SYMMETRY AND POSITIVE DEFINITENESS IN ORIENTED MATROIDS

by

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Abstract

We show the properties of a class of oriented matroids that properly generalizes the class of oriented matroids that can be represented by matrices of the form \((I, -A)\), where \(A\) is a real \(n \times n\) symmetric matrix and \(I\) is the \(n \times n\) identity matrix. Several results from linear algebra about positive definiteness, symmetry, and eigenvalues are shown to have natural generalizations in the context of oriented matroids. The relationships among the oriented matroid generalizations of the linear algebraic concepts are seen to be analogous to the relationships among the original linear algebraic concepts.
I. Introduction

Let \( E \) be a finite set. A signed set in \( E \) is a pair \( X = (X^+, X^-) \) with \( X^+ \subseteq E \), \( X^- \subseteq E \), and \( X^+ \cap X^- = \emptyset \). The opposite of \( X \) is the signed set \(-X = (X^-, X^+)\), and the set underlying \( X \) is \( X = X^+ \cup X^- \). A signed set \( X \) contains a signed set \( Y \), \( Y \subseteq X \), if \( Y^+ \subseteq X^+ \), \( Y^- \subseteq X^- \), and \( X \) contains an unsigned set \( Z \) if \( Z \subseteq X \).

An oriented matroid \( M \) is a pair \((E, C)\), where \( E \) is a finite set and \( C \) is a collection of signed sets in \( E \), called circuits, satisfying

\[
\begin{align*}
\text{C}_1 & : (\emptyset, \emptyset) \notin C \\
\text{C}_2 & : C \in C \Rightarrow -C \in C \\
\text{C}_3 & : C_1, C_2 \in C \text{ and } C_1 \subseteq C_2 \Rightarrow C_1 = C_2 \text{ or } C_1 = -C_2. \\
\text{C}_4 & : C_1, C_2 \in C \text{ and } e \in (C_1^+ \cap C_2^-) \cup (C_1^- \cap C_2^+) \text{ imply that there exists } C_3 \in C \text{ with } C_3^+ \subseteq (C_1^+ \cup C_2^+) \setminus \{e\}, C_3^- \subseteq (C_1^- \cup C_2^-) \setminus \{e\}.
\end{align*}
\]

We write \( C = C(M) \).

The collection \( C = \{C : C \in C \} \) of underlying sets of circuits is the family of circuits of an (unoriented) matroid \( M \) on \( E \), called the underlying matroid of \( M \). We will assume that the reader is familiar with the basic definitions and properties of matroids (see [12]). The independent and dependent sets, bases, loops and coloops, and rank of \( M \) are those of the underlying matroid \( M \).

A signed set \( X \) in \( E \) has a conformal decomposition into circuits of an oriented matroid if \( X^+ = C_1^+ \cup \ldots \cup C_m^+ \) and \( X^- = C_1^- \cup \ldots \cup C_m^- \) for circuits \( C_1, \ldots, C_m \) in \( C(M) \). We call signed sets having conformal decomposition into circuits of \( M \) cycles of \( M \). We write \( K(M) \) for the set of cycles of \( M \).
The main motivation for oriented matroid theory comes from the case when the cycles of an oriented matroid are signed supports of vectors in a subspace of $\mathbb{R}^n$. Let $V$ be a subspace of $\mathbb{R}^n$, and let $E = \{e_1, \ldots, e_n\}$. For every $x \in V$, define a signed set $K_x = \{\{e_i : x_i > 0\}, \{e_i : x_i < 0\}\}$, called the signed support of $x$. Then the set $\{K_x : x \in V\}$ is the set of cycles of an oriented matroid. If $V$ is the null space of an $m \times n$ matrix $A$, we call this oriented matroid the oriented matroid represented by $A$, written $M(A)$. Thus $M(A)$ captures sign properties of vectors in the null space of $A$, but ignores their numerical values.

Oriented matroids give an abstract combinatorial setting for studying the properties of matrices that represent them. On the other hand, since there are oriented matroids that cannot be represented by any matrix, the results obtained properly generalize those of linear algebra. Previously, notions of arrangements of hyperplanes and topological extensions, face lattices and polarity of polyhedra, convexity, and signs of determinants of matrices have been studied in the context of oriented matroids [1], [3], [6], [7], [8]. Linear and quadratic programming have been shown to have natural generalizations in the context of oriented matroids, which capture many of the important features of these problems [2], [4], [5], [11]. In the study of quadratic programming for oriented matroids, one would like to have generalizations of the properties of quadratic forms associated with symmetric matrices. This is the subject of the present paper.
We would like to have oriented matroid generalizations of the notions of symmetry, positive (semi)definiteness, and eigenvalues. These generalizations should ideally satisfy many properties that the original linear algebraic notions satisfy. In particular:

1. Oriented matroids represented by matrices of the form \((I, -A)\), where \(A\) is square and symmetric positive (semi)definite are symmetric and positive (semi)definite. Note that \(y = Ax\) iff \((y, x)\) is in the null space of \((I, -A)\). Therefore, to study the sign properties of \(y\) and \(x\), we consider the oriented matroid \(M(I, -A)\).

