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A PROOF OF THE CONJECTURE THAT
THE TUKEY-KRAMER MULTIPLE COMPARISONS
PROCEDURE IS CONSERVATIVE

by

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In this paper we present a first general proof of Tukey's (1953) conjecture concerning the extension of the Tukey multiple comparisons procedure to the case of unequal sample sizes, thereby proving that the so-called Tukey-Kramer procedure is conservative in all cases. Also a brief history of the conjecture is given and some extensions and further problems concerning the procedure are discussed.

1. Introduction

Consider the usual one-way fixed effects analysis of variance (ANOVA) model

\[ X_{ij} = \mu_i + \epsilon_{ij}, \quad (1 \leq j \leq n_i, \ 1 \leq i \leq k) \]

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where the $\varepsilon_{ij}$ are independent $N(0, \sigma^2)$ random variables and $\mu_i$ is the mean of the $i^{th}$ treatment ($1 \leq i \leq k$). The $\mu_i$ and $\sigma^2$ are unknown parameters. Let $\bar{X}_i$ be the sample mean of the $i^{th}$ treatment based on $n_i$ observations ($1 \leq i \leq k$), and let $S^2$ be an unbiased estimate of $\sigma^2$ which is distributed independently of the $\bar{X}_i$ as a $\sigma^2 \chi^2_{\nu}/\nu$ random variable. Usually the ANOVA mean square error with $\nu = \sum_{i=1}^{k} (n_i - k)$ degrees of freedom is used as the estimate $S^2$.

A commonly occurring inference problem in practice is that of making simultaneous pairwise comparisons between the treatment means $\mu_i$. Tukey (1953) proposed his celebrated T-procedure to do this in the special case when all the $n_i$ are equal to a common sample size $n$ (say). This procedure can be summarized by the following probability statement which gives exact $(1-\alpha)$-level joint confidence intervals for all the differences $\mu_i - \mu_j$:

\[
(1.1) \quad P[\mu_i - \mu_j \in [\bar{X}_i - \bar{X}_j \pm q_{(\alpha)} \frac{S}{\sqrt{n}} ]; \quad 1 \leq i,j \leq k] = 1-\alpha
\]

where $q_{(\alpha)}$ is the upper $\alpha$ point of the Studentized range distribution with parameter $k$ and $\nu$ degrees of freedom (Miller, 1966, p. 38).

When the $n_i$ are unequal, Tukey (1953) suggested that (1.1) be modified by replacing $1/\sqrt{n}$ by $\{(1/n_i + 1/n_j)/2 \}^{1/2}$ in the confidence interval for $\mu_i - \mu_j$ ($1 \leq i,j \leq k$). He made the conjecture (which we shall refer to as the Tukey conjecture) that this procedure "...is apparently in the conservative direction..." (Tukey, 1953, p. 39), i.e.

\[
(1.2) \quad P[\mu_i - \mu_j \in [\bar{X}_i - \bar{X}_j + q_{(\alpha)} \frac{S}{\sqrt{\frac{1}{n_i} + \frac{1}{n_j}}}; \quad 1 \leq i,j \leq k] \geq 1-\alpha
\]

for all values of the $n_i$. If the $n_i$ are equal then, of course, we have
equality in (1.2) because of (1.1). Kramer (1956) independently proposed the same modification, though in the slightly different context of a multiple range test procedure, and so (1.2) is referred to as the **Tukey-Kramer (TK)-procedure**.

Over the past thirty years many statisticians have attempted to prove or disprove the Tukey conjecture but its general resolution has remained an unsolved problem. Nevertheless, despite the uncertainty associated with the Tukey conjecture, the TK-procedure is widely preferred to many other procedures because of its intuitive appeal and because it provides shorter intervals. In this paper we offer a first general proof of the validity of the TK-procedure.

We now give a brief history of the problem. In a doctoral dissertation under Tukey's supervision, Kurtz (1956) proved the inequality (1.2) when $k = 3$ and for nearly equal $n_i$'s when $k = 4$. (The case $k = 2$ trivially gives equality in (1.2).) He also studied certain limiting cases (involving highly unequal $n_i$'s) for arbitrary $k$, and found the conjecture to be true in these cases. Based on these results he expressed "a strong feeling" for the truth of the conjecture for all $k$. Later, Miller (1966, p. 87) advised against the use of the TK-procedure, describing it as "inexact" and as having "no mathematical proof or numerical substantiation." Instead Miller suggested using Scheffé's (1953) S-procedure or the classical Bonferroni procedure, neither of which requires the assumption of equal $n_i$'s. However, these procedures are rather conservative - especially the former if interest is confined to pairwise comparisons of the means. This situation prompted many authors to develop other procedures to deal with the case of unequal sample sizes,
e.g. Spjøtvoll and Stoline (1973), Dunn (1974) and Hochberg (1974a, 1975). However, the TK-procedure provides shorter intervals than these other procedures and so it became crucial to settle the validity of (1.2). For this purpose Dunnett (1980) carried out an extensive simulation study which provided a quite strong indication of the truth of the conjecture for all k. Inspired by these simulation results, Brown (1979) succeeded in proving the conjecture for the cases k = 3, 4 and 5. Based on all this evidence, Stoline (1981) concluded that "For all practical purposes, the TK method is conservative." The proof presented in this paper finally removes any uncertainty concerning the use of the TK-procedure.

The proof of the Tukey conjecture is given in Section 2 of this paper and a technical lemma required in this proof is given in the Appendix. Some extensions and further problems relating to the TK-procedure are discussed in Section 3.

2. The Main Result

By conditioning on S it is apparent that the inequality (1.2) follows from the theorem stated below.

