\partial \rho}(s, s\rho, c)$ is decreasing in $s$, for fixed $\rho$. This implies, by (27), that $\rho^*_{s,c}$ is increasing in $s$, for fixed $c \in [1, +\infty)$.

(iii) The fact that $\rho^*_{s,\infty} = 1$ follows from (28). Let $c \in [1, +\infty)$, then from (29), and using Lemma 3(iii) and Lemma 2(iv), it follows that

\[ \left[ \frac{\partial D^{-1}}{\partial \rho}(1, \rho, c) \right]_{\rho=1} = 1 - \frac{c - 1}{2c} - \frac{1}{c} = 0, \]

so that $\rho^*_{1,c} \equiv 1$. This, by (ii), implies that

$\rho^*_{s,c} < (=, >) 1 \iff s < (=, >) 1.$

(iv) From (iii) it suffices to consider $c \in [1, +\infty)$. Suppose thus that $c \in [1, +\infty)$. To prove the statement we need, by Theorem 2(i) and (27), to check that

\[ \left[ \frac{\partial D^{-1}}{\partial \rho}(s, s\rho, c) \right]_{\rho=2} > 0. \]  
\[ (30) \]

From equation (29), Lemma 3(i) and (iii), and Lemma 2(iv) we have:

\[ \left[ \frac{\partial D^{-1}}{\partial \rho}(s, s\rho, c) \right]_{\rho=2} > 1 + 2 \frac{\partial f}{\partial \rho}(c, 2) - f(c, 2) = \frac{2^c}{(2^c - 1)^2} [2^c - (c + 1)] \geq 0, \]

thus proving (30). \qed

**Corollary 1** For $s, \rho \in (0, +\infty)$ and $c \in [1, +\infty],$

\[ \exists \rho^*_{s,c} \in (0, +\infty) \text{ s.t. } \frac{\partial}{\partial \rho} D(s, s\rho, c) > (<) 0 \iff \rho < (>) \rho^*_{s,c}. \]  
\[ (31) \]

Moreover $\rho^*_{s,c}$ is the same value as in equation (27) and has therefore all the properties stated in Theorem 3.

**Proof:** From (27) and the fact that

\[ \frac{\partial D}{\partial \rho}(s, s\rho, c) = -D^{-2}(s, s\rho, c) \frac{\partial D^{-1}}{\partial \rho}(s, s\rho, c), \]

the result follows. \qed

The implications of Corollary 1 for the properties of the steady state delay probability in $M/M/s/s + c$ systems are given in Corollary 2 (which follows directly from Corollary 1).
Corollary 2 The steady state delay probability in M/M/s/s + c systems with c ≥ 1 is increasing in the traffic intensity ρ if ρ is below some ρ^*_s,c and decreasing if ρ is greater than ρ^*_s,c. Moreover,

(i) 1 ≤ ρ^*_s,c < 2.
(ii) ρ^*_1,c = ρ^*_s,+∞ = 1.
(iii) ρ^*_s,c is increasing in s.

4 Numerical Illustrations

In this section we consider ρ^*_s,c as defined in Theorem 1. Tables 1 and 2 give the values ρ^*_s,c for a broad range of number of servers s and queue capacity c, spaced appropriately with a view to giving a good idea as to how ρ^*_s,c changes as we change the number of servers (table 1) and the queue capacity (table 2).
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Table 1: Zeros of GED’s derivative with respect to $\rho$.  

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Table 2: Zeros of GED’s derivative with respect to ρ.
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