2. Certain minors of symmetric (positive (semi)definite) oriented matroids, corresponding to principal submatrices in the representable case, are symmetric (positive (semi)definite) oriented matroids.

3. The inverse of a symmetric (positive definite) oriented matroid is symmetric (positive definite) iff it exists.

4. A symmetric positive semidefinite oriented matroid is positive definite iff it is nonsingular.

5. A symmetric positive definite oriented matroid has a positive eigencycle, and a symmetric positive semidefinite oriented matroid has a positive eigencycle iff it is not representable by the matrix \((I, 0)\).

6. A symmetric oriented matroid is positive semidefinite iff it has no negative eigencycles.

These properties will all be shown to hold for natural definitions of the concepts for oriented matroids.
II. Preliminaries

Let $M$ be an oriented matroid on a set $E$. If $M$ is represented by a matrix $A$, then $M^*$, the oriented matroid that has as its cycles the signed supports of vectors in the row space of $A$, is called the dual of $M$. The null space of $A$ and the row space of $A$ are orthogonal complements, so $M$ and $M^*$ must satisfy the following orthogonality property:

$(\perp)$ For any cycles $K$ of $M$ and $L$ of $M^*$

$$(K^+ \cap L^+) \cup (K^- \cap L^-) \neq \emptyset \text{ iff } (K^+ \cap L^-) \cup (K^- \cap L^+) \neq \emptyset.$$ 

In general, for every oriented matroid $M$ on $E$, there is a unique oriented matroid $M^*$ on $E$ such that the cycles of $M^*$ are all the signed sets on $E$ that satisfy $(\perp)$ with respect to all the cycles of $M$. $M^*$ is the dual of $M$ (see [3]).

Let $M$ be an oriented matroid on $E$ with $K = K(M)$ its set of cycles. Let $F$ and $G$ be disjoint subsets of $E$. Let $K(F) = \{(K^+ \setminus F) \cup (K^- \setminus F), (K^- \setminus F) \cup (K^+ \setminus F) : K \in K\}$. Then $K(F)$ is the family of cycles of an oriented matroid; we say that oriented matroid is obtained by reversing signs on $F$. For a signed set $X$ on $E$, and any subset $H$ of $E$, let the signed set $X \setminus H$ be $(X^+ \setminus H, X^- \setminus H)$. Let $K \setminus F / G$ denote the collection of signed sets $\{K \setminus G : K \in K, K \setminus F = \emptyset\}$. Then $K \setminus F / G$ is the set of cycles of an oriented matroid $M \setminus F / G$ on $E \setminus (F \cup G)$ [3, p. 114]. For the sake of brevity, $M \setminus F (M / G)$ is written for $M \setminus F / \emptyset (M \setminus \emptyset / G)$. We say $M \setminus F / G$ is obtained from $M$ by deleting $F$ and contracting $G$. If $M$ is represented by a matrix $A$, then $M \setminus F / G$ is represented by the matrix $A'$, obtained from $A$ by deleting the
columns corresponding to $F$, then pivoting on successive nonzero columns corresponding to $G$, followed by deleting the pivot row and column, and finally deleting zero columns corresponding to $G$.

If $B$ is a base of $M$ and $e \in E \setminus B$, then there is exactly one circuit $C$ of $M$ with $e \in C^+$ and $C \subseteq B \cup \{e\}$. It is called the fundamental circuit $C(B,e)$ associated with $B$ and $e$. For every base $B$ of $M$, define the $B$-tableau to be the set of fundamental circuits of $M$ associated with $e_i$ and $B$ for $e_i \notin B$.

If $K_1, K_2$ are two cycles of an oriented matroid $M$, then $K_1 \circ K_2 = (K_1^+ \cup (K_2^+ \setminus K_1)), K_1^- \cup (K_2^- \setminus K_1^+) \in K(M)$. This operator $\circ$, Bland's composition operator, corresponds in the representable case to taking the signed support of $x^1 + \varepsilon x^2$ for some suitably small $\varepsilon > 0$, where $K_1$ and $K_2$ are the signed supports of $x^1$ and $x^2$, respectively.