**Theorem**

Let \( X_i \sim N(0, \sigma_i^2) \), \( 1 \leq i \leq k \), be independent where \( 0 < \sigma_i < \infty \), and let \( \xi_{i,j} = q(\sigma_i^2 + \sigma_j^2)^{1/2} \) for some fixed \( q > 0 \) (\( 1 \leq i, j \leq k \)). Then the function

\[
F = F(\sigma_1, \ldots, \sigma_k) = P\{ |X_i - X_j| \leq \xi_{i,j}; 1 \leq i, j \leq k \}
\]

is strictly minimized when the \( \sigma_i \) are equal.
Remark

Note that when the $\sigma_i$ are equal, $F$ is independent of their value. Also, since $F$ is strictly minimized, there will be strict inequality in (1.2) when the $n_i$ are not equal.

The proof of the theorem is rather long and so it is broken into several lemmas. We first explain the main idea and the steps involved in the proof so that it will be easier to follow. The proof depends upon the fact that if we take the partial derivative of $F$ with respect to one of the $\sigma_i$'s, $\sigma_k$ say, then this may be written as the sum of $k-1$ terms, $G_i$, $1 \leq i \leq k-1$ (Lemma 1) where the sign of each $G_i$ depends only on the sign of $\sigma_i - \sigma_k$ (Lemma 4). Lemma 2 shows that the $G_i$, as defined in Lemma 1, can be written as the integral over $\mathbb{R}$ of an odd function multiplied by another function $H_i(r)$ (see (2.6) in Lemma 4). The functions $H_i(r)$ involve integrals over intervals $R_{ij}$ which are investigated in Lemma 3. In particular, Lemma 3 shows that the sign of the midpoints of these intervals depend only on the sign of $\sigma_i - \sigma_k$, and using this result, Lemma 5 in the Appendix shows that the sign of $H_i(r) - H_i(-r)$ is the same for all $r > 0$ and depends only on the sign of $\sigma_i - \sigma_k$. Lemma 4 then follows from this last result and Lemma 2.

Proof of the Theorem

Let

$$\psi_k(x_1, \ldots, x_k) = \prod_{i=1}^{k} f_{\sigma_i}(x_i) \mathbb{I}[|x_i - x_j| \leq \xi_{ij}; 1 \leq i, j \leq k]$$

where $f_{\sigma_i}(\cdot)$ is the density function of a $N(0, \sigma_i^2)$ random variable and $\mathbb{I}\{A\}$ is the indicator function of the set $A$. The subscript $k$ on $\psi$ refers to the dimension of its domain. Then
\[ F = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \psi_k(x_1, \ldots, x_k) \, dx_1 \cdots dx_k \]

\[ = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \psi_{k-1}(x_1, \ldots, x_{k-1}) f_{\sigma_k}(x_k) \]

\[ \times \left( \prod_{i=1}^{k-1} I\{ |x_i - x_i^*| < \xi_{ik} \} \right) \, dx_1 \cdots dx_k. \]

Substituting \( y_k = x_k / \sigma_k \) gives

\[ (2.1) \quad F = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (2\pi)^{-1/2} \exp\left\{ -\frac{y^2_k}{2} \right\} \]

\[ \times \psi_{k-1}(x_1, \ldots, x_{k-1}) \, dx_1 \cdots dx_{k-1} \, dy_k. \]

Lemma 1

For \( 1 \leq i \leq k-1 \), define

\[ x_i^p = y_i \sigma_k + \xi_{ik}, \]

\[ N_i = \{ j : 1 \leq j \leq k-1, j \neq i \}, \]

and the set \( V_i(y_k) \subset \mathbb{R}^{k-2} \) as

\[ V_i(y_k) = \{ (x_j : j \in N_i) : |x_j - y_i \sigma_k| < \xi_{jk}, \forall j \in N_i \}. \]

Then
\[
\frac{\partial F}{\partial \sigma_k} = \left(\frac{2}{\pi}\right)^{1/2} k^{-1} \sum_{i=1}^{k-1} G_i
\]

where

\[
G_i = \int_{y_k^{\infty}}^{\infty} \int_{V_1(y_k)} \cdots \int_{V_1(y_k)} \exp\left\{-\frac{y_k^2}{2}\right\} \psi_{k-1}(x_1, \ldots, x_i^*, \ldots, x_{k-1})
\]

\[
\times [y_k + \frac{\partial \xi_{ik}}{\partial \sigma_k} (\prod_{j \in N_i} dx_j)] \, dy_k.
\]

**Proof of Lemma 1**

It follows from (2.1) that

\[(2.2) \quad \frac{\partial F}{\partial \sigma_k} = (2\pi)^{-1/2} \int_{y_k^{\infty}}^{\infty} \exp\left\{-\frac{y_k^2}{2}\right\} D \, dy_k
\]

where

\[
D = \frac{\partial}{\partial \sigma_k} \int_{y_k^{\infty}}^{y_k+\xi_{1k}} \psi_{k-1}(x_1, \ldots, x_{k-1}) \, dx_1 \cdots dx_{k-1}.
\]

Then if we let \(x_{i}^{**} = y_k \sigma_k - \xi_{ik}\), we obtain

\[
D = \sum_{i=1}^{k-1} \left\{ \int_{V_1(y_k)} \psi_{k-1}(x_1, \ldots, x_i^*, \ldots, x_{k-1}) [y_k + \frac{\partial \xi_{ik}}{\partial \sigma_k} (\prod_{j \in N_i} dx_j)]
\]

\[- \int_{V_1(y_k)} \psi_{k-1}(x_1, \ldots, x_i^{**}, \ldots, x_{k-1}) [y_k - \frac{\partial \xi_{ik}}{\partial \sigma_k} (\prod_{j \in N_i} dx_j)] \right\}
\]

\[= \sum_{i=1}^{k-1} \{A_i - B_i\} \text{ (say).}
\]
If in $B_i$ we make the substitutions $y_k' = -y_k$ and $x_j' = -x_j$, $j \in N_i$; and since $\psi_{k-1}(x_1, \ldots, x_{k-1}) = \psi_{k-1}(-x_1, \ldots, -x_{k-1})$, we see that $B_i = -A_i$, $1 \leq i \leq k-1$. Therefore

$$D = 2 \sum_{i=1}^{k-1} A_i$$

$$= 2 \sum_{i=1}^{k-1} \int_{V_i(y_k')} \psi_{k-1}(x_1, \ldots, x_i, \ldots, x_{k-1}) [y_k + \frac{3 \xi_{ik}}{\sigma_k}] (\prod_{j \in N_i} dx_j).$$

Lemma 1 now follows by putting this expression for $D$ in (2.2) and exchanging the order of the summation over $i$ and the integration over $y_k$.  