Henceforth, we make the following blanket assumption:

$M$ is an oriented matroid on a set $E = S \cup T$, where $S = \{s_1, \ldots, s_n\}$, $T = \{t_1, \ldots, t_m\}$, $S \cap T = \emptyset$, and $S$ is a base of $M$. Such an oriented matroid is called square.

The particular ordering of $S$ and $T$ is significant. We define the switch of a signed set $X$ on $E$ to be the signed set $swX = \{(s_i : t_i \in X^-) \cup (t_i : s_i \in X^+) \cup (s_i : t_i \in X^+) \cup (t_i : s_i \in X^-)\}$, obtained by reversing signs on $T$ and then interchanging occurrences of $s_i$ and $t_i$ in $X$ for all $i = 1, \ldots, n$. The switch of an oriented matroid, $swM$, is the matroid that has as its cycles the switches of the cycles of $M$. The $(S,T)$-transpose of $M$, denoted $M_{ST}^t$, is the
switch of $M^*$. This is motivated by observing that if $M$ is represented by the matrix $(I,-A)$, where $S$ corresponds to the columns of $I$ and $T$ to the columns of $-A$, then $M^*$ is represented by $(A^t, I)$ and $M_{ST}^*$ by $(I, -A^t)$.

Definition. A square oriented matroid $M$ is symmetric (with respect to $S$ and $T$) iff $M = M_{ST}^*$.

It is possible to replace this indirect definition with another one that does not explicitly involve the dual oriented matroid.

Theorem II.1. A square oriented matroid $M$ is symmetric (with respect to $S$ and $T$) iff the following property holds:

$(\dagger)'$: For any two cycles $K_1$ and $K_2$ of $K(M)$, (I) holds iff (II) holds.

(I) There exists an $i$ such that one of the cases below holds

- $s_i \in K_1^+$, $t_i \in K_2^-$
- $s_i \in K_1^-$, $t_i \in K_2^+$
- $t_i \in K_1^+$, $s_i \in K_2^+$
- $t_i \in K_1^-$, $s_i \in K_2^-$

(II) There exists a $j$ such that one of the cases below holds

- $s_j \in K_1^+$, $t_j \in K_2^+$
- $s_j \in K_1^-$, $t_j \in K_2^-$
- $t_j \in K_1^+$, $s_j \in K_2^-$
- $t_j \in K_1^-$, $s_j \in K_2^+$

Proof. The condition $(\dagger)'$ is equivalent to the requirement that $M$ and $swM$ satisfy $(\dagger)$. If $M$ and $swM$ satisfy $(\dagger)$, then the cycles of $swM$ are cycles of $M^*$, and independent sets of $M^*$ are independent in $swM$. In particular, bases of $M^*$ are bases of $swM$, since both
oriented matroids are of rank $n$. $M$ and $swM$ have the same number of bases, as do $M$ and $M^*$. Therefore, the bases of $M^*$ are exactly the bases of $swM$, and $M^*$ and $swM$ have the same underlying matroid. From [3], we then get that $swM = M^*$, implying that $M = swM^* = M^t_{ST}$.

Clearly, if $M = M^t_{ST}$, then $swM = M^*$, and thus $M$ and $swM$ satisfy (1).

The following theorem is an easy application of (1)''. We call a subset $F$ of $E = S \cup T$, where $S = \{s_1, \ldots, s_n\}$, $T = \{t_1, \ldots, t_n\}$, $S \cap T = \emptyset$, complementary iff $|F \cap \{s_i, t_i\}| \leq 1$ for $i = 1, \ldots, n$.

**Theorem II.2.** Let $B$ be a complementary base of $M$, a symmetric oriented matroid. Then the following hold:

a) If $s_i, s_j \in B$, then $s_j \in C(B, t_i)^+$ iff $s_i \in C(B, t_j)^+$, and $s_j \in C(B, t_i)^-$ iff $s_i \in C(B, t_j)^-$.

b) If $s_i, t_j \in B$, then $s_i \in C(B, s_j)^+$ iff $t_j \in C(B, t_i)^-$, and $s_i \in C(B, s_j)^-$ iff $t_j \in C(B, t_i)^+$.

c) If $t_i, t_j \in B$, then $t_i \in C(B, s_j)^+$ iff $t_j \in C(B, s_i)^+$, and $t_i \in C(B, s_j)^-$ iff $t_j \in C(B, s_i)^-$.

**Corollary II.2.1.** The $S$-tableau of a symmetric oriented matroid is symmetric:

$$s_j \in C(S, t_i)^+ \text{ iff } s_i \in C(S, t_j)^+ \text{ and } s_j \in C(S, t_i)^- \text{ iff } s_i \in C(S, t_j)^-.$$ 

This is a special case of Theorem II.2, with $B = S$. When $M$ is represented by $(I, -A)$, with $A$ symmetric, the circuits of the $S$-
tableau are the signed sets corresponding to the rows of the matrix 
\((A^t, I)\). This corollary, then, generalizes the property that \(a_{ij}\) and 
\(a_{ji}\) agree in sign, for all \(i, j\).

If \(T\) is a base of a square oriented matroid \(M\), then we will 
call \(M\) nonsingular. The observation that the matrices \((I, -A)\) and 
\((-A^{-1}, I)\) have the same null spaces, for nonsingular \(A\), prompts the 
following definition. Let \(M\) be a nonsingular oriented matroid. 
The \((S, T)\) inverse of \(M\), \(M_{ST}^{-1}\), is the oriented matroid obtained from 
\(M\) by interchanging the labels \(s_i\) and \(t_i\), for all \(i\).

Proposition II.3. Let \(M\) be a square nonsingular oriented matroid. 
Then \(M_{ST}^{-1}\) is symmetric iff \(M\) is.

Proof. It is easily seen that \(M_{ST}^{-1}\) satisfies \(\bot\)' iff \(M\) does, and 
since \(T\) is a base of \(M\), we have that \(S\) is a base of \(M_{ST}^{-1}\). \qed

Principal submatrices of symmetric oriented matrices are also 
symmetric. It is therefore natural to expect that the corresponding 
minors of symmetric oriented matroids would stay symmetric.

Theorem II.4. Let \(M\) be a symmetric oriented matroid. For any \(i\), 
\(1 \leq i \leq n\), \(M \setminus t_i/s_i\) is a symmetric oriented matroid with respect to 
\(S \setminus s_i\) and \(T \setminus t_i\), and if \(s_i \in \mathcal{C}(S, t_i)\), then so is \(M \setminus s_i/t_i\).

Proof. First, note that \(s_i\) is not a loop and \(t_i\) is not a coloop, so 
\(\text{rank}(M \setminus t_i/s_i) = \text{rank}(M) - 1\). The existence of a circuit of \(M \setminus t_i/s_i\)
contained in \( S \setminus s_i \) would imply the existence of a circuit of \( M \) in \( S \), which is impossible. Therefore, \( S \setminus s_i \) is a base of \( M \setminus t_i / s_i \). Let 
\( K_j , j = 1,2 \) be any two cycles of \( M \setminus t_i / s_i \). There exist two cycles \( K'_j \) of \( M \) such that \( K'_j \setminus s_i = K_j , t_i \not\in K'_j \). If, say, alternative I of (1)' holds for \( K_1, K_2 \) of \( M \setminus t_i / s_i \), then it must also hold for \( K'_1 \) and \( K'_2 \). Thus alternative II holds for \( K'_1 \) and \( K'_2 \) and consequently for \( K_1, K_2 \), since \( t_i \not\in K'_1, K'_2 \). Thus \( M \setminus t_i / s_i \) is symmetric with respect to \( S \setminus s_i \) and \( T \setminus t_i \). If \( s_i \in C(S,t_i) \), then \( s_i \) cannot be a coloop and \( t_i \) cannot be a loop of \( M \), so \( \text{rank}(M \setminus s_i / t_i) = \text{rank}(M) - 1 \). The only circuit of \( M \) contained in \( S \cup t_i \) is \( C(S,t_i) \), which contains \( s_i \). Thus \( S \setminus s_i \) is a base of \( M \setminus s_i / t_i \). The same argument as above proves that \( M \setminus s_i / t_i \) is symmetric with respect to \( S \setminus s_i \) and \( T \setminus t_i \). \( \square \)

All oriented matroids that can be represented by matrices of the form \((I,-A)\), where \( A \) is symmetric, are symmetric. The class of symmetric oriented matroids is much larger than this, though. As an example, we give a symmetric orientation of the nonrepresentable Vamos Matroid. Consider the oriented matroid represented by the matrix

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & -1 & 1 & -1 \\
0 & 1 & 0 & 0 & -1 & 0 & 1 & -1 \\
0 & 0 & 1 & 0 & 1 & 1 & -4 & 5 \\
0 & 0 & 0 & 1 & -1 & -1 & 5 & -6
\end{pmatrix}.
\]