Lemma 2

For $1 \leq i \leq k-1$, the quantity $G_i$ defined in Lemma 1 can be expressed as

$$G_i = a_i \int_{r=-\infty}^{\infty} r \exp\{-\frac{1}{2} (\frac{1}{\sigma_i^2} + \frac{1}{\sigma_k^2}) r^2\} \int_{x_j \in R_{ij}} \psi_{k-2}(x_j : j \in N_i) (\prod_{j \in N_i} dx_j) dr$$

where $a_i > 0$ is some constant and the set $R_{ij} \subset \mathbb{R}$ is defined by

$$\{x_j : |x_j - r| + \frac{\sigma_i^2 \xi_{ik}}{\sigma_i^2 + \sigma_k^2} \leq \xi_{jk}\} \cap \{x_j : |x_j - r - \frac{\sigma_i^2 \xi_{ik}}{\sigma_i^2 + \sigma_k^2}| \leq \xi_{ij}\}$$

$$\equiv \{x_j : x_j - r \in R_{ij}\}.$$

Proof of Lemma 2

Notice that:

(i) $\psi_{k-1}(x_1, \ldots, x_i, \ldots, x_{k-1}) = \psi_{k-2}(x_j : j \in N_i) \times (2\pi \sigma_i^2)^{-1/2}$

$$\times \exp\{-\frac{1}{2\sigma_i^2} (y_k - \sigma_k \xi_{ik})^2\} (\prod_{j \in N_i} I_{\{x_j - y_k \sigma_k \xi_{ik} < \xi_{ij}\}}).$$
(ii) \( \exp\left\{ -\frac{1}{2} y_k^2 \right\} \times \exp\left\{ -\frac{1}{2} \sigma_1^2 (y_k \sigma_k + \xi_{ik})^2 \right\} \]
\[= \exp\left\{ -\frac{1}{2} \left( 1 + \frac{\sigma_k^2}{\sigma_1^2} \right) \sigma_1^2 (y_k \sigma_k + \xi_{ik})^2 \right\} \times \exp\left\{ -\frac{1}{2} \sigma_k^2 \right\} ; \]

and

(iii) \( \frac{\partial \xi_{ik}}{\partial \sigma_k} = \frac{3}{\sigma_k^2} \{ q(\sigma_1^2 + \sigma_k^2) \}^{1/2} \]
\[= \frac{\sigma_k \xi_{ik}}{\sigma_1^2 + \sigma_k^2} . \]

Then, writing

\[ z = y_k + \frac{\sigma_k \xi_{ik}}{\sigma_1^2 + \sigma_k^2} \]

we use (i), (ii), (iii), and the definition of \( G_i \) in Lemma 1 to obtain,

for \( 1 \leq i \leq k-1 \)

\[ (2.3) \quad G_i = \exp\left\{ -\frac{q_2^2}{2} \right\} \int_{y_k=-\infty}^{\infty} \int \ldots \int (2\pi \sigma_1^2)^{-1/2} \left\{ \text{I}\left( \frac{z}{\sigma_1^2} \right) \right\} \exp\left\{ -\frac{1}{2} \left( 1 + \frac{\sigma_k^2}{\sigma_1^2} \right) z^2 \right\} \]
\[\times \psi_{k-2}(x_j : j \in N_1)( \prod_{j \in N_1^i} \text{I}\left( |x_j - y_k \sigma_k - \xi_{ik}| \leq \xi_{ij} \right) ) \left( \prod_{j \in N_1} dx_j \right) dy_k . \]

If we make the further substitution \( r = \sigma_k z \), we have:

\[ (1 + \frac{\sigma_k^2}{\sigma_1^2}) z^2 = r^2 (\frac{1}{\sigma_1^2} + \frac{1}{\sigma_k^2}) , \]
\[ z \left( dx_k \right) = \frac{r \left( dr \right)}{\sigma_k^2} \]
\[ x_j - y_k \sigma_k^0 = x_j - r + \frac{\sigma_1^2 \xi_{1k}}{\sigma^2_1 + \sigma^2_k}, \]

and

\[ x_j - y_k \sigma_k^0 - \xi_{1k} = x_j - r - \frac{\sigma_1^2 \xi_{1k}}{\sigma^2_1 + \sigma^2_k}. \]

Substituting these expressions in (2.3) we obtain

\[ G_i = (2\pi)^{-1/2}(\sigma_k^2 \sigma_i^2)^{-1} \exp\left\{ -\frac{q^2}{2} \right\} \int_{-\infty}^{\infty} r \exp\left\{ -\frac{1}{2} \left( \frac{1}{\sigma_i^2} + \frac{1}{\sigma_k^2} \right) r^2 \right\} \]

\[ \times \int \ldots \int \psi_{k-2}(x_j: j \in N_i) \]

\[ \times \left( \prod_{j \in N_i} \left[ \{ |x_j - r + \frac{\sigma_1^2 \xi_{1k}}{\sigma^2_1 + \sigma^2_k}| < \xi_{ij} \} \cap \{ |x_j - r - \frac{\sigma_1^2 \xi_{1k}}{\sigma^2_1 + \sigma^2_k}| < \xi_{ij} \} \right] \right) \]

\[ \times \left( \prod_{j \in N_i} \, dx_j \right) \, dr. \]

Then for \( a_i = (2\pi)^{-1/2}(\sigma_k^2 \sigma_i^2)^{-1} \exp\left\{ -\frac{q^2}{2} \right\} > 0 \), the expression for \( G_i \) given in this Lemma follows by the definition of \( R_{ij} \).