The circuits of this oriented matroid are those given below and their negatives.
Here a circuit \([\{1,6\}, \{2,5\}]\) is written \(1256\). A symmetric orientation of the Vamos matroid is then obtained by replacing the circuit \(25\overline{78}\) with the circuits \(1\overline{2}5\overline{78}, 23\overline{5}7\overline{8}, 256\overline{78}, \) and \(245\overline{78}\). The result is an orientation that is symmetric with respect to \(S = (1,2,3,4), T = (5,6,7,8)\). By relabeling the elements \((1,2,3,4,5,6,7,8)\) as \((1,3,6,4,5,2,7,8)\), one can see that the underlying matroid is the same as that given in [3, p. 111]. For a more detailed explanation, see [9].
III. Positive Definiteness and Eigenvalues

III.1 Positive definiteness

All of the oriented matroids in this section will be symmetric unless otherwise noted. The definition of positive definiteness in oriented matroids given below, however, applies to any square oriented matroid. An oriented matroid $M$, represented by $(I,-A)$ where $A$ is symmetric, will be positive (semi)definite iff $A$ is. The correspondence is not as direct when $A$ is not symmetric. For example, the same symmetric oriented matroid is represented by both 
\[
\begin{pmatrix}
1 & 0 & -5 & -1 \\
0 & 1 & -4 & -1
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
1 & 0 & -2 & -1 \\
0 & 1 & -1 & -2
\end{pmatrix},
\]
even though \((2,1)\) is positive definite and \((5,1)\) is not.

If a vector space $V$ is the null space of an $n \times n$ matrix $(I,-A)$, then $V$ is the space of solutions $(y,x)$ to the equation $(I,-A)(y,x) = 0$, which is $y = Ax$. The matrix $A$ is positive definite iff $x^T y \geq 0$ for all nonzero $(y,x) \in V$, and positive semidefinite iff $x^T y \geq 0$ for all $(y,x) \in V$.

Definitions. A cycle $K$ of a symmetric oriented matroid $M$ is called \textbf{sign reversing} if $\{s_i, t_i\} \notin K^+$ and $\{s_i, t_i\} \notin K^-$ for every $i$. $K$ is called \textbf{strictly sign reversing} if, in addition, $s_i \in K^+$, $t_i \in K^-$ or $s_i \in K^-$, $t_i \in K^+$ for some $i$. $M$ is called \textbf{positive definite} if it has no sign reversing cycles. $M$ is called \textbf{positive semidefinite} if it has no strictly sign reversing cycles.

Note that if $M$ is represented by $(I,-A)$, then a sign reversing cycle implies the existence of a vector $(y,x)$ in the null space of $(I,-A)$.
such that $y_i x_i \leq 0$ for all $i$. A strictly sign reversing cycle implies that,
in addition, $y_i x_i < 0$ for some $i$. Thus, if $A$ is positive (semi)definite, then
$M$ is positive (semi)definite. The converse holds when $A$ is symmetric.
If a symmetric $A$ is not positive (semi)definite, then it has an
eigenvector $x$ corresponding to a nonpositive (negative) eigenvalue.
The signed support of the vector $(Ax, x)$ will then be a (strictly)
sign reversing cycle of $M$.

**Lemma III.1.1.** If $M$ is a symmetric positive semidefinite oriented
matroid, then $s_i \not\in C(S, t_i)$ implies that $s_j \not\in C(S, t_i)$ for all $j$, so that
t_i is a loop.

**Proof.** Suppose $s_j \in C(S, t_i)^+$ and $s_i \not\in C(S, t_i)$ for some $i, j$. Then
$s_i \in C(S, t_j)^+$ by the symmetry of the $S$-tableau. Hence $K = C(S, t_i) \circ
-C(S, t_j)$ has $\{s_i, t_j\} \subseteq K^-, \{s_j, t_i\} \subseteq K^+$, and $\{s_k, t_k\} \not\in K$ for all other
$k$. Then $K$ is strictly sign reversing, contradicting the assumption
that $M$ is positive semidefinite. A strictly sign reversing cycle
can be constructed in an analogous way if $s_j \in C(S, t_i)^-$ for some $j$. □

**Corollary III.1.2.** Let $M$ be a symmetric, positive semidefinite, non-
singular oriented matroid. Then $s_i \not\in C(S, t_i)$ and $t_i \in C(T, s_i)$ for all $i$.