\( \square \)

**Lemma 3**

For \( 1 \leq i \leq k-1, j \in N_i \), the set \( R_{ij} \) defined in Lemma 2 is the interval

\[ R_{ij} = \left[ \frac{\sigma_1^2 \xi_{1k} \sigma_1^2}{\sigma^2_1 + \sigma^2_k} - \xi_{ij}, \frac{-\sigma_1^2 \xi_{1k} \sigma_1^2}{\sigma^2_1 + \sigma^2_k} + \xi_{jk} \right] \]
which has length \( > 0 \).

Also, if the midpoint of \( R_{ij} \) is denoted by \( m_{ij} \), then \( \forall j \in N_i \), \( 1 \leq i \leq k-1 \),

\[
\sigma_i < \sigma_k \iff m_{ij} < 0.
\]

Proof of Lemma 3

Let

\[
I_1 = \left[ \frac{\sigma_i^2 \xi_{ik}}{\sigma_i^2 + \sigma_k^2} - \xi_{ij}, \frac{\sigma_i^2 \xi_{ik}}{\sigma_i^2 + \sigma_k^2} + \xi_{ij} \right]
\]

and

\[
I_2 = \left[ \frac{-\sigma_k^2 \xi_{ik}}{\sigma_i^2 + \sigma_k^2} - \xi_{jk}, \frac{-\sigma_k^2 \xi_{ik}}{\sigma_i^2 + \sigma_k^2} + \xi_{jk} \right].
\]

Then \( R_{ij} = I_1 \cap I_2 \). Observe that:

\[
(1) \quad (\sigma_i^2 + \sigma_j^2)^{1/2} > |\sigma_i - \sigma_k| > |(\sigma_i^2 + \sigma_j^2)^{1/2} - (\sigma_k^2 + \sigma_j^2)^{1/2}|
\]

and so

\[
(2.4) \quad \xi_{ik} > |\xi_{ij} - \xi_{jk}|.
\]

Suppose that \( I_1 \supset I_2 \), then

\[
\frac{\sigma_i^2 \xi_{ik}}{\sigma_i^2 + \sigma_k^2} - \xi_{ij} \leq \frac{-\sigma_k^2 \xi_{ik}}{\sigma_i^2 + \sigma_k^2} - \xi_{jk}
\]

\[
\iff \xi_{ik} \leq \xi_{ij} - \xi_{jk}
\]
which contradicts (2.4). Similarly, if $I_2 \supseteq I_1$, then

$$\frac{-\sigma_k^2 \xi_{1k}}{\sigma_1^2 + \sigma_k^2} + \frac{\sigma_k^2 \xi_{1k}}{\sigma_1^2 + \sigma_k^2} > \frac{\sigma_k^2 \xi_{1k}}{\sigma_1^2 + \sigma_k^2} + \xi_{ij}$$

$$\xi_{jk} - \xi_{ij} > \xi_{ik}$$

which again contradicts (2.4). So neither of $I_1$ or $I_2$ is contained in the other. Also,

(ii) $$(\sigma_1^2 + \sigma_j^2)^{1/2} + (\sigma_k^2 + \sigma_j^2)^{1/2} > \sigma_1 + \sigma_k > (\sigma_1^2 + \sigma_k^2)^{1/2}$$

$$\Rightarrow \quad \xi_{ij} + \xi_{jk} > \xi_{ik}$$

$$\Rightarrow \quad \frac{\sigma_1^2 \xi_{ik}}{\sigma_1^2 + \sigma_k^2} - \frac{-\sigma_k^2 \xi_{ik}}{\sigma_1^2 + \sigma_k^2} > \frac{\sigma_k^2 \xi_{ik}}{\sigma_1^2 + \sigma_k^2} + \xi_{jk}$$

and so $I_1$ and $I_2$ are not disjoint.

Together (i) and (ii) imply that $R_{ij}$ is as stated in this Lemma, and the strict inequality in (ii) means that $R_{ij}$ has length $> 0$.

Turning now to the second half of the Lemma, we have

$$2m_{ij} = \left(\frac{\sigma_1^2 - \sigma_k^2}{\sigma_1^2 + \sigma_k^2}\right) \xi_{ik} - \xi_{ij} + \xi_{jk}$$

so that

$$\frac{2m_{ij}}{q} = \frac{\sigma_1^2 - \sigma_k^2}{(\sigma_1^2 + \sigma_k^2)^{1/2}} - [(\sigma_1^2 + \sigma_j^2)^{1/2} - (\sigma_k^2 + \sigma_j^2)^{1/2}].$$
Immediately we see that if $\sigma_i = \sigma_k$, then $m_{ij} = 0 \quad \forall j \in N_i$.

If $\sigma_i \neq \sigma_k$, then a small amount of algebra leads to the following inequality:

$$\left| \frac{\sigma_i^2 - \sigma_k^2}{(\sigma_i^2 + \sigma_k^2)^{1/2}} \right| > \left| (\sigma_i^2 + \sigma_j^2)^{1/2} - (\sigma_k^2 + \sigma_j^2)^{1/2} \right|.$$ 

Therefore it follows from (2.5) that $m_{ij}$ has the same sign as $(\sigma_i^2 - \sigma_k^2)$, i.e. $m_{ij} > 0$ for $\sigma_i > \sigma_k, \forall j \in N_i$. □

**Lemma 4**

For the quantity $G_i$ in Lemma 2, $1 \leq i \leq k-1$, we have

$$\sigma_k > \sigma_i \iff G_i > 0.$$ 

**Proof of Lemma 4**

From Lemma 2 we have that

$$G_i = a_i \int_{-\infty}^{\infty} r \exp\left\{-\frac{1}{2} \left(\frac{1}{\sigma_i} + \frac{1}{\sigma_k}r^2\right)\right\} H_i(r) \, dr$$

where

$$H_i(r) = \int \cdots \int \psi_{k-2}(x_j; j \in N_i)(\prod_{j \in N_i} dx_j)$$

$$= \int \cdots \int (\prod_{j \in N_i} f_j(x_j))I\{||x_j - x_m| < \sigma_j, \forall l, m \in N_i\}(\prod_{j \in N_i} dx_j).$$

$$j \in N_i$$
By transforming from $x_j$ to $x_j + r$ we can rewrite this last equation as

$$
(2.7) \quad H_i(r) = \int \ldots \int \left( \prod_{j \in N_i} f_{\sigma_j} (r+x_j) \right) I \left\{ \left| x_j - x_j^{m} \right| \leq \varepsilon_j ; \forall \varepsilon, m \in N_i \right\} \prod_{j \in N_i} dx_j.
$$

Also by transforming from $x_j$ to $-x_j$ in (2.7), and since $f_{\sigma_j} (x) = f_{\sigma_j} (-x)$, it follows that

$$
(2.8) \quad H_i(-r) = \int \ldots \int \left( \prod_{j \in N_i} f_{\sigma_j} (r+x_j) \right) I \left\{ \left| x_j - x_j^{m} \right| \leq \varepsilon_j ; \forall \varepsilon, m \in N_i \right\} \prod_{j \in N_i} dx_j.
$$

Note that $H_i(r) > 0$.

To prove this Lemma we now use the results of Lemma 3 above, and Lemma 5 and its Corollary which are contained in the Appendix. In the case $\sigma_i = \sigma_k$ we have by Lemma 3 that $m_{ij} = 0 \quad \forall j \in N_i$, so $R_{ij} = -R_{ij} \quad \forall j \in N_i$, and hence we see from (2.7) and (2.8) that $H_i(r) = H_i(-r)$. It then follows from (2.6) that $G_i = 0$.

In the case $\sigma_i > \sigma_k$ we have by Lemma 3 that $m_{ij} > 0 \quad \forall j \in N_i$. Then a direct application of Lemma 5 to equations (2.7) and (2.8) gives

$$
H_i(r) < H_i(-r) \quad \text{for} \quad r > 0.
$$

In fact we have strict inequality, $H_i(r) < H_i(-r)$, because the conditions of the Corollary to Lemma 5 are satisfied, namely:

(a) $m_{ij} > 0$ and $R_{ij}$ has length $> 0$, $\forall j \in N_i$ (see Lemma 3),
and

(b) for $\forall \varepsilon, m \in N_i$ we have
\[- \frac{\sigma^2_{ik}}{\sigma^2_i + \sigma^2_k} \xi_{ik} + \frac{\sigma^2_{kj}}{(\sigma^2_i + \sigma^2_k)} \xi_{jk} \right) - \left( \frac{\sigma^2_{ik}}{\sigma^2_i + \sigma^2_k} + \xi_{mk} \right) \]

\[= |\xi_{ik} - \xi_{mk}| < \xi_{m} \text{ (see (2.4)).} \]

Then since \(0 \leq H_i(r) < H_i(-r)\) for \(r > 0\), and \(a_i > 0\), it follows from (2.6) that \(G_i < 0\).

When \(\sigma_i < \sigma_k\) we have by Lemma 3 that \(m_{ij} < 0 \forall j \in N_i\), and so in a similar way it follows from Lemma 5 that \(0 \leq H_i(-r) < H_i(r)\) for \(r > 0\), and hence from (2.6) it follows that \(G_i > 0\). \(\square\)

We now complete the proof of the theorem. It follows from Lemmas 1 and 4 that \(\frac{\partial F}{\partial \sigma_k}\) can be expressed as

\[\frac{\partial F}{\partial \sigma_k} = \sum_{i=1}^{k-1} b_{ki}\]

for some \(b_{ki}\) which satisfy

\[b_{ki} > 0 \iff \sigma_k > \sigma_i.\]

But \(\sigma_k\) was arbitrarily chosen from among the \(\sigma_i's\), so more generally, for \(1 \leq i \leq k\), \(\frac{\partial F}{\partial \sigma_i}\) can be expressed as

\[\frac{\partial F}{\partial \sigma_i} = \sum_{j=1}^{k} b_{ij},\]

for some \(b_{ij}\) which satisfy
\[ b_{ij} \leq 0 \iff \sigma_i \leq \sigma_j. \]

This leads to the required result that \( F \) has a strict minimum when all the \( \sigma_i \) are equal, as we now show.

Let \( \sigma_1 \leq \sigma_2 \leq \ldots \leq \sigma_k \) denote the ordered \( \sigma_i \)'s. If we

(1) increase \( \sigma_1 \) to \( \sigma_2 \),
(2) increase \( \sigma_1 \) and \( \sigma_2 \) to \( \sigma_3 \) (keeping \( \sigma_1 = \sigma_2 \)),

\[ \vdots \]

\[ \vdots \]

(k-1) increase \( \sigma_1, \ldots, \sigma_{k-1} \) to \( \sigma_k \) (keeping \( \sigma_1 = \ldots = \sigma_{k-1} \)),

then \( F \) will be strictly decreased at each step where an increase is necessary (and there will be such a step unless the \( \sigma_i \) are all equal). This completes the proof of the theorem. \( \square \)

3. Extensions of the Tukey-Kramer Procedure

It is sometimes of interest to have a procedure which gives joint confidence intervals for all contrasts of the treatment means, i.e. for all parametric functions \( \sum_{i=1}^{k} c_i \mu_i \) where \( \sum_{i=1}^{k} c_i = 0 \). This is useful as it provides confidence intervals for any group of contrasts which are selected for consideration after looking at the data ("data-snooping").

We can extend the TK-procedure to do this by using the following result of Tukey (1953).