**Proof.** If $s_i \not\in C(S, t_i)$ for some $i$, then $t_i$ is a loop, contradicting the
assumption that $T$ is a base. If $M$ is nonsingular, it is immediate
that $M_{ST}^{-1}$ is a symmetric positive semidefinite oriented matroid. Thus,
if $t_i \not\in C(T, s_i)$ in $M$, then $s_i \not\in C(S, t_i)$ in $M_{ST}^{-1}$, which is impossible. □
Lemma III.1.3. If $M$ is a positive (semi)definite symmetric oriented matroid, then $M \setminus t_i / s_i$ is a positive (semi)definite symmetric oriented matroid for all $i$, and if $s_i \in C(S, t_i)$, then $M \setminus s_i / t_i$ is a positive (semi)definite symmetric oriented matroid.

Proof. By Theorem II.2.1, $M \setminus t_i / s_i$ is a symmetric oriented matroid for any $i$, and if $s_i \in C(S, t_i)$, then $M \setminus s_i / t_i$ is a symmetric oriented matroid. Suppose that $M \setminus t / s_i$ has a (strictly) sign reversing cycle $K$; then this cycle, together with a possibile additional element $s_i$, is a (strictly) sign reversing cycle of $M$. The same argument applies to (strictly) sign reversing cycles of $M \setminus s_i / t_i$.

Theorem III.1.4. A positive semidefinite oriented matroid is positive definite iff it is nonsingular.

Proof. One implication is trivial. If $M$ has a circuit $C$ contained in $T$, then $\{s_i, t_i\} \notin C$ for all $i$, so $C$ is a sign reversing cycle. The opposite implication is proved by induction on $|S|$. Suppose that Theorem III.1.4 is true for symmetric oriented matroids with $|S| \leq n-1$. Suppose also that $M$ is a nonsingular symmetric positive semidefinite oriented matroid with $|S| = n$. If $M$ is not positive definite, then there exists a nonempty cycle $K$ of $M$ such that $\{s_i, t_i\} \notin K$ for all $i$. $T$ is a base of $M$, so there is an $i$ such that $t_i \notin K$. The oriented matroid $M \setminus t_i / s_i$ is symmetric and positive
semidefinite, by Lemma III.1.3. \( T \) is a base of \( M \), and by Corollary III.1.2, we have \( t_i \in C(T, s_i) \). Therefore, there can be no circuit of \( M \backslash t_i/s_i \) in \( T \backslash t_i \), so \( T \backslash t_i \) is a base of \( M \backslash t_i/s_i \). By the induction hypothesis, \( M \backslash t_i/s_i \) is positive definite. However, \( K \not\subseteq \{s_i\} \), since \( S \) is a base of \( M \), so \( K \backslash \{s_i, t_i\} \) is a nonempty cycle of \( M \backslash t_i/s_i \). Moreover, \( K \backslash \{s_i, t_i\} \) is sign reversing. Thus we have a contradiction. To establish the basis for the induction, note that there are three symmetric oriented matroids with \( |S| = 1 \). The oriented matroid represented by \((1,0)\) is positive semidefinite, singular, and not positive definite. \( M(1,-1) \) is both positive definite and positive semidefinite, and it is nonsingular. \( M(1,1) \) is neither positive definite nor positive semidefinite.

From this result we can also conclude that if \( M \) is a symmetric nonsingular oriented matroid, then \( M \) is positive definite iff \( M^{-1} \) is positive definite.

The theorem and lemmas from this section correspond to well known theorems about symmetric matrices. If \( A \) is a positive semidefinite symmetric matrix and \( a_{ii} = 0 \) for some \( i \), then \( a_{ij} = a_{ji} = 0 \) for all \( j \) (Lemma III.1.1). If \( A \) is positive definite, then so is \( A^{-1} \), and the diagonal elements of both \( A \) and \( A^{-1} \) are positive (Lemma III.1.2). Deleting row \( i \) and column \( i \) from a symmetric positive (semi) definite matrix \( A \) yields a symmetric positive (semi) definite matrix, and pivoting on a nonzero diagonal element of \( A \) followed by deleting the pivot row and column yields another symmetric positive (semi)
definite matrix (Lemma III.1.3). Finally, a symmetric positive (semi)definite matrix is positive definite iff it is nonsingular (Theorem III.1.4).