Let \( C^k \) be the set of all \( k \)-dimensional contrasts,

\[ C^k = \{ c = (c_1, \ldots, c_k) : \sum_{i=1}^{k} c_i = 0 \} \subset \mathbb{R}^k. \]
Then for any real vector \((y_1, \ldots, y_k)\) and non-negative numbers \(\xi_{ij}\) satisfying \(\xi_{ij} = \xi_{ji}, 1 \leq i, j \leq k,\)

\[
|y_i - y_j| \leq \xi_{ij}
\]

\(1 \leq i, j \leq k\)

(3.1)

\[
\Ll| \sum_{i=1}^{k} c_i y_i \Ll| \leq \frac{\sum_{i=1}^{k} \sum_{j=1}^{k} c_i^+ c_j^- \xi_{ij}}{\frac{1}{2} \sum_{i=1}^{k} |c_i|} \quad \forall \xi \in C^k
\]

where \(c_i^+ = \max(c_i, 0)\) and \(c_j^- = -\min(c_j, 0)\). Therefore if we let

\[
\xi_{ij} = q_k(\alpha) S\left[\frac{1}{2} \left( \frac{1}{n_i} + \frac{1}{n_j} \right) \right]^{1/2}
\]

and

\[
y_i = \overline{x}_i - \mu_i
\]

it follows from (1.2) that

(3.2)

\[
P\left\{ \sum_{i=1}^{k} c_i \overline{x}_i \in \left[ \sum_{i=1}^{k} c_i \overline{x}_i \pm q_k(\alpha) S \left( \sum_{i=1}^{k} \sum_{j=1}^{k} c_i^+ c_j^- \left( \frac{1}{n_i} + \frac{1}{n_j} \right) \right)^{1/2} \right] \right\}; \forall \xi \in C^k \geq 1-\alpha.
\]

However, while it is true that when only pairwise differences of the treatment means are considered, the TK-procedure is preferred to other procedures because it provides shorter intervals, this may no longer be true when we consider all contrasts of the treatment means. In particular, Scheffé's (1953) S-procedure (Miller, 1966, p. 49) which provides exact \((1-\alpha)\)-level joint confidence intervals for all contrasts may be preferred in this case.
A further extension of the TK-procedure is to designs other than the one-way layout, when estimates of the $\mu_i$, $\hat{\mu}_i$ say, are not independent (e.g. the one-way analysis of covariance model). Suppose that the vector $\hat{\bar{\mathbf{u}}} = (\hat{\mu}_1, \ldots, \hat{\mu}_k)$ has a multivariate normal distribution with mean vector $\bar{\mathbf{u}} = (\mu_1, \ldots, \mu_k)$ and covariance matrix $\sigma^2 \bar{\mathbf{V}}$, where $\sigma^2$ is unknown and $\bar{\mathbf{V}} = (v_{ij})$ is a known $(k \times k)$ positive-definite, symmetric matrix. It was proposed independently by Tukey (1953) and Kramer (1957) that in this case, $1/\sqrt{n}$ in (1.1) be replaced by $[(v_{ii} + v_{jj} - 2v_{ij})/2]^{1/2}$ in the confidence interval for $\mu_i - \mu_j$. Tukey conjectured that this procedure is also conservative, i.e.

$$\Pr \{ \mu_i - \mu_j \in [\hat{\mu}_i - \hat{\mu}_j + t_{k-1}^{(\alpha)} S[\frac{1}{2} (v_{ii} + v_{jj} - 2v_{ij})]^{1/2}] ; 1 \leq i, j \leq k \} \geq 1 - \alpha.$$  

Notice that if $\bar{\mathbf{V}}$ is diagonal and if $v_{ii} = 1/n_i$ then (3.3) reduces to (1.2).

Also, using (3.1), (3.3) can be extended to the case of all contrasts of the $\mu_i$.

In the case $k = 3$, (3.3) has been proved by Brown (1982). Also Hochberg (1974b) has shown that whenever $v_{ii} + v_{jj} - 2v_{ij} = v, \forall i, j$, for some constant $v$ (i.e. the variance of $\hat{\mu}_i - \hat{\mu}_j$ is the same for all $i, j$), then there is equality in (3.3) for all $k$. The question of the general validity of (3.3) remains an unsolved problem.
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Appendix

In Lemma 5 and its proof we use the following notation. For any set $A$, $-A = \{x: -x \in A\}$; and if $B \subseteq A$ then $A-B = \{x: x \in A, x \notin B\}$. Also if $A$ is a finite set then $|A|$ denotes the number of elements in $A$. Finally $\mathbb{N}$ is the set of natural numbers $\{1, 2, 3, \ldots\}$.

Lemma 5.

For a finite set $J \subseteq \mathbb{N}$, and for some fixed $r > 0$, $\delta_{ij} > 0$ and $\tau_i > 0$, define

$$g_J \left( x_i: i \in J \right) = \left( \prod_{i \in J} f_{\tau_i} \left( x_i + r \right) \right) I_{\{x_i - x_j \leq \delta_{ij}; \forall i, j \in J\}}$$

where $f_{\tau_i}(x)$ is the density of a $N(0, \tau_i^2)$ random variable. Also for some fixed $d_i > 0$ and $m_i \in \mathbb{R}$, define for $n \in \mathbb{N}$,

$$A_n = \{(x_1, \ldots, x_n): m_i - d_i \leq x_i \leq m_i + d_i, \ 1 \leq i \leq n\} \subseteq \mathbb{R}^n$$
so that

\[-A_n = \{(x_1, \ldots, x_n) : -m_i - d_i \leq x_i \leq -m_i + d_i, \ 1 \leq i \leq n\} \subset \mathbb{R}^n.\]

Then if \(m_i > 0\) for \(1 \leq i \leq n\) we have

\[(A1) \quad \int_{A_n} g_n(x_1, \ldots, x_n) dx_1 \ldots dx_n \leq \int_{A_n} g_n(x_1, \ldots, x_n) dx_1 \ldots dx_n \]

for all \(n \in \mathbb{N}\).

Proof of Lemma 5.

We shall prove \((A1)\) by induction on \(n\). For \(m_1, d_1 > 0\),

\[
\int_{m_1 - d_1}^{m_1 + d_1} f_{\tau_1}(x_1 + r) dx_1 \leq \int_{-m_1 - d_1}^{-m_1 + d_1} f_{\tau_1}(x_1 + r) dx_1,
\]

so \((A1)\) is true for \(n = 1\).

Now assume that \((A1)\) is true for \(n = 1, 2, \ldots, k-1\) and we will show that this implies that \((A1)\) is true for \(n = k\). The proof is divided into four parts:

(i) Let

\[B_k = A_k - (A_k \cap -A_k),\]

so

\[-B_k = -A_k - (A_k \cap -A_k).\]

Then a necessary and sufficient condition for \((A1)\) in the case \(n = k\) is that

\[(A2) \quad \int_{B_k} g_k(x_1, \ldots, x_k) dx_1 \ldots dx_k \leq \int_{B_k} g_k(x_1, \ldots, x_k) dx_1 \ldots dx_k.\]

(Note: If \(m_i - d_i > 0\) for any \(i, 1 \leq i \leq k\), then \(A_k \cap -A_k = \emptyset\) and \(B_k = A_k\).)
Also if \( m_i = 0 \) for each \( i, 1 \leq i \leq k \), then \( A_k = -A_k \) and \( B_k = \emptyset \). Throughout this proof, for convenience, we define an integral over an empty set to be 0.)

(ii) Define \( D_k = \{(x_1, \ldots, x_k) \in B_k : x_i > -m_i + d_i, 1 \leq i \leq k\} \) and define \( C_k = B_k - D_k \) (so \( -C_k = -B_k - (-D_k) \)).

First note that \((x_1, \ldots, x_k) \in B_k\)

\[ \Rightarrow (x_1, \ldots, x_k) \in A_k \]

\[ \Rightarrow x_i < m_i + d_i, 1 \leq i \leq k. \]

Therefore \( D_k = \emptyset \) if any \( m_i = 0 \).

If \( D_k \neq \emptyset \) then \((x_1, \ldots, x_k) \in D_k \Rightarrow x_i > 0, 1 \leq i \leq k, \) since

\[ x_i > -m_i + d_i \] by the definition of \( D_k \)

and

\[ x_i > m_i - d_i \] by the definition of \( A_k \) (and \( D_k \subseteq A_k \)).

Therefore, because \( f_{t_1}(x_1 + r) < f_{t_1}(-x_1 + r) \) for \( x_i > 0 \), we have

\[ (A3) \quad g_k(x_1, \ldots, x_k) \leq g_k(-x_1, \ldots, -x_k) \text{ for } (x_1, \ldots, x_k) \in D_k. \]

Hence either \( D_k = \emptyset \) or

\[ \int_{D_k} \ldots \int g_k(x_1, \ldots, x_k)dx_1 \ldots dx_k < \int_{-D_k} \ldots \int g_k(x_1, \ldots, x_k)dx_1 \ldots dx_k. \]

So to show (A2) it is sufficient to show that

\[ (A4) \quad \int_{C_k} \ldots \int g_k(x_1, \ldots, x_k)dx_1 \ldots dx_k < \int_{-C_k} \ldots \int g_k(x_1, \ldots, x_k)dx_1 \ldots dx_k. \]
(iii) Let $J$ be the set of all nonempty, proper subsets of \{1, 2, ..., k\}. For each $I \in J$ let $I^\# = \{1, 2, ..., k\} - I$ and define

$$M(I) = \{(x_i: i \in I): m_i - d_i \leq x_i \leq m_i + d_i, x_i > -m_i + d_i; \forall i \in I\} \subseteq \mathbb{R}^{|I|}$$

and

$$L((x_i: i \in I)) = \{(x_j: j \in I^\#): m_j - d_j \leq x_j \leq m_j + d_j, x_j < -m_j + d_j,$$

$$|x_i - x_j| \leq \delta_{ij}; \forall i \in I, j \in I^\#\} \subseteq \mathbb{R}^{|I^\#|}.$$  

For each $(x_1, ..., x_k) \in C_k$, there exists $1 \leq i \leq k$, such that $x_i > -m_i + d_i$.

(Otherwise $(x_1, ..., x_k) \in A_k \cap -A_k$), and there exists $1 \leq j \leq k$, such that $x_j < -m_j + d_j$. (Otherwise $(x_1, ..., x_k) \in D_k$). Therefore it follows from the definitions of $M$, $L$, and $J$ that

$$\int_{C_k} \cdots \int_{C_k} g_k(x_1, ..., x_k) dx_1 ... dx_k$$

$$= \sum_{I \in J} \int_{M(I)} g_{|I|}(x_i: i \in I) \int_{L((x_i: i \in I))} g_{|I^\#|}(x_j: j \in I^\#) dx_1 ... dx_k$$

and

$$\int_{C_k} \cdots \int_{C_k} g_k(x_1, ..., x_k) dx_1 ... dx_k$$

$$= \sum_{I \in J} \int_{-M(I)} g_{|I|}(x_i: i \in I) \int_{-L((x_i: i \in I))} g_{|I^\#|}(x_j: j \in I^\#) dx_1 ... dx_k.$$ 

Suppose $(x_i: i \in I) \in M(I)$ for some $I \in J$. Then using similar arguments to those in part (ii) of this proof, we have $x_i > 0$ for each $i \in I$, and hence

$$g_{|I|}(x_i: i \in I) \leq g_{|I|}(x_i: i \in I).$$
It follows from (A5), (A6) and (A7) that to show (A4) it is sufficient to show that for each \((x_i : i \in I) \in M(I)\) for any \(I \in J\), either \(L((x_i : i \in I)) = \emptyset\) or

\[
\int_{L((x_i : i \in I))} g|_{I^*}(x_j : j \in I^*) \left( \prod_{j \in I^*} dx_j \right) \leq \int_{-L((x_i : i \in I))} g|_{I^*}(x_j : j \in I^*) \left( \prod_{j \in I^*} dx_j \right).
\]