III.2 Eigencycles

Let A be a real n×n matrix. A real number λ is an eigenvalue of A iff there is a nonzero x ∈ R^n such that λx = Ax, or equivalently, iff there is a nonzero x such that (I,-A)(λx,x) = 0. This fact motivates the following definitions. Let M be an oriented matroid on E = S ∪ T, S = {s_1, ..., s_n}, T = {t_1, ..., t_n}, S a base of M. A nonempty cycle K of M is called a positive eigencycle of M iff we have s_i ∈ K⁺ ↔ t_i ∈ K⁺ and s_i ∈ K⁻ ↔ t_i ∈ K⁻ for all i. K is called a negative eigencycle iff we have s_i ∈ K⁺ ↔ t_i ∈ K⁻ and s_i ∈ K⁻ ↔ t_i ∈ K⁺. K is a zero eigencycle iff K ⊆ T. A positive (negative, zero) eigencycle K will be called a minimal eigencycle if there is no positive (negative, zero) eigencycle K' contained in K and different from it. In the representable case, where M = M(I,-A), if (y,x) is in the null space of (I,-A), x must have the same signed support as y for x to be a positive eigenvector. This is, of course, not a sufficient condition. Consider, for example, the matrix

\[
A = \begin{pmatrix}
19 & 10 & 4 \\
10 & 10 & -14 \\
4 & -14 & 7
\end{pmatrix}
\]

The eigenvectors of this matrix are (1,-2,-2), (-2,1,-2), and (-2,-2,1), corresponding to the eigenvalues -9, +18, and +27 respectively. The
vector \((52,16,1,2,1,1)\) is in the null space of \((I,-A)\). Thus \((\{s_1,s_2,s_3,t_1,t_2,t_3\},\emptyset)\) is a positive eigencycle of \(M(I,-A)\). There are, however, no eigenvectors of \(A\) that are positive on all coordinates. These qualifications must be kept in mind when one uses the term eigencycle.

**Theorem III.2.1.** If \(M\) is a symmetric positive definite oriented matroid, then it has a positive eigencycle.

**Proof.** Consider the fundamental circuit \(C(S,t_i)\). \(M\) is positive definite, so for each \(i\), \(s_i \in C(S,t_i)^+\) (otherwise \(C(S,t_i)\) would be sign reversing). The following algorithm produces a positive eigencycle \(K\).

1. \(K = C(S,t_1)\), \(i = 1\)
2. If \(i = n\), stop
   - If \(s_{i+1} \in K^-\), then \(K = K \circ -C(S,t_{i+1})\)
     - else \(K = K \circ C(S,t_{i+1})\)
   - \(i = i+1\)
   - go to (1)

In step (1), \(K\) does not contain \(t_{i+1}\), though it may contain \(s_{i+1}\). The composition in step (1) ensures that \(\{s_{i+1},t_{i+1}\} \subseteq K^+\) or \(\{s_{i+1},t_{i+1}\} \subseteq K^-\). The algorithm will stop with a positive eigencycle \(K\) such that \(K = E\). \(\square\)
The proof must be modified for the positive semidefinite case, for it may happen that \( s_i \notin C(S,t_i) \) for some \( i \).

**Theorem III.2.2.** If \( M \) is a symmetric positive semidefinite oriented matroid, and there is an element of \( T \) that is not a loop of \( M \), then \( M \) has a positive eigencycle.

**Proof.** Let \( t_i \) be an element of \( T \) that is not a loop of \( M \). Then \( s_i \in C(S,t_i)^+ \) by Lemma III.1.1. The following algorithm will produce a positive eigencycle.

1. \( K \leftarrow C(S,t_i) \)
2. If there is a \( j \) such that \( s_j \in K \) but \( t_j \notin K \) then do:
   - If \( s_j \in K^+ \), let \( K \leftarrow K \circ C(S,t_j) \)
   - If \( s_j \in K^- \), let \( K \leftarrow K \circ -C(S,t_j) \)
   - go to 1

Else stop.

At each step of the algorithm, a \( t_j \) is admitted into \( K \) so that it has the same sign as \( s_j \), which is already in \( K \). \( \square \)

Note that if every element of \( T \) is a loop, then every cycle of \( M \) is in \( T \), so there can be no positive eigencycle. In that case \( M \) is represented by \((I,0)\).

From the definition, a symmetric oriented matroid with a negative eigencycle is not positive semidefinite. In fact, the converse is also true.
Theorem III.2.3. A symmetric oriented matroid $M$ is positive semidefinite iff it has no negative eigencycle.

Proof. Let $M$ be symmetric and not positive semidefinite. It suffices to show that $M$ contains a negative eigencycle. By definition, $M$ contains a strictly sign reversing cycle $K$. If \(|\{s_i, t_i\} \cup K| = 1\) for some $i$, then the member of $\{s_i, t_i\}$ that is in $K$ is called a violator in $K$. The following algorithm creates a negative eigencycle from a strictly sign reversing cycle $K$ with all of its violators in $S$.