(iv) For \((x_j : j \in I^*)\) to be in the set \(L((x_i : i \in I))\) it is necessary, by the definition of \(L\), that \(m_j - d_j \leq x_j\) and \(x_j \leq -m_j + d_j\) for each \(j \in I^*\). So if \(m_j - d_j > 0\) for any \(j \in I^*\), then \(L((x_i : i \in I)) = \emptyset\). Otherwise

\((x_j : j \in I^*) \in L((x_i : i \in I)) \iff x_j \in E_j \ \forall j \in I^*\)

where

\[
E_j = \cap_{i \in I} [x_i - \delta_{ij}, x_i + \delta_{ij}] \cap [m_j - d_j, -m_j + d_j].
\]

Now \(E_j\) is the intersection of intervals each with mid-point \(\geq 0\), so either it is empty, or it is an interval with mid-point \(> 0\). If \(E_j = \emptyset\) for any \(j \in I^*\) then \(L((x_i : i \in I)) = \emptyset\), and if \(E_j\) is an interval with mid-point \(\geq 0\) for each \(j \in I^*\), then (A8) is true by the induction assumption since \(1 \leq |I^*| \leq k - 1\).

So in all cases either \(L((x_i : i \in I)) = \emptyset\) or (A8) is true, which prove s (A1) is true at the \(k^{th}\) stage. That (A1) is true for all \(n \in \mathbb{N}\) follows by induction. \(\square\)
Corollary to Lemma 5

Suppose that (a) \( d_i, m_i > 0 \) \( \forall i \),
and (b) \( |(m_i + d_i) - (m_j + d_j)| < \delta_{ij} \) \( \forall i, j \).

Then

\[
(A9) \quad \int_{A_n} g_n(x_1, \ldots, x_n) dx_1 \ldots dx_n < \int_{-A_n} g_n(x_1, \ldots, x_n) dx_1 \ldots dx_n
\]

for all \( n \in \mathbb{N} \).

Proof of Corollary

If \( m_1, d_1 > 0 \) then

\[
\int_{m_1-d_1}^{m_1+d_1} f_{\tau_1}(x_1 + r) dx_1 < \int_{-m_1-d_1}^{-m_1+d_1} f_{\tau_1}(x_1 + r) dx_1,
\]

so (A9) is true for \( n = 1 \).

For \( n = k > 2 \), conditions (a) and (b) \( \Rightarrow \exists \delta x_i > 0, 1 \leq i \leq k \), such that

(i): \( F_k = \{(x_1, \ldots, x_k): m_i + d_i - \delta x_i < x_i < m_i + d_i, 1 \leq i \leq k\} \subseteq D_k \)
and
(ii): \((x_1, \ldots, x_k) \in F_k \Rightarrow |x_i - x_j| < \delta_{ij}, 1 \leq i, j \leq k\).

To see this, notice that (i) is satisfied if \( m_i + d_i - \delta x_i > -m_i + d_i \) and \( m_i + d_i - \delta x_i > m_i - d_i \), i.e. \( \delta x_i < 2m_i \) and \( \delta x_i < 2d_i \) for each \( i \) (which we can satisfy by condition (a)); and (ii) is satisfied if
\[ \delta x_i < \frac{1}{2} \min_{1 \leq i, j \leq k} \{ \delta_{ij} - |(m_i + d_i) - (m_j + d_j)| \} \text{ for each } i, \]

(and the right-hand side of this inequality is > 0 by condition (b)).

Now (i) \( \Rightarrow \) \( (m_1 + d_1 - \delta x_1, \ldots, m_k + d_k - \delta x_k) \in D_k \)

\( \Rightarrow m_i + d_i - \delta x_i > 0, 1 \leq i \leq k \) (see part (ii) of the proof of Lemma 5)

\[ \int_{m_i + d_i - \delta x_i}^{m_i + d_i} f_{\tau_i} (x_i + r) dx_i < \int_{-(m_i + d_i)}^{-(m_i + d_i)} f_{\tau_i} (x_i + r) dx_i, 1 \leq i \leq k. \]

Therefore, because the indicator function term of \( g_k \) is 1 everywhere in \( F_k \)

by condition (ii) above, we have

\[ (A10) \int_{F_k} \ldots \int_{F_k} g_k(x_1, \ldots, x_k) dx_1 \ldots dx_k \]

\[ = \prod_{i=1}^{k} \int_{m_i + d_i - \delta x_i}^{m_i + d_i} f_{\tau_i} (x_i + r) dx_i \]

\[ < \prod_{i=1}^{k} \int_{-(m_i + d_i)}^{-(m_i + d_i)} f_{\tau_i} (x_i + r) dx_i \]

\[ = \int_{-F_k} \ldots \int_{-F_k} g_k(x_1, \ldots, x_k) dx_1 \ldots dx_k. \]

Define \( G_k = D_k - F_k \). Since \( G_k \subseteq D_k \) it follows from (A3) that

\[ (A11) \int_{G_k} \ldots \int_{G_k} g_k(x_1, \ldots, x_k) dx_1 \ldots dx_k \leq \int_{-G_k} \ldots \int_{-G_k} g_k(x_1, \ldots, x_k) dx_1 \ldots dx_k. \]
Then (A10) and (All) imply that

\[\int_{D_k} g_k(x_1, \ldots, x_k) \, dx_1 \cdots dx_k < \int_{-D_k} g_k(x_1, \ldots, x_k) \, dx_1 \cdots dx_k.\]

Therefore the inequality (A4), which was verified as part of the proof of Lemma 5, is sufficient to show that

\[\int_{A_k} g_k(x_1, \ldots, x_k) \, dx_1 \cdots dx_k < \int_{-A_k} g_k(x_1, \ldots, x_k) \, dx_1 \cdots dx_k.\]

This completes the proof of the Corollary. \(\square\)
References


_______ (1982). Personal communication.