(0) Let $K$ be any strictly sign reversing cycle with all of its violators in $S$.

(1) If there exists an $i$ such that $s_i \in K$ but $t_i \notin K$, then
   \begin{align*}
   &\text{if } s_i \in K^+, \text{ let } K \leftarrow K \circ -C(S,t_i) \\
   &\text{if } s_i \in K^-, \text{ let } K \leftarrow K \circ C(S,t_i)
   \end{align*}
   go to (1)

   If there is no such $i$, stop; $K$ is a negative eigencycle.

After every step, a strictly sign reversing $K$ is maintained with all of its violators in $S$. We need such a $K$ before we can apply the above algorithm. First, note that if $s_i \in C(S,t_i)^-$ for any $i$, then $C(S,t_i)$ suffices. Next, suppose that $s_i \notin C(S,t_i)$, $s_j \in C(S,t_j)$ for some $i,j$. By the symmetry of the $S$-tableau, $s_i \in -C(S,t_j)$. If $s_j \in C(S,t_j)^+$, then $s_i \in C(S,t_j)^+$ and $C(S,t_i) \circ -C(S,t_j)$ will be a strictly sign reversing cycle with all of its violators in $S$, and if $s_j \in C(S,t_j)^-$, $s_i \in C(S,t_j)^-$ then $C(S,t_i) \circ C(S,t_j)$ will be one.
Suppose that Theorem III.2.4 is true for symmetric oriented matroids with \(|S| = n-1\). Let \(K\) be a strictly sign reversing cycle of \(M\). If, for some \(i\), we have \(t_i \notin K\), then we have a strictly sign reversing cycle \(K \setminus \{s_i, t_i\}\) in the symmetric oriented matroid \(M \setminus t_i / s_i\). By the induction hypothesis, there must be a negative eigencycle of \(M \setminus t_i / s_i\) and therefore a strictly sign reversing cycle of \(M\) for which the only possible violator is \(s_i\), which is in \(S\). Apply the algorithm to this cycle.

Now suppose that \(t_i \in K\) for all \(i\). If \(s_i \in K\) for all \(i\), then \(K\) is a negative eigencycle. Suppose \(s_i \notin K\) for some \(i\). As shown earlier, if \(s_i \in C(S, t_i)^-\) or if \(s_i \notin C(S, t_i)\) when \(t_i\) is not a loop, we get a strictly sign reversing cycle with all of the violators in \(S\). If \(t_i\) is a loop, then \(K \setminus t_i\) is a strictly sign reversing cycle of \(M\) with \(t_i \notin K\); this was treated earlier. The last case is when \(s_i \in C(S, t_i)^+\). In that case, \(M \setminus s_i / t_i\) is a symmetric oriented matroid containing a strictly sign reversing cycle \(K \setminus t_i\). By the induction hypothesis, there then exists a negative eigencycle in \(M \setminus s_i / t_i\), implying the existence of a strictly sign reversing cycle \(K\) in \(M\) with \(t_i\) being its only possible violator. If \(t_i \notin K\), then \(K\) is a negative eigencycle. If \(t_i \in K\), then either \(K \circ C(S, t_i)\) or \(K \circ -C(S, t_i)\) is a strictly sign reversing cycle with all of its violators in \(S\). Thus, one can always find such a strictly sign reversing cycle. For the ground case of the induction, note that when \(|S| = |T| = 1\), any strictly sign-reversing cycle is necessarily a negative eigencycle.

From this result and Theorem II.1.4, we obtain
Corollary III.2.4. A symmetric oriented matroid is positive definite iff it has no nonpositive eigencycle.

The results in this section are again direct analogs to theorems of linear algebra. A symmetric positive definite matrix has a real positive eigenvalue (Theorem III.2.1). A symmetric positive semi-definite matrix has a real positive eigenvalue iff it is not the zero matrix. A symmetric matrix is positive (semi)definite iff it has no nonpositive (negative) eigenvalues (Theorem III.2.3 and Corollary III.2.4).

As noted earlier, there are limitations to what one can say about eigenvectors from only looking at the sign patterns of vectors in a subspace of $\mathbb{R}^n$. There is no way yet known of finding out which eigencycles of $M(I,-A)$ correspond to real eigenvectors of $A$ by looking only at the oriented matroid. However, the results given in this section demonstrate the possibility of extending some significant basic properties of quadratic forms to the combinatorial setting of oriented matroids.
References


References